ON A NEW FIXED POINT THEOREM ON HILBERT SPACES VIA THE MOUNTAIN PASS LEMMA AND APPLICATIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract: In this paper we present a new fixed point theorem in Hilbert spaces. Using a suitable critical point theorem we provide conditions under which potential operators have a fixed point. An application is given to illustrate the theory.

Key Words: fixed point theorem, potential operators

1. Introduction

Let \((H, \langle ., . \rangle)\) be a real Hilbert space and let \(\langle ., . \rangle\) denote the scalar product on
$H$ and $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ the norm. A functional $\varphi : H \to \mathbb{R}$ is said to be Gâteaux differentiable on $H$ if

$$\lim_{t \to 0} \frac{\varphi(u + th) - \varphi(u)}{t} = \langle \text{Grad} \varphi(u), h \rangle \quad \forall u, h \in H.$$ 

The operator $T : H \to H$, is called a potential operator (or a gradient operator) on $H$, if there exists a Gâteaux differentiable functional $\varphi$ such that $\text{Grad} \varphi(u) = T(u)$, for all $u \in H$.

An operator $T : H \to H$ is said to have a fixed point in $H$ if there exists $u_0 \in H$ such that $T(u_0) = u_0$. In [6] the authors rephrase the problem of searching for fixed points of a potential operator $T$ to searching for critical points of the functional $J$ with $J' = I - T$. Recall that a critical point for $J$ on $H$ is an element $u_0 \in H$ such that, $J'(u_0) = 0$. We give conditions on the potential operator $T$ so that the Mountain Pass Theorem guarantees the existence of critical points of $J$, so as a result guarantees the existence of fixed points of $T$. First, we recall some concepts from critical point theory.

**Definition 1.1.** [2, 1, 3] Let $J \in C^1(H, \mathbb{R})$. If any sequence $(u_n) \subset H$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ as $n \to +\infty$ in $H$ possesses a convergent subsequence, then we say that $J$ satisfies the Palais-Smale condition (PS condition for short).

**Theorem 1.1.** (Mountain Pass Theorem [5, 3]) Let $H$ be a Banach space and let $J \in C^1(H, \mathbb{R})$, satisfy the Palais-Smale condition. Assume $J(0) = 0$ and there exist positive numbers $\rho$, and $\alpha$ such that:

1. $J(u) \geq \alpha$ if $\|u\| = \rho$,

2. There exists $\vartheta \in H$ such that $\|\vartheta\| > \rho$ and $J(\vartheta) < \alpha$.

Then $c$ is a critical value of $J$ with $c \geq \alpha$, where $c = \min_{A \in \Gamma} \max_{u \in A} J(u)$, and

$$\Gamma = \{ \gamma([0, 1]); \gamma \in C([0, 1], H) \; \gamma(0) = 0, \; \gamma(1) = \vartheta \}.$$ 

**Theorem 1.2.** [1] Let $H$ and $Y$ be two Banach spaces, $\Omega$ an open subset of $H$, and $\varphi : \Omega \to Y$ a mapping of class $C^1$. Given $x, y \in \Omega$, if $x + ty \in \Omega$ for all $t \in (0, 1)$, then

$$\varphi(x + y) = \varphi(x) + \int_0^1 < D\varphi(x + ty), y > dt.$$ 

This result connects the potential operator $T$ and the Gâteaux differentiable functional $\varphi$ since

$$\varphi(x) = \int_0^1 \langle T(sx), x \rangle ds.$$
Following the ideas in [6], the authors in [7] used a suitable minimization principle to prove the following fixed point theorem

**Theorem 1.3.** Let $H$ be a Hilbert space and $T : H \to H$ a compact potential operator such that there exists a bounded linear operator $B$ on $H$ and $v^* \in H$ satisfying

$$\left( T(su), u \right) \leq \left( B(su), u \right) + (v^*, u), \quad \forall s \in [0, 1], \quad \forall u \in H \text{ with } \|B\| < 1. \quad (1.1)$$

Then, the operator $T$ has a fixed point in $H$.

In this paper we prove a new fixed point theorem which weakens the condition in the above theorem.

### 2. Existence Result

**Theorem 2.1.** Let $H$ be a Hilbert space and $T : H \to H$ a compact potential operator such that there exists a bounded linear operator $B$ on $H$ and a $v^* \in H'$ satisfying

1. $\left( T(su), u \right) \leq \left( B(su), u \right) + (v^*, u), \quad \forall s \in (0, 1), \quad \forall u \in H \text{ with } \|u\| = \rho_0, \text{ for some } \rho_0 > \frac{2\|v^*\|}{1-\|B\|}, \quad \|B\| < 1,$

2. $\exists e \in H \text{ with } \|e\| = 1 \text{ and } \left( B(e), e \right) > 1$ such that,

$$\left( T(se), e \right) \geq \left( B(se), e \right), \quad \forall s \in (0, 1),$$

Then, the operator $T$ has a fixed point in $H$.

**Proof.** Since $T$ is a potential operator, there exists a Gâteaux differentiable functional $\varphi : H \to \mathbb{R}$ such that $\varphi' = T$. Let $J = I - \varphi$. Then $J \in C^1(H, \mathbb{R})$ with $J' = I - \varphi' = I - T$, i.e., $\forall u \in H$, $J'(u) = u - Tu$. Since $T$ is a potential operator, it can be represented in the form

$$\varphi(u) = \int_0^1 \left( T(su), u \right) ds.$$

Consider the functional $J : H \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 \left( T(su), u \right) ds.$$
Step 1. $J$ satisfies the Palais-Smale condition. Let $(u_n)$ be a sequence in $H$ such that $\lim_{n \to +\infty} J'(u_n) = 0$ and $(J(u_n))$ is bounded. i.e., there exists some positive constant $K$ such that $|J(u)| \leq K$, for all positive integer $n$. From assumption (1), we have

$$K \geq J(u_n) \geq \frac{1}{2} ||u_n||^2 - \int_0^1 \left[ (B(su_n), u_n) + \langle v^*, u_n \rangle \right] ds$$

$$\geq \frac{1}{2} ||u_n||^2 - \int_0^1 s||u_n||^2 ||B|| ds - \int_0^1 \langle v^*, u_n \rangle ds$$

$$\geq \frac{1}{2} ||u_n||^2 - \frac{1}{2} ||u_n||^2 ||B|| - ||v^*|| ||u_n||$$

$$= \frac{1}{2} \left( 1 - ||B|| \right) ||u_n||^2 - ||v^*|| ||u_n||,$$

which implies that $(u_n)$ is bounded in $H$. We note that, $J'(u_n) = u_n - T(u_n)$, with $\lim_{n \to +\infty} J'(u_n) = 0$. Since the sequence $(u_n)$ is bounded and the operator $T$ is compact, the sequence $(T(u_n))$ is relatively compact, and as a consequence there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $T(u_{n_k}) \to w$. Hence $u_{n_k} \to w$ in $H$ (note

$$||u_{n_k} - w|| \leq ||u_{n_k} - T(u_{n_k})|| + ||T(u_{n_k}) - w|| \to 0, \text{ as } k \to +\infty.$$)

Thus, the (PS) condition is satisfied.

Step 2. $J$ satisfies assumption (1) of Theorem 1.1. Consider any $\rho_0 > \frac{2||v^*||}{1-||B||}$, and let $\alpha_0 = \frac{1}{2} \rho_0 \left( (1 - ||B||) \rho_0 - 2||v^*|| \right)$. From assumption (1), if $||u|| = \rho_0$ we have

$$J(u) \geq \frac{1}{2} ||u||^2 - \int_0^1 \left[ (B(su), u) + \langle v^*, u \rangle \right] ds$$

$$\geq \frac{1}{2} ||u||^2 - \int_0^1 s||u||^2 ||B|| ds - \int_0^1 \langle v^*, u \rangle ds$$

$$\geq \frac{1}{2} ||u||^2 - \frac{1}{2} ||u||^2 ||B|| - ||v^*|| ||u||$$

$$= \frac{1}{2} \left( 1 - ||B|| \right) ||u||^2 - ||v^*|| ||u||$$

$$= \frac{1}{2} ||u|| \left( (1 - ||B||)||u|| - 2||v^*|| \right)$$

$$= \frac{1}{2} \rho_0 \left( (1 - ||B||) \rho_0 - 2||v^*|| \right)$$
Step 3. $J$ satisfies assumption (2) of Theorem 1.1.

From assumption (2), there exists $e \in H$ such that,

$$\|e\| = 1 \text{ and } 1 - (B(e), e) < 0.$$ 

Let $\vartheta = \xi e$, with $\xi > 0$. Then

$$J(\xi e) = \frac{1}{2} \|\xi e\|^2 - \int_0^1 \left( T(s\xi e), \xi e \right) ds$$

$$\leq \frac{1}{2} \|\xi e\|^2 - \int_0^1 \left( B(s\xi e), \xi e \right) ds$$

$$= \frac{1}{2} \xi^2 \left( \|e\|^2 - (B(e), e) \right)$$

$$= \frac{1}{2} \xi^2 \left( 1 - (B(e), e) \right) \longrightarrow -\infty \text{ as } \xi \longrightarrow +\infty.$$

The functional $J$ satisfies the hypotheses of the Mountain Pass Theorem, so $J$ has a critical point. Thus $T$ has a fixed point.

3. Application

Consider the problem

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega, \\ u(x) = 0, & x \in \partial \Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ a Carathéodory function satisfying the following growth condition:

there exist $a, b > 0$ and $0 \leq \sigma \leq \frac{N+2}{N-2}$ if $N \geq 3$ ($0 \leq \sigma < \infty$ if $N = 1, 2$) such that

$$|f(x, u)| \leq a|u|^\sigma + b, \quad \forall x \in \Omega, \quad \forall u \in \mathbb{R}.$$ 

One can easily see that a weak solution of the problem (3.1) is a solution of the problem,

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f(x, u(x))v(x) dx = 0, \quad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega), \end{cases}$$

(3.2)
so, a weak solution of problem (3.1) satisfies
\[(u, v) - \int_{\Omega} f(x, u(x)).v(x)dx = 0, \quad \forall v \in H^1_0(\Omega).\]

Set \(J'(u).v = (u, v) - \int_{\Omega} f(x, u(x)).v(x)dx,\) and let \(T\) be the operator defined by
\[(Tu, v) = \int_{\Omega} f(x, u(x)).v(x)dx, \quad \forall v \in H^1_0(\Omega).\] (3.3)

We have
\[J'(u).v = (u - Tu, v) = ((I - T)u, v), \quad \text{and} \quad T = I - J' = (I - J)'
\]
which implies that \(T\) is a potential operator.

**Lemma 1.** ([8], Remark 2.1)
The operator \(T : H^1_0(\Omega) \rightarrow H^1_0(\Omega)\) is compact.

Let \(\lambda_1\) be the first eigenvalue of the linear Dirichlet problem
\[
\begin{cases}
-\Delta(x) = \lambda u(x), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}
\]

Recall \(\|u\|_{\infty} = \text{ess sup}_{x \in \Omega}|u(x)|, \quad \forall u \in L^\infty(\Omega).\)

**Theorem 3.1.** Assume the following conditions hold:

1. There exist functions \(a, b \in L^\infty(\Omega)\) such that,
\[f(x, u(x)) \leq a(x)u(x) + b(x), \quad \forall x \in \Omega, \forall u \in L^2(\Omega)\text{ with } \|u\|_{L^2} \leq r_0,\]

for some \(r_0 > \frac{1}{\sqrt{\lambda_1|\Omega|}}\) and \(\lambda_1 > \frac{\|a\|_{L^\infty}}{\lambda_1}, \quad 2\|b\|_{\infty}|\Omega| > \frac{1}{\sqrt{\lambda_1}}.\)

2. There exists \(w_0 \in H^1_0\) with \(\|w_0\| = 1\), such that
\[
\frac{1}{\|w_0\|_{L^2}^2} < a(x) < \lambda_1, \quad \forall x \in \Omega \text{ and } f(x, sw_0(x)) \geq a(x)sw_0(x), \quad \forall x \in \Omega, \forall s \in (0, 1).
\]

Then problem (3.1) has a weak solution.

**Proof.** We will apply Theorem 2.1. First we show assumption (1) of Theorem 2.1 is satisfied. Let \(B\) be the bounded linear operator defined on \(H^1_0(\Omega)\) by
\[(Bu, v) = \int_{\Omega} a(x)u(x).v(x)dx, \quad \forall u, v \in H^1_0(\Omega)\]
and \( v^* \in (H_0^1(\Omega))' \) be such that

\[
\langle v^*, v \rangle = \int_\Omega b(x)v(x)dx, \quad \forall v \in H_0^1(\Omega).
\]

Let \( \rho_0 = r_0\sqrt{\lambda_1} \), and \( u \in H_0^1(\Omega) \) such that \( \|u\| = \rho_0 \). Note \( \|u\|_{L^2} \leq r_0 \) and \( \|su\|_{L^2} \leq r_0 \) for all \( s \in (0, 1) \). From assumption (1) of Theorem 3.1 we have

\[
\left( T(su), u \right) = \int_\Omega f(x,(su)(x)).u(x)dx \leq \int_\Omega a(x)(su)(x).u(x) + \int_\Omega b(x).u(x)dx = (B(su), u) + \langle v^*, u \rangle.
\]

Note (using the Poincaré inequality)

\[
\|Bu\| = \sup_{\|v\| \leq 1} |\langle Bu, v \rangle| = \sup_{\|v\| \leq 1} \left| \int_0^1 a(x)u(x)v(x)dx \right| \leq \|a\|_{\infty} \sup_{\|v\| \leq 1} \int_0^1 |u(x)v(x)|dx \leq \|a\|_{\infty} \sup_{\|v\| \leq 1} \|u\|_{L^2} \|v\|_{L^2} = \|a\|_{\infty} \|u\|_{L^2} \sup_{\|v\| \leq 1} \|v\|_{L^2} \leq \|a\|_{\infty} \frac{1}{\sqrt{\lambda_1}} \|u\| \sup_{\|v\| \leq 1} \frac{1}{\sqrt{\lambda_1}} \|v\| \leq \frac{1}{\lambda_1} \|a\|_{\infty} \|u\|.
\]

Since \( B \) is a linear operator we have

\[
\|B\| \leq \frac{1}{\lambda_1} \|a\|_{\infty} < 1.
\]

Now we show \( \rho_0 > \frac{2\|v^*\|}{1 - \|B\|} \). To see this notice (using the Cauchy-Schwarz and the Poincaré inequalities)

\[
\|v^*\| = \sup_{\|v\| \leq 1} |\langle v^*, v \rangle|.
\]
\[ \sup_{\|v\| \leq 1} \int_{\Omega} b(x)v(x)dx \]
\[ \leq \|b\|_\infty \sup_{\|v\| \leq 1} \int_{\Omega} |v(x)|dx \]
\[ \leq \|b\|_\infty \sup_{\|v\| \leq 1} \sqrt{|\Omega|} \|v\|_{L^2} \]
\[ = \|b\|_\infty \sqrt{|\Omega|} \sup_{\|v\| \leq 1} \|v\|_{L^2} \]
\[ \leq \|b\|_\infty \sqrt{|\Omega|} \frac{1}{\sqrt{\lambda_1}} \sup_{\|v\| \leq 1} \|v\| \]
\[ \leq \|b\|_\infty \sqrt{|\Omega|} \frac{1}{\sqrt{\lambda_1}}. \]

Since \( \frac{1}{\sqrt{\lambda_1}} < r_0 \sqrt{|\Omega|} \) then \( \|v^*\| < \|b\|_\infty |\Omega|r_0 \) and so, \( r_0 > \frac{\|v^*\|}{\|b\|_\infty |\Omega|} \). Also since \( \rho_0 = r_0 \sqrt{\lambda_1} \), we have
\[ \rho_0 > \frac{\|v^*\|\sqrt{\lambda_1}}{\|b\|_\infty |\Omega|}. \]

Next since \( \|B\| \leq \frac{\|a\|_\infty}{\lambda_1} \), then \( 1 - \frac{\|a\|_\infty}{\lambda_1} \leq 1 - \|B\| \). Also since \( 2\|b\|_\infty |\Omega| \frac{1}{\sqrt{\lambda_1}} < 1 - \frac{\|a\|_\infty}{\lambda_1} \) one has
\[ 2\|b\|_\infty |\Omega| \frac{1}{\sqrt{\lambda_1}} < 1 - \|B\|, \]
so
\[ \frac{\sqrt{\lambda_1}}{2\|b\|_\infty |\Omega|} > \frac{1}{1 - \|B\|} \]
i.e. \( \frac{2\|v^*\|\sqrt{\lambda_1}}{2\|b\|_\infty |\Omega|} > \frac{2\|v^*\|}{1 - \|B\|} \), so \( \rho_0 > \frac{2\|v^*\|}{1 - \|B\|} \).

Next we show assumption (2) of Theorem 2.1 is satisfied. First we show there exists \( e \in H^1_0 \) with \( \|e\| = 1 \) such that \( (Be, e) > 1 \). From assumption (2) of Theorem 3.1, there exists \( w_0 \in H^1_0(\Omega) \) with \( \|w_0\| = 1 \), and \( a(x) > \frac{1}{\|w_0\|^2_{L^2}} \), \( \forall x \in \Omega \). Put \( e = w_0 \) and note
\[ (Be, e) = \int_{\Omega} a(x)(e(x))^2dx \]
\[ > \frac{1}{\|e\|^2_{L^2}} \int_{\Omega} |e(x)|^2dx \]
\[ = 1. \]
Next we show \( (T(se), e) \geq (B(se), e) \) for all \( s \in (0, 1) \). From assumption (2) of Theorem 3.1, for \( e = w_0 \) we have \( f(x, se) \geq a(x)se \), and so

\[
(T(se), e) = \int_{\Omega} f(x, se).e(x)dx, \quad \forall s \in (0, 1)
\]

\[
\geq \int_{\Omega} a(x)(se)(x).e(x)dx
\]

\[
= (B(se), e).
\]

From Theorem 2.1, the potential operator \( T \) has a fixed point \( u_0 \) in \( H^1_0(\Omega) \), which implies that \( J'(u_0) = 0 \). Thus problem (3.1) has a weak solution.

References


