ON A NEW FIXED POINT THEOREM ON HILBERT
SPACES VIA THE MOUNTAIN PASS LEMMA AND
APPLICATIONS TO ELLIPTIC BOUNDARY
VALUE PROBLEMS

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Abstract: In this paper we present a new fixed point theorem in Hilbert spaces. Using a suitable critical point theorem we provide conditions under which potential operators have a fixed point. An application is given to illustrate the theory.

Key Words: fixed point theorem, potential operators

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and let $\langle \cdot, \cdot \rangle$ denote the scalar product on

H and $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ the norm. A functional $\varphi : H \rightarrow \mathbb{R}$ is said to be Gâteaux differentiable on H if

$$\lim_{t \rightarrow 0} \frac{\varphi(u + th) - \varphi(u)}{t} = (\text{Grad } \varphi(u), h) \quad \forall u, h \in H.$$

The operator $T : H \rightarrow H$, is called a potential operator (or a gradient operator) on H , if there exists a Gâteaux differentiable functional φ such that $\text{Grad } \varphi(u) = T(u)$, for all $u \in H$.

An operator $T : H \rightarrow H$ is said to have a fixed point in H if there exists $u_0 \in H$ such that $T(u_0) = u_0$. In [6] the authors rephrase the problem of searching for fixed points of a potential operator T to searching for critical points of the functional J with $J' = I - T$. Recall that a critical point for J on H is an element $u_0 \in H$ such that, $J'(u_0) = 0$. We give conditions on the potential operator T so that the Mountain Pass Theorem guarantees the existence of critical points of J , so as a result guarantees the existence of fixed points of T . First, we recall some concepts from critical point theory.

Definition 1.1. [2, 1, 3] Let $J \in C^1(H, \mathbb{R})$. If any sequence $(u_n) \subset H$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \rightarrow 0$ as $n \rightarrow +\infty$ in H possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (PS condition for short).

Theorem 1.1. (Mountain Pass Theorem [5, 3]) Let H be a Banach space and let $J \in C^1(H, \mathbb{R})$, satisfy the Palais-Smale condition. Assume $J(0) = 0$ and there exist positive numbers ρ , and α such that:

1. $J(u) \geq \alpha$ if $\|u\| = \rho$,
2. There exists $\vartheta \in H$ such that $\|\vartheta\| > \rho$ and $J(\vartheta) < \alpha$.

Then c is a critical value of J with $c \geq \alpha$, where $c = \min_{A \in \Gamma} \max_{u \in A} J(u)$, and $\Gamma = \{\gamma([0, 1]); \gamma \in C([0, 1], H) \text{ } \gamma(0) = 0, \gamma(1) = \vartheta\}$.

Theorem 1.2. [1] Let H and Y be two Banach spaces, Ω an open subset of H , and $\varphi : \Omega \rightarrow Y$ a mapping of class C^1 . Given $x, y \in \Omega$, if $x + ty \in \Omega$ for all $t \in (0, 1)$, then

$$\varphi(x + y) = \varphi(x) + \int_0^1 \langle D\varphi(x + ty), y \rangle dt.$$

This result connects the potential operator T and the Gâteaux differentiable functional φ since

$$\varphi(x) = \int_0^1 (T(sx), x) ds.$$

Following the ideas in [6], the authors in [7] used a suitable minimization principle to prove the following fixed point theorem

Theorem 1.3. *Let H be a Hilbert space and $T : H \rightarrow H$ a compact potential operator such that there exists a bounded linear operator B on H and $v^* \in H$ satisfying*

$$\left(T(su), u \right) \leq \left(B(su), u \right) + \langle v^*, u \rangle, \quad \forall s \in [0, 1], \quad \forall u \in H \quad \text{with} \quad \|B\| < 1. \quad (1.1)$$

Then, the operator T has a fixed point in H .

In this paper we prove a new fixed point theorem which weakens the condition in the above theorem.

2. Existence Result

Theorem 2.1. *Let H be a Hilbert space and $T : H \rightarrow H$ a compact potential operator such that there exists a bounded linear operator B on H and a $v^* \in H'$ satisfying*

1. $\left(T(su), u \right) \leq \left(B(su), u \right) + \langle v^*, u \rangle, \quad \forall s \in (0, 1), \quad \forall u \in H \quad \text{with} \quad \|u\| = \rho_0, \quad \text{for some} \quad \rho_0 > \frac{2\|v^*\|}{1-\|B\|}, \quad \|B\| < 1,$
2. $\exists e \in H \quad \text{with} \quad \|e\| = 1 \quad \text{and} \quad \left(B(e), e \right) > 1 \quad \text{such that,}$
 $\left(T(se), e \right) \geq \left(B(se), e \right), \quad \forall s \in (0, 1),$

Then, the operator T has a fixed point in H .

Proof. Since T is a potential operator, there exists a Gâteaux differentiable functional $\varphi : H \rightarrow \mathbb{R}$ such that $\varphi' = T$. Let $J = I - \varphi$. Then $J \in C^1(H, \mathbb{R})$ with $J' = I - \varphi' = I - T$, i.e., $\forall u \in H, J'(u) = u - Tu$. Since T is a potential operator, it can be represented in the form

$$\varphi(u) = \int_0^1 \left(T(su), u \right) ds.$$

Consider the functional $J : H \rightarrow \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 \left(T(su), u \right) ds.$$

Step 1. J satisfies the Palais-Smale condition.

Let (u_n) be a sequence in H such that $\lim_{n \rightarrow +\infty} J'(u_n) = 0$ and $(J(u_n))$ is bounded. i.e., there exists some positive constant K such that $|J(u_n)| \leq K$, for all positive integer n . From assumption (1), we have

$$\begin{aligned} K \geq J(u_n) &\geq \frac{1}{2}\|u_n\|^2 - \int_0^1 \left[(B(su_n), u_n) + \langle v^*, u_n \rangle \right] ds \\ &\geq \frac{1}{2}\|u_n\|^2 - \int_0^1 s\|u_n\|^2\|B\| ds - \int_0^1 \langle v^*, u_n \rangle ds \\ &\geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u_n\|^2\|B\| - \|v^*\|\|u_n\| \\ &= \frac{1}{2}(1 - \|B\|)\|u_n\|^2 - \|v^*\|\|u_n\|, \end{aligned}$$

which implies that (u_n) is bounded in H . We note that, $J'(u_n) = u_n - T(u_n)$, with $\lim_{n \rightarrow +\infty} J'(u_n) = 0$. Since the sequence (u_n) is bounded and the operator T is compact, the sequence $(T(u_n))$ is relatively compact, and as a consequence there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $T(u_{n_k}) \rightarrow w$. Hence $u_{n_k} \rightarrow w$ in H (note

$$\|u_{n_k} - w\| \leq \|u_{n_k} - T(u_{n_k})\| + \|T(u_{n_k}) - w\| \rightarrow 0, \text{ as } k \rightarrow +\infty).$$

Thus, the (PS) condition is satisfied.

Step 2. J satisfies assumption (1) of Theorem 1.1.

Consider any $\rho_0 > \frac{2\|v^*\|}{1-\|B\|}$, and let $\alpha_0 = \frac{1}{2}\rho_0((1 - \|B\|)\rho_0 - 2\|v^*\|)$. From assumption (1), if $\|u\| = \rho_0$ we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - \int_0^1 \left[(B(su), u) + \langle v^*, u \rangle \right] ds \\ &\geq \frac{1}{2}\|u\|^2 - \int_0^1 s\|u\|^2\|B\| ds - \int_0^1 \langle v^*, u \rangle ds \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u\|^2\|B\| - \|v^*\|\|u\| \\ &= \frac{1}{2}(1 - \|B\|)\|u\|^2 - \|v^*\|\|u\| \\ &= \frac{1}{2}\|u\|((1 - \|B\|)\|u\| - 2\|v^*\|) \\ &= \frac{1}{2}\rho_0((1 - \|B\|)\rho_0 - 2\|v^*\|) \end{aligned}$$

$$= \alpha_0.$$

Step 3. J satisfies assumption (2) of Theorem 1.1.

From assumption (2), there exists $e \in H$ such that,

$$\|e\| = 1 \text{ and } 1 - (B(e), e) < 0.$$

Let $\vartheta = \xi e$, with $\xi > 0$. Then

$$\begin{aligned} J(\xi e) &= \frac{1}{2}\|\xi e\|^2 - \int_0^1 (T(s\xi e), \xi e) dx \\ &\leq \frac{1}{2}\|\xi e\|^2 - \int_0^1 (B(s\xi e), \xi e) ds \\ &= \frac{1}{2}\xi^2 (\|e\|^2 - (B(e), e)) \\ &= \frac{1}{2}\xi^2 (1 - (B(e), e)) \longrightarrow -\infty \text{ as } \xi \longrightarrow +\infty. \end{aligned}$$

The functional J satisfies the hypotheses of the Mountain Pass Theorem, so J has a critical point. Thus T has a fixed point. $\square$

3. Application

Consider the problem

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (3.1)$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ a Carathéodory function satisfying the following growth condition:

there exist $a, b > 0$ and $0 \leq \sigma \leq \frac{N+2}{N-2}$ if $N \geq 3$ ($0 \leq \sigma < \infty$ if $N = 1, 2$) such that

$$|f(x, u)| \leq a|u|^\sigma + b, \quad \forall x \in \Omega, \quad \forall u \in \mathbb{R}.$$

One can easily see that a weak solution of the problem (3.1) is a solution of the problem,

$$\begin{cases} \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} f(x, u(x)) \cdot v(x) dx = 0, & \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega), \end{cases} \quad (3.2)$$

so, a weak solution of problem (3.1) satisfies

$$(u, v) - \int_{\Omega} f(x, u(x)) \cdot v(x) dx = 0, \quad \forall v \in H_0^1(\Omega).$$

Set $J'(u) \cdot v = (u, v) - \int_{\Omega} f(x, u(x)) \cdot v(x) dx$, and let T be the operator defined by

$$(Tu, v) = \int_{\Omega} f(x, u(x)) \cdot v(x) dx, \quad \forall v \in H_0^1(\Omega). \quad (3.3)$$

We have

$$J'(u) \cdot v = (u - Tu, v) = ((I - T)u, v), \quad \text{and } T = I - J' = (I - J)'$$

which implies that T is a potential operator.

Lemma 1. ([8], Remark 2.1)

The operator $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ is compact.

Let λ_1 be the first eigenvalue of the linear Dirichlet problem

$$\begin{cases} -\Delta(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Recall $\|u\|_{\infty} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$, $\forall u \in L^{\infty}(\Omega)$.

Theorem 3.1. Assume the following conditions hold:

1. There exist functions $a, b \in L^{\infty}(\Omega)$ such that,

$$f(x, u(x)) \leq a(x)u(x) + b(x), \quad \forall x \in \Omega, \forall u \in L^2(\Omega) \text{ with } \|u\|_{L^2} \leq r_0,$$

$$\text{for some } r_0 > \frac{1}{\sqrt{\lambda_1}|\Omega|} \text{ and } 1 - \frac{\|a\|}{\lambda_1} > 2\|b\|_{\infty}|\Omega|^{\frac{1}{\sqrt{\lambda_1}}}.$$

2. There exists $w_0 \in H_0^1$ with $\|w_0\| = 1$, such that

$$\frac{1}{\|w_0\|_{L^2}^2} < a(x) < \lambda_1, \quad \forall x \in \Omega \text{ and } f(x, sw_0(x)) \geq a(x)sw_0(x), \quad \forall x \in \Omega, \forall s \in (0, 1).$$

Then problem (3.1) has a weak solution.

Proof. We will apply Theorem 2.1. First we show assumption (1) of Theorem 2.1 is satisfied. Let B be the bounded linear operator defined on $H_0^1(\Omega)$ by

$$(Bu, v) = \int_{\Omega} a(x)u(x) \cdot v(x) dx, \quad \forall u, v \in H_0^1(\Omega)$$

and $v^* \in (H_0^1(\Omega))'$ be such that

$$\langle v^*, v \rangle = \int_{\Omega} b(x)v(x)dx, \quad \forall v \in H_0^1(\Omega).$$

Let $\rho_0 = r_0\sqrt{\lambda_1}$, and $u \in H_0^1(\Omega)$ such that $\|u\| = \rho_0$. Note $\|u\|_{L^2} \leq r_0$ and $\|su\|_{L^2} \leq r_0$ for all $s \in (0, 1)$. From assumption (1) of Theorem 3.1 we have

$$\begin{aligned} (T(su), u) &= \int_{\Omega} f(x, (su)(x)) \cdot u(x) dx \\ &\leq \int_{\Omega} a(x)(su)(x) \cdot u(x) + \int_{\Omega} b(x) \cdot u(x) dx \\ &= (B(su), u) + \langle v^*, u \rangle. \end{aligned}$$

Note (using the Poincaré inequality)

$$\begin{aligned} \|Bu\| &= \sup_{\|v\| \leq 1} |\langle Bu, v \rangle| \\ &= \sup_{\|v\| \leq 1} \left| \int_0^1 a(x)u(x)v(x) dx \right| \\ &\leq \|a\|_{\infty} \sup_{\|v\| \leq 1} \int_0^1 |u(x)v(x)| dx \\ &\leq \|a\|_{\infty} \sup_{\|v\| \leq 1} \|u\|_{L^2} \|v\|_{L^2} \\ &= \|a\|_{\infty} \|u\|_{L^2} \sup_{\|v\| \leq 1} \|v\|_{L^2} \\ &\leq \|a\|_{\infty} \frac{1}{\sqrt{\lambda_1}} \|u\| \sup_{\|v\| \leq 1} \frac{1}{\sqrt{\lambda_1}} \|v\| \\ &\leq \frac{1}{\lambda_1} \|a\|_{\infty} \|u\|. \end{aligned}$$

Since B is a linear operator we have

$$\|B\| \leq \frac{1}{\lambda_1} \|a\|_{\infty} < 1.$$

Now we show $\rho_0 > \frac{2\|v^*\|}{1 - \|B\|}$. To see this notice (using the Cauchy-Schwarz and the Poincaré inequalities)

$$\|v^*\| = \sup_{\|v\| \leq 1} |\langle v^*, v \rangle|$$

$$\begin{aligned}
&= \sup_{\|v\| \leq 1} \left| \int_{\Omega} b(x)v(x)dx \right| \\
&\leq \|b\|_{\infty} \sup_{\|v\| \leq 1} \int_{\Omega} |v(x)|dx \\
&\leq \|b\|_{\infty} \sup_{\|v\| \leq 1} \sqrt{|\Omega|} \|v\|_{L^2} \\
&= \|b\|_{\infty} \sqrt{|\Omega|} \sup_{\|v\| \leq 1} \|v\|_{L^2} \\
&\leq \|b\|_{\infty} \sqrt{|\Omega|} \frac{1}{\sqrt{\lambda_1}} \sup_{\|v\| \leq 1} \|v\| \\
&\leq \|b\|_{\infty} \sqrt{|\Omega|} \frac{1}{\sqrt{\lambda_1}}.
\end{aligned}$$

Since $\frac{1}{\sqrt{\lambda_1}} < r_0 \sqrt{|\Omega|}$ then $\|v^*\| < \|b\|_{\infty} |\Omega| r_0$ and so, $r_0 > \frac{\|v^*\|}{\|b\|_{\infty} |\Omega|}$. Also since $\rho_0 = r_0 \sqrt{\lambda_1}$, we have

$$\rho_0 > \frac{\|v^*\| \sqrt{\lambda_1}}{\|b\|_{\infty} |\Omega|}.$$

Next since $\|B\| \leq \frac{\|a\|_{\infty}}{\lambda_1}$, then $1 - \frac{\|a\|_{\infty}}{\lambda_1} \leq 1 - \|B\|$. Also since $2\|b\|_{\infty} |\Omega| \frac{1}{\sqrt{\lambda_1}} < 1 - \frac{\|a\|_{\infty}}{\lambda_1}$ one has

$$2\|b\|_{\infty} |\Omega| \frac{1}{\sqrt{\lambda_1}} < 1 - \|B\|,$$

so $\frac{\sqrt{\lambda_1}}{2\|b\|_{\infty} |\Omega|} > \frac{1}{1 - \|B\|}$ i.e. $\frac{2\|v^*\| \sqrt{\lambda_1}}{2\|b\|_{\infty} |\Omega|} > \frac{2\|v^*\|}{1 - \|B\|}$, so $\rho_0 > \frac{2\|v^*\|}{1 - \|B\|}$.

Next we show assumption (2) of Theorem 2.1 is satisfied. First we show there exists $e \in H_0^1$ with $\|e\| = 1$ such that $(Be, e) > 1$. From assumption (2) of Theorem 3.1, there exists $w_0 \in H_0^1(\Omega)$ with $\|w_0\| = 1$, and $a(x) > \frac{1}{\|w_0\|_{L^2}^2}$, $\forall x \in \Omega$. Put $e = w_0$ and note

$$\begin{aligned}
(Be, e) &= \int_{\Omega} a(x)(e(x))^2 dx \\
&> \frac{1}{\|e\|_{L^2}^2} \int_{\Omega} |e(x)|^2 dx \\
&= 1.
\end{aligned}$$

Next we show $\left(T(se), e\right) \geq \left(B(se), e\right)$ for all $s \in (0, 1)$. From assumption (2) of Theorem 3.1, for $e = w_0$ we have $f(x, se) \geq a(x)se$, and so

$$\begin{aligned} \left(T(se), e\right) &= \int_{\Omega} f(x, se) \cdot e(x) dx, \quad \forall s \in (0, 1) \\ &\geq \int_{\Omega} a(x)(se)(x) \cdot e(x) dx \\ &= \left(B(se), e\right). \end{aligned}$$

From Theorem 2.1, the potential operator T has a fixed point u_0 in $H_0^1(\Omega)$, which implies that $J'(u_0) = 0$. Thus problem (3.1) has a weak solution. $\square$

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