INFINITE GAMES VIA COVERING PROPERTIES IN IDEAL TOPOLOGICAL SPACES

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Abstract: First, we propound a comment about the Meneger (X) game. We show that player TWO has a winning strategy always per contra that player ONE. So, we define a new game, say $G(\mathcal{C})$, by using the same data of the Meneger (X) without any winning strategy for both players in general.

In this paper, we are using the concept of ideal topological spaces with its covering properties and I -compactness to introduce infinitely long games like: $G(\mathcal{C}, \mathcal{I})$, $\mathcal{G}_{\mathcal{D}}(\mathcal{C}, \mathcal{I})$, $\mathcal{G}_{\mathcal{O}}(\mathcal{C}, \mathcal{I})$ and $G(\mathcal{C}^*, \mathcal{C})$. So, we show some results that explain many conditions to make anyone of players have winning strategy. Also, the efficacies of some types of ideals on the strategies for players are studied. Finally, comparisons among player's strategies through the equivalent ideal topological spaces are showed.

These have been important topics of research and have been crucial in the development of game theory especially of topological game theory.

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1. Introduction and Preliminaries

Throughout the present paper (X, τ) and (Y, σ) denote topological spaces with no separation properties assumed. For a subset A of a topological space (X, τ) , $cl(A)$ and $int(A)$ denote the closure and interior of A in (X, τ) respectively.

The subject of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [13]. An ideal I on a set X is a nonempty collection of subsets of X which satisfies the conditions: (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and $\mathcal{P}(X)$ is the set of all subsets of X . A set operator $(\cdot)^* : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is called a local function [13] of A with respect to τ and I iff for $A \subseteq X$, $A^*(I, \tau) = \{x \in X : A \cap U \notin I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(\cdot)$ [13] for a topology $\tau^*(I, \tau)$, called the $*$ -topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$. When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* for $\tau^*(I, \tau)$. If I is an ideal on X , then (X, τ, I) is called an ideal topological space.

In an ideal topological space (X, τ, I) , if $A \subseteq X$, $int^*(A)$ denote the interior of A in (X, τ^*) . Closed subsets in (X, τ^*) are called τ^* -closed sets. A subset A of an ideal topological space (X, τ, I) is τ^* -closed [13] if and only if $A^* \subseteq A$.

For any ideal topological space (X, τ, I) , the collection $\{V - J : V \in \tau \text{ and } J \in I\}$ is a basis for τ^* . The elements of τ^* are called τ^* -open sets. A subset A of an ideal topological space (X, τ, I) is said to be τ^* -dense if $cl^*(A) = X$. It is clear that, in a space (X, τ, I) if $I = \{\emptyset\}$, then $\tau = \tau^*$ [6] [9].

Recall that if (X, τ, I) is an ideal topological space and A is a subset of X , then (A, τ_A, I_A) is an ideal topological space where τ_A is the relative topology on A and $I_A = \{A \cap J : J \in I\}$.

It is interesting to note that if I is a proper ideal (i.e. $X \notin I$), then the collection of the complements of the members of I form a filter on X . This is why sometimes ideals are called dual filters.

The following collections form important ideals [9] on a topological space (X, τ) :

- (i) $\{\emptyset\}$ or $I_{\{\emptyset\}}$: the trivial ideal.
- (ii) $\mathcal{P}(X)$: the improper ideal.
- (iii) $\mathcal{F} : (\mathcal{I}_f)$ the ideal of all finite sets.
- (iv) $\mathcal{C} : (I_c)$ the ideal of all countable subsets of X .
- (v) I_K : the ideal of all relatively compact subsets of X ; $I_K = \{A \subseteq X : cl(A) \text{ is compact}\}$.
- (iv) I_n : the ideal of all nowhere dense subsets of X ; $I_n = \{A \subseteq X : int(cl(A)) = \emptyset\}$.

$\emptyset\}$.

A function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be $*$ -continuous with respect to τ, I, σ and J if $f : (X, \tau^*) \rightarrow (Y, \sigma^*)$ is continuous [8].

A subset A of an ideal topological space (X, τ, I) is said to be I -compact [11] if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by open sets of X , there exists a finite subset Λ_0 of Λ such that $A - \cup\{U_\lambda : \lambda \in \Lambda_0\} \in I$. An ideal topological space (X, τ, I) is said to be I -compact if X is I -compact as a subset.

A family $\{F_i : i \in \Lambda\}$ of subsets of X is said to have the "finite intersection property", denoted by $(f.i.p.)$, if for every finite subset Λ_0 of Λ the intersection $\cap\{F_i : i \in \Lambda_0\}$ is non-empty [1].

A family of subsets of a topological space (X, τ) is said to be locally finite family if every point has a neighborhood which intersects only a finite number of them [2].

Many mathematical notions are defined, or characterized, in terms of selection principles of the following two sorts: Let S be an infinite set and $\mathcal{A}, \mathcal{B}$ both be sets whose members are families of subsets of S . Then $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection principle. That is to say;

For each sequence $(A_n : n \in N)$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in N)$ of finite sets such that for each $n \in N, B_n \subset A_n$ and $\cup\{B_n : n \in N\}$ is an element of $\mathcal{B}$.

There is a natural game, denoted $G_{fin}(\mathcal{A}, \mathcal{B})$, associated with $S_{fin}(\mathcal{A}, \mathcal{B})$. This game is played as follows: There is an inning per positive integer. In the n -th inning ONE chooses an $A_n \in \mathcal{A}$ and TWO responds with a finite set $B_n \subset A_n$. A play $A_1, B_1, \dots, A_n, B_n, \dots$ is won by TWO if $\cup\{B_n : n \in N\}$ is an element of $\mathcal{B}$. Otherwise ONE wins.

$S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle in the following meaning: For each sequence $(A_n : n \in N)$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in N)$ such that for each $n \in N, b_n \in A_n$ and $\{b_n : n \in N\}$ is an element of $\mathcal{B}$ [3].

A topological space (X, τ) is said to have Menger property if for every sequence $(A_n : n \in N)$ of open covers of X there is a sequence $(B_n : n \in N)$ such that for each n, B_n is a finite subset of A_n , and such that $\{B_n : n \in N\}$ is an open cover of X [7].

Hurewicz discovered a very useful description of spaces which have the Menger property. This description is given most economically in the language of game - theory. Let (X, τ) be a topological space. Players ONE and TWO play the following infinitely long game: They play an inning for each positive integer. In the n -th inning ONE chooses an open cover A_n of X and TWO responds by selecting a finite subset B_n of A_n . TWO wins the play $A_1, B_1, \dots, A_n, B_n, \dots$ of this game if $\{B_n : n \in N\}$ is an open cover of X . Otherwise, ONE wins. We

call this game the Meneger game. It is denoted Meneger (X) [12].

For a set X of real numbers, X has the Meneger property iff ONE does not have a winning strategy in the game Meneger (X) [12].

In what follows we are mostly concerned with the question as to whether either player has a winning strategy in a game. A strategy for one of the players is an algorithm that specifies each move of the player in every possible situation. More precisely, in our work we will study infinitely long game for two players. In the n -th inning player ONE chooses a set A_n (in some restricted collection of sets defined by the "rules" of the game). Player TWO responds by choosing a set B_n , i.e.,

ONE: $A_1, A_2, \dots, A_n, \dots$

TWO: $B_1, B_2, \dots, B_n, \dots$

A strategy for player ONE is a function $\delta = \{\delta_n : A^{n-1} \times B^{n-1} \rightarrow A_n\}$ such that $\delta_n(A_1, B_1, \dots, A_{n-1}, B_{n-1}) = A_n$. Similarly, one can consider that a strategy for player TWO is a function $T = \{T_n : A^n \times B^{n-1} \rightarrow B_n\}$ such that $T_n(A_1, B_1, \dots, A_{n-1}, B_{n-1}, A_n) = B_n$.

For any game, say G , we denote the fact that player ONE has a winning strategy in the game G by " $\text{ONE} \uparrow G$ ". If he does not have a winning strategy we write " $\text{ONE} \nmid G$ ". So we denote that player ONE has a losing strategy in the game G by " $\text{ONE} \downarrow G$ ". If he does not have a losing strategy we write " $\text{ONE} \nmid G$ ". The meaning of " $\text{TWO} \uparrow G$ ", " $\text{TWO} \nmid G$ ", " $\text{TWO} \downarrow G$ " and " $\text{TWO} \nmid G$ " is defined analogously with the same notation used for the game G .

Our models of this work are special types of infinitely alternating topological games with perfect information on an infinite set X [4], [5].

2. Games in Ideal Topological Spaces

In Meneger (X), we note that player TWO has a winning strategy always per contra that player ONE. As the following proposition shows;

Proposition 2.1. *In Meneger (X), player TWO has a winning strategy.*

Proof. Let (X, τ) be a topological space. Players ONE and TWO play infinitely meneger (X). We define a strategy T for TWO in the game Meneger (X). In the first move, ONE will choose an open cover A_1 for X , TWO responds by fix $x_1 \in X$ and selecting an open neighborhood B_1 of x_1 such that $B_1 = T(A_1)$. In the second move, ONE will choose an open cover A_2 for X , TWO responds by fix $x_2 \in X - B_1$ and selecting an open neighborhood B_2 of x_2 such that $B_2 = T(A_1, B_1, A_2)$. In the n -th inning ONE chooses an open cover A_n of X ,

TWO responds by fix $x_n \in X - (B_1 \cup B_2 \cup \dots \cup B_{n-1})$ and selecting an open neighborhood B_n of x_n such that $B_n = T(A_1, B_1, \dots, A_{n-1}, B_{n-1}, A_n)$. Since $X = \cup\{B_n : n \in N\}$. Then T wins for TWO in Meneger (X) . $\square$

In the following definition we introduce a new game, say $G(C)$, by using the same data of the Meneger (X) with change the winning rule of it.

Definition 2.2. Let (X, τ) be a topological space and $\mathcal{C}$ a collection of all open covers for X . we define the infinitely long game $G(\mathcal{C})$ as follows: They play an inning for each positive integer. In the n -th inning, ONE chooses $A_n \in \mathcal{C}$ and TWO responds by selecting a non-empty finite subset M_n of A_n such that $B_n = \cup\{U : U \in M_n\}$. ONE wins the play $A_1, B_1, \dots, A_n, B_n, \dots$ of this game if $\{B_n : n \in N\}$ is an open cover of X . Otherwise, TWO wins.

Proposition 2.3. If (X, τ) is a compact topological space, then $\text{TWO} \downarrow G(\mathcal{C})$.

In the following, we introduce new games by using the concepts of ideal topological spaces.

Definition 2.4. Let (X, τ, I) be an ideal topological space and $\mathcal{C}$ a collection of all open covers for X . We define the infinitely long game $G(\mathcal{C}, \mathcal{I})$ as follows: They play an inning for each positive integer. In the n -th inning, ONE chooses $A_n \in \mathcal{C}$ and TWO responds by selecting a non-empty finite subset M_n of A_n such that $B_n = \cup\{U : U \in M_n\}$. ONE wins the play $A_1, B_1, \dots, A_n, B_n, \dots$ of this game if $X - \cup\{B_n : n \in N\} \in I$. Otherwise, TWO wins.

Proposition 2.5. Let (X, τ, I) be an ideal topological space. The following statements are hold.

- (i) If X is I -compact, then $\text{TWO} \downarrow G(\mathcal{C}, \mathcal{I})$.
- (ii) If $\text{ONE} \uparrow G(\mathcal{C})$, then $\text{ONE} \uparrow G(\mathcal{C}, \mathcal{I})$.
- (iii) If $I = \{\emptyset\}$, then $\text{ONE} \uparrow G(\mathcal{C})$ if and only if $\text{ONE} \uparrow G(\mathcal{C}, \mathcal{I})$.
- (iv) If $I = \mathcal{P}(\mathcal{X})$, then $\text{TWO} \nrightarrow G(\mathcal{C}, \mathcal{I})$.

Definition 2.6. Let (X, τ, I) be an ideal topological space and $\mathcal{C}$ a collection of all open covers for X . We define the infinitely long game $G_D(\mathcal{C}, \mathcal{I})$ (resp. $G_O(\mathcal{C}, \mathcal{I})$) as follows: They play an inning for each positive integer. In the n -th inning, ONE chooses $A_n \in \mathcal{C}$ and TWO responds by selecting a non-empty finite subset M_n of A_n such that $B_n = \cup\{U : U \in M_n\}$. ONE wins the play $A_1, B_1, \dots, A_n, B_n, \dots$ of the game $G_D(\mathcal{C}, \mathcal{I})$ (resp. $G_O(\mathcal{C}, \mathcal{I})$) if $X - cl(\cup\{B_n : n \in N\}) \in I$ (resp. $X - \cup\{cl(B_n) : n \in N\} \in I$). Otherwise, TWO wins

Proposition 2.7. Let (X, τ, I) be an ideal topological space. Then the

following statements are hold.

- (i) $\text{ONE} \uparrow G(\mathcal{C}, \mathcal{I})$ implies $\text{ONE} \uparrow G_O(\mathcal{C}, \mathcal{I})$.
- (ii) $\text{ONE} \uparrow G_O(\mathcal{C}, \mathcal{I})$ implies $\text{ONE} \uparrow G_D(\mathcal{C}, \mathcal{I})$.

Proof. It follows from the fact that $\cup\{cl(B_n) : n \in \Lambda\} \subseteq cl(\cup\{B_n : n \in \Lambda\})$ and by the hereditary property of an ideal. $\square$

Proposition 2.8. *If (X, τ, I) is an ideal topological space such that every open subset of X is closed, then the following are equivalents.*

- (i) ONE has a winning strategy in $G(\mathcal{C}, \mathcal{I})$.
- (ii) ONE has a winning strategy in $G_D(\mathcal{C}, \mathcal{I})$.
- (iii) ONE has a winning strategy in $G_O(\mathcal{C}, \mathcal{I})$.

Proposition 2.9. *If (X, τ, I) is an ideal topological space such that every open subset of X is dense, then the following statements are hold.*

- (i) TWO has no winning strategy in $G_D(\mathcal{C}, \mathcal{I})$.
- (ii) TWO has no winning strategy in $G_O(\mathcal{C}, \mathcal{I})$.

Proposition 2.10. *Let Y be a clopen subset of an ideal topological space (X, τ, I) and $\mathcal{C}_Y$ a collection of all open covers for Y with respect to the relative topology. The following statements are hold;*

- (i) $\text{ONE} \uparrow G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y implies $\text{ONE} \uparrow G(\mathcal{C}, \mathcal{I})$.
- (ii) $\text{ONE} \uparrow G_D(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y implies $\text{ONE} \uparrow G_D(\mathcal{C}, \mathcal{I})$.
- (iii) $\text{ONE} \uparrow G_O(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y implies $\text{ONE} \uparrow G_O(\mathcal{C}, \mathcal{I})$.

Proof. (i): Let δ' be a winning strategy for ONE in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y . We define δ to be a winning strategy for ONE in $G(\mathcal{C}, I)$ on X . Player ONE looks to the first move $A'_1 = \delta'(\emptyset)$ of player ONE in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y and take $A_1 = \delta(\emptyset) = A'_1 \cup (X - Y)$ to be the first move in $G(\mathcal{C}, I)$ on X . TWO responds by selecting a non-empty finite subset M_1 of A_1 such that $B_1 = \cup\{U : U \in M_1\}$. Now $M'_1 = \{U \cap Y : U \in M_1\} \subseteq A'_1$ where $B'_1 = \cup\{V : V \in M'_1\}$ is a legitimate move for TWO in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y . Player ONE looks to the second move $A'_2 = \delta'(A'_1, B'_1)$ of player ONE in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y and take $A_2 = \delta(A_1, B_1) = A'_2 \cup (X - Y)$ to be the second move in $G(\mathcal{C}, \mathcal{I})$ on X . TWO responds by selecting a non-empty finite subset M_2 of A_2 such that $B_2 = \cup\{U : U \in M_2\}$. Now $M'_2 = \{U \cap Y : U \in M_2\} \subseteq A'_2$ where $B'_2 = \cup\{V : V \in M'_2\}$ is a legitimate move for TWO in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y . In n -th inning, Player ONE looks to the n -th move $A'_n = \delta'(A'_1, B'_1, \dots, A'_{n-1}, B'_{n-1})$ of player ONE in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y and take $A_n = \delta(A_1, B_1, \dots, A_{n-1}, B_{n-1}) = A'_n \cup (X - Y)$ to be the n -th move in $G(\mathcal{C}, \mathcal{I})$ on X . TWO responds by selecting a non-empty finite subset

M_n of A_n such that $B_n = \cup\{U : U \in M_n\}$. Now $M'_n = \{U \cap Y : U \in M_n\} \subseteq A'_n$ where $B'_n = \cup\{V : V \in M'_n\}$ is a legitimate move for TWO in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y . Now, since δ' is a winning strategy for ONE in $G(\mathcal{C}_Y, \mathcal{I}_Y)$ on Y implies $X - \cup\{B'_n : n \in N\} \in I_Y$. So, $X - \cup\{B_n : n \in N\} \in I$. Hence, δ is a winning strategy for ONE in $G(\mathcal{C}, \mathcal{I})$ on X .

In similar way we can prove (ii) and (iii). $\square$

Theorem 2.11. *If $f : (X, \tau) \rightarrow (Y, \tau')$ is a homeomorphism function, I is an ideal on X and $\mathcal{C}$ (resp. $\mathcal{C}'$) a family of all open covers on X (resp. Y), then the following statements are hold.*

- (i) $\text{ONE} \uparrow G(\mathcal{C}, \mathcal{I})$ on X if and only if $\text{ONE} \uparrow G(\mathcal{C}', \{(\mathcal{I})\})$ on Y .
- (ii) $\text{ONE} \uparrow G_D(\mathcal{C}, \mathcal{I})$ on X if and only if $\text{ONE} \uparrow G_D(\mathcal{C}', \{(\mathcal{I})\})$ on Y .
- (iii) $\text{ONE} \uparrow G_O(\mathcal{C}, \mathcal{I})$ on X if and only if $\text{ONE} \uparrow G_O(\mathcal{C}', \{(\mathcal{I})\})$ on Y .

Proposition 2.12. *If (X, τ, I) is an ideal topological space satisfying the Meneger property, then the following statements are hold.*

- (i) $\text{TWO} \downarrow G(\mathcal{C}, \mathcal{I})$.
- (ii) $\text{TWO} \downarrow G_D(\mathcal{C}, \mathcal{I})$.
- (iii) $\text{TWO} \downarrow G_O(\mathcal{C}, \mathcal{I})$.

Lemma 2.13. *If (X, τ) is a topological space and $\tau - \{\emptyset\}$ has a finite intersection property, then the interior of every proper closed subset F of X is empty.*

Proposition 2.14. *If (X, τ, I_n) is an ideal topological space such that $\tau - \{\emptyset\}$ has a finite intersection property, then the following statements are hold.*

- (i) $\text{TWO} \nmid G(\mathcal{C}, \mathcal{I})$.
- (ii) $\text{TWO} \nmid G_D(\mathcal{C}, \mathcal{I})$.
- (iii) $\text{TWO} \nmid G_O(\mathcal{C}, \mathcal{I})$.

Proof. (i): Let T be a strategy for player TWO in $G(\mathcal{C}, \mathcal{I})$. In the n -th inning, ONE chooses $A_n \in \mathcal{C}$ and TWO responds by selecting a non-empty finite subset M_n of A_n such that $B_n = \cup\{U : U \in M_n\}$. By Lemma 2.13, $X - \cup\{B_n : n \in N\} \in I_n$. Hence, ONE wins the play $A_1, B_1, \dots, A_n, B_n, \dots$ and so TWO has no winning strategy.

The proof of (ii) and (iii) Follows from (i) and the hereditary property of an ideal. $\square$

Proposition 2.15. *If (X, τ, I_K) is an ideal topological space such that X is compact space, then the following statements are hold.*

- (i) $TWO \nrightarrow G(\mathcal{C}, \mathcal{I})$.
- (ii) $TWO \nrightarrow G_D(\mathcal{C}, \mathcal{I})$.
- (iii) $TWO \nrightarrow G_O(\mathcal{C}, \mathcal{I})$.

3. Games and τ^* -Open Covers

In this section, we introduce new games by using some types of covering for ideal topological spaces.

Definition 3.1. Let (X, τ, I) be an ideal topological space and $\mathcal{C}$ (resp. $\mathcal{C}^*$) a collection of all open (resp. τ^* -open) covers for X . We define the infinitely long game $G(\mathcal{C}^*, \mathcal{C})$ as follows: They play an inning for each positive integer. In the n -th inning, ONE chooses $A_n \in \mathcal{C}^*$ and TWO responds by selecting a non-empty finite subset M_n of A_n such that $B_n = \cup\{U : U \in M_n\}$ is an open set. TWO wins the play $A_1, B_1, \dots, A_n, B_n, \dots$ of this game if $\{B_n : n \in \mathbb{N}\}$ is an open cover for X . Otherwise, ONE wins.

Remark 3.2. Let (X, τ, I) be an ideal topological space. If $I = \{\emptyset\}$, then the two games Meneger (X) and $G(\mathcal{C}^*, \mathcal{C})$ are equivalent.

Proposition 3.3. Let (X, τ, I) be an ideal topological space. If TWO has a winning strategy in $G(\mathcal{C}^*, \mathcal{C})$, then ONE has a winning strategy in $G(\mathcal{C})$.

Proof. Let T be a winning strategy for TWO in $G(\mathcal{C}^*, \mathcal{C})$. We define a strategy δ for ONE to be a winning strategy in $G(\mathcal{C})$. Let A_1 be the first move for ONE in $G(\mathcal{C}^*, \mathcal{C})$. Player ONE in $G(\mathcal{C})$ looks to the first move for player TWO in $G(\mathcal{C}^*, \mathcal{C})$, say M_1 , which is finite subset of A_1 such that $B_1 = T(A_1) = \cup\{U : U \in M_1\}$ is an open subset of X . Then take any open cover $\mathcal{U}_\infty$ for X and choose $A'_1 = \{B_1 \cup V : V \in \mathcal{U}_\infty\}$ to be the first move for player ONE in $G(\mathcal{C})$, $\delta(\emptyset) = \mathcal{A}'_\infty$. TWO responds by selecting a non-empty finite subset M'_1 of A'_1 such that $B'_1 = \cup\{V : V \in M'_1\}$. Let A_2 be the second move for ONE in $G(\mathcal{C}^*, \mathcal{C})$. Player ONE in $G(\mathcal{C})$ looks to the second move for player TWO in $G(\mathcal{C}^*, \mathcal{C})$, say M_2 which is finite subset of A_2 such that $B_2 = T(A_1, B_1, A_2) = \cup\{U : U \in M_2\}$ is an open subset of X . Then take any open cover $\mathcal{U}_\epsilon$ for X and choose $A'_2 = \{B_2 \cup V : V \in \mathcal{U}_\epsilon\}$ to be the second move for player ONE in $G(\mathcal{C})$ such that $\delta(A'_1, B'_1) = A'_2$. TWO responds by selecting a non-empty finite subset M'_2 of A'_2 such that $B'_2 = \cup\{V : V \in M'_2\}$. In n -th inning, let A_n be the n -th move for ONE in $G(\mathcal{C}^*, \mathcal{C})$. Player ONE in $G(\mathcal{C})$ looks to the n -th move for player TWO in $G(\mathcal{C}^*, \mathcal{C})$, say M_n which is finite subset of A_n such that $B_n = T(A_1, B_1, \dots, A_{n-1}, B_{n-1}, A_n) = \cup\{U : U \in M_n\}$ is an open subset of X .

Then take any open cover $\mathcal{U}$ for X and choose $A'_n = \{B_n \cup V : V \in \mathcal{U}\}$ to be the n -th move for player ONE in $G(\mathcal{C})$ such that $\delta(A'_1, B'_1, \dots, A'_{n-1}, B'_{n-1}) = A'_n$. TWO responds by selecting a non-empty finite subset M'_n of A'_n such that $B'_n = \cup\{V : V \in M'_n\}$. Since T is a winning strategy for TWO in $G(\mathcal{C}^*, \mathcal{C})$, then $\{B_n : n \in N\}$ is an open cover for X . Implies, $X = \cup\{B'_n : n \in N\}$. So δ is a winning strategy for ONE in $G(\mathcal{C})$. $\square$

Proposition 3.4. *Let (X, τ, I) be an ideal topological space such that $I = \{\emptyset\}$ and $\mathcal{C}$ has a finite intersection property. If ONE has a winning strategy in $G(\mathcal{C})$, then TWO has a winning strategy in $G(\mathcal{C}^*, \mathcal{C})$.*

Proof. Let δ be a winning strategy for ONE in $G(\mathcal{C})$. We define a strategy T for TWO to be a winning strategy in $G(\mathcal{C}^*, \mathcal{C})$. Let $A_1 \in \mathcal{C}^*$ be the first move for ONE in $G(\mathcal{C}^*, \mathcal{C})$ and A'_1 be the first move for ONE in $G(\mathcal{C})$ where $\delta(\emptyset) = A'_1$. Since $I = \{\emptyset\}$ and $\mathcal{C}$ has a finite intersection property, then $A_1 \cap A'_1$ is non-empty. Player TWO in $G(\mathcal{C}^*, \mathcal{C})$ choose M_1 to be a finite subset of $A_1 \cap A'_1$ such that $T(A_1) = B_1 = \cup\{U : U \in M_1\}$ is an open subset of X which is a legitimate move for TWO in $G(\mathcal{C})$. Let $A_2 \in \mathcal{C}^*$ be the second move for ONE in $G(\mathcal{C}^*, \mathcal{C})$ and A'_2 be the second move for ONE in $G(\mathcal{C})$ where $\delta(A'_1, B'_1) = A'_2$. Player TWO in $G(\mathcal{C}^*, \mathcal{C})$ choose M_2 to be a finite subset of $A_2 \cap A'_2$ such that $T(A_1, B_1, A_2) = B_2 = \cup\{U : U \in M_2\}$ is an open subset of X which is a legitimate move for TWO in $G(\mathcal{C})$. In n -th inning. Let $A_n \in \mathcal{C}^*$ be the n -th move for ONE in $G(\mathcal{C}^*, \mathcal{C})$ and A'_n be the n -th move for ONE in $G(\mathcal{C})$ where $\delta(A'_1, B'_1, \dots, A'_{n-1}, B'_{n-1}) = A'_n$. Player TWO in $G(\mathcal{C}^*, \mathcal{C})$ choose M_n to be a finite subset of $A_n \cap A'_n$ such that $T(A_1, B_1, \dots, A_{n-1}, B_{n-1}, A_n) = B_n = \cup\{U : U \in M_n\}$ is an open subset of X which is a legitimate move for TWO in $G(\mathcal{C})$. Since δ is a winning strategy for ONE in $G(\mathcal{C})$, then $X = \cup\{B_n : n \in N\}$. Hence, T is a winning strategy for TWO in $G(\mathcal{C}^*, \mathcal{C})$. $\square$

Proposition 3.5. *Let (X, τ, I) be an ideal topological space. If $I = \mathcal{P}(\mathcal{X})$ and $\tau \subset \mathcal{P}(\mathcal{X})$, then $\text{ONE} \uparrow G(\mathcal{C}^*, \mathcal{C})$.*

Proof. Since $\tau \subset \mathcal{P}(\mathcal{X})$, this means there exists $x_0 \in X$ such that the only open set containing x_0 is X . So $I = \mathcal{P}(\mathcal{X})$, implies $\tau^* = \mathcal{P}(\mathcal{X})$. Now, in every n -th inning, ONE must choose a τ^* -open cover $A_n = \{\{x\} : x \in X\}$. Therefore, for every choosing non-empty finite subset M_n of A_n for player TWO $\{x_0\} \notin M_n$ such that $x_0 \notin B_n = \cup\{U : U \in M_n\}$ is an open subset of X . Hence, $\{B_n : n \in N\} \notin \mathcal{C}$. So, ONE wins the play. $\square$

Proposition 3.6. *If (X, τ, I_f) is an ideal indiscrete topological space, then $\text{TWO} \uparrow G(\mathcal{C}^*, \mathcal{C})$.*

Proof. In n -th inning, for any choosing A_n for ONE there are two probabilities. Either $A_n \in \mathcal{C}$ or $A_n \notin \mathcal{C}$. If $A_n \in \mathcal{C}$, then TWO can choose a finite subcover for X . If $A_n \notin \mathcal{C}$, then this implies there exists a τ^* -open set $U \in A_n$ which its complement is finite. Let $X - U = \{x_1, x_2, \dots, x_n\}$. TWO must choose $M_n = \{U, U_1, U_2, \dots, U_n\}$ such that $x_i \in U_i, i = 1, 2, \dots, n$ and $B_n = \cup\{U : U \in M_n\} = X$. Hence, TWO wins the play. $\square$

Proposition 3.7. Let (X, τ_{co}, I) be an ideal topological space, where τ_{co} is the co-finite topology on X . If $I = \{\emptyset\}$, then $TWO \uparrow G(\mathcal{C}^*, \mathcal{C})$.

Proposition 3.8. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, f(I))$ be a bijective, open and $*$ -continuous function. If $TWO \uparrow G(\mathcal{C}^*, \mathcal{C})$ in (X, τ, I) , then $TWO \uparrow G(\mathcal{C}^*, \mathcal{C})$ in $(Y, \sigma, f(I))$.

4. Games in I-Compact Spaces

In this section, we use the concept of I -compact spaces to define new game say $G(X, I)$ and study the relationships between this game and the terms of selection principles.

Definition 4.1. Let (X, τ, I) be an ideal topological space. We define the infinitely long game $G(X, I)$ as follows: They play an inning for each positive integer. In the n -th inning, ONE chooses I -compact subset H_n of X and TWO responds by selecting an I -compact subset M_n of X such that $H_n \cap M_n = \emptyset$. ONE wins the play $H_1, M_1, \dots, H_n, M_n, \dots$ of this game if $\{M_n : n \in \mathbb{N}\}$ is locally finite and $X - \cup\{M_n : n \in \mathbb{N}\} \in I$. Otherwise, TWO wins.

Proposition 4.2. If (X, τ, I) is an I -compact space and $I = \mathcal{P}(\mathcal{X})$, then $ONE \uparrow G(X, I)$.

Proposition 4.3. If (X, τ, I) is an I -compact space and $I \subset \mathcal{P}(\mathcal{X})$, then $ONE \downarrow G(X, I)$.

In the next Theorems, we can use the natural game $G_1(\mathcal{A}, \mathcal{B})$ to explain the win state of chances for both players in the game $G(X, I)$, where $\mathcal{A}$ and $\mathcal{B}$ are defined as follows:

Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ where A_n is a family of I -compact subsets of X satisfying that for each I -compact subsets F of X there exists $D \in \mathcal{A}_n$ such that $F \cap D = \emptyset$.

If X is non I -compact space then, for each I -compact subset F of X there exists $\{x\} \subseteq X - F$ such that $\{x\} \cap F = \emptyset$.

Let $\mathcal{B} = \{\mathcal{B}_\lambda : \lambda \in *\}$ where $\mathcal{B}_n$ is an infinite locally finite family of I -compact subsets of X .

Theorem 4.4. *Let (X, τ, I) be a non I -compact space such that $I = \mathcal{P}(\mathcal{X})$. Then $\text{ONE} \uparrow G(X, I)$ if and only if $\text{TWO} \uparrow G_1(\mathcal{A}, \mathcal{B})$.*

Proof. Necessity, let δ be a winning strategy for ONE in $G(X, I)$. We define T to be a winning strategy for TWO in $G_1(\mathcal{A}, \mathcal{B})$. In the first move of $G_1(\mathcal{A}, \mathcal{B})$, ONE choose $A_1 \in \mathcal{A}$. Player TWO in $G_1(\mathcal{A}, \mathcal{B})$ looks to the first move for player ONE in $G(X, I)$, say $H_1 = \delta(\emptyset)$, and choose $T(A_1) = M_1$ such that $M_1 \in A_1$ and $H_1 \cap M_1 = \emptyset$ which is possible by definition of A . M_1 is a legitimate move for TWO in $G(X, I)$. In the second move of $G_1(\mathcal{A}, \mathcal{B})$, ONE choose $A_2 \in \mathcal{A}$. Player TWO in $G_1(\mathcal{A}, \mathcal{B})$ looks to the second move for player ONE in $G(X, I)$, say $H_2 = \delta(H_1, M_1)$, and choose $T(A_1, M_1, A_2) = M_2$ such that $M_2 \in A_2$ and $H_2 \cap M_2 = \emptyset$. M_2 is a legitimate move for TWO in $G(X, I)$. In the n -th inning of $G_1(\mathcal{A}, \mathcal{B})$, ONE choose $A_n \in \mathcal{A}$. Player TWO in $G_1(\mathcal{A}, \mathcal{B})$ looks to the n -th move for player ONE in $G(X, I)$, say $H_n = \delta(H_1, M_1, \dots, H_{n-1}, M_{n-1})$, and choose $T(A_1, M_1, \dots, A_{n-1}, M_{n-1}, A_n) = M_n$ such that $M_n \in A_n$ and $H_n \cap M_n = \emptyset$. M_n is a legitimate move for TWO in $G(X, I)$. Since δ is a winning strategy for ONE in $G(X, I)$, then $\{M_1, M_2, \dots, M_n\}$ is locally finite family and $X - \cup\{M_n : n \in N\} \in I$. Hence, $\{M_n : n \in N\} \in \mathcal{B}$. So, T is a winning strategy for TWO in $G_1(\mathcal{A}, \mathcal{B})$.

Sufficient; let $W_1 = \{K : K \text{ is } I\text{-compact subset of } X \text{ such that } T(A_1) \neq K \text{ for each } A_1 \in \mathcal{A}\}$. We claim that $W_1 \notin \mathcal{A}$. If $W_1 \in \mathcal{A}$, then W_1 is a legitimate move for ONE in $G_1(\mathcal{A}, \mathcal{B})$. Consider that TWO responds by selecting $D = T(W_1)$, implies $D \in W_1$. This is a contradiction of definition of W_1 . Since $W_1 \notin \mathcal{A}$, then there exists a non-empty I -compact subset H_1 of X satisfying that for each $K \in W_1$ implies $K \cap H_1 \neq \emptyset$. Now, in the first move of $G(X, I)$, ONE choose $H_1 = \delta(\emptyset)$. TWO responds by selecting M_1 such that $H_1 \cap M_1 = \emptyset$, implies $M_1 \notin W_1$. Let $A_1 \in \mathcal{A}$ be the first move of ONE in $G_1(\mathcal{A}, \mathcal{B})$ such that $M_1 = T(A_1)$. In the second move, let $W_2 = \{K : K \text{ is } I\text{-compact subset of } X, \text{ such that } T(A_1, M_1, A_2) \neq K \text{ for each } A_2 \in \mathcal{A}\}$. Since $W_2 \notin \mathcal{A}$, then there exists a non-empty I -compact subset H_2 of X satisfying that for each $K \in W_2$ implies $K \cap H_2 \neq \emptyset$. Now, in the second move of $G(X, I)$, ONE choose $H_2 = \delta(H_1, M_1)$. TWO responds by selecting M_2 such that $H_2 \cap M_2 = \emptyset$, implies $M_2 \notin W_2$. Let $A_2 \in \mathcal{A}$ be the second move of ONE in $G_1(\mathcal{A}, \mathcal{B})$ such that $M_2 = T(A_1, M_1)$. In n -th inning, let $W_n = \{K : K \text{ is } I\text{-compact subset of } X \text{ such that } T(A_1, M_1, \dots, A_{n-1}, M_{n-1}, A_n) \neq K \text{ for each } A_n \in \mathcal{A}\}$. Since $W_n \notin \mathcal{A}$, then there exists a non-empty I -compact subset H_n of X satisfying that for each $K \in W_n$ implies $K \cap H_n \neq \emptyset$. Now, in the n -th move of $G(X, I)$,

ONE choose $H_n = \delta(H_1, M_1, \dots, H_{n-1}, M_{n-1})$. TWO responds by selecting M_n such that $H_n \cap M_n = \emptyset$, implies $M_n \notin W_n$. Let $A_n \in \mathcal{A}$ be the n -th move of ONE in $G_1(\mathcal{A}, \mathcal{B})$ such that $M_n = T(A_1, M_1, \dots, A_{n-1}, M_{n-1}, A_n)$. Since T is winning strategy for TWO in $G_1(\mathcal{A}, \mathcal{B})$, then $\{M_1, M_2, \dots, M_n\}$ is locally finite family. Hence, δ is a winning strategy for ONE in $G(X, I)$. $\square$

Theorem 4.5. *For any non I -compact space (X, τ, I) , $\text{TWO} \uparrow G(X, I)$ if and only if $\text{ONE} \uparrow G_1(\mathcal{A}, \mathcal{B})$.*

Proof. Necessity, let T be a winning strategy for TWO in $G(X, I)$. We define δ to be a winning strategy for ONE in $G_1(\mathcal{A}, \mathcal{B})$. Let H_1 be the first move of ONE in $G(X, I)$ and $A_1 = \{K : K \text{ is } I\text{-compact subset of } X \text{ such that } T(H_1) = K\}$. Implies, $A_1 \in \mathcal{A}$ and A_1 is a legitimate move to be the first selecting of ONE in $G_1(\mathcal{A}, \mathcal{B})$. TWO, in $G_1(\mathcal{A}, \mathcal{B})$, responds by selecting $M_1 = \delta(A_1)$ since $M_1 \in A_1$, then $H_1 \cap M_1 = \emptyset$ and $T(H_1) = M_1$ is the first move for TWO in $G(X, I)$. In the second move, let H_2 be the second move of ONE in $G(X, I)$ and $A_2 = \{K : K \text{ is } I\text{-compact subset of } X \text{ such that } T(H_1, M_1, H_2) = K\}$. Implies, $A_2 \in \mathcal{A}$ and A_2 is a legitimate move to be the second selecting of ONE in $G_1(\mathcal{A}, \mathcal{B})$. TWO, in $G_1(\mathcal{A}, \mathcal{B})$, responds by selecting $M_2 = \delta(A_1, M_1, A_2)$ since $M_2 \in A_2$, then $T(H_1, M_1, H_2) = M_2$ and $H_2 \cap M_2 = \emptyset$. Now, in the n -th move of $G(X, I)$, ONE choose H_n . Let $A_n = \{K : K \text{ is } I\text{-compact subset of } X \text{ such that } T(H_1, M_1, \dots, H_{n-1}, M_{n-1}, H_n) = K\}$. Implies, $A_n \in \mathcal{A}$ and A_n is a legitimate move to be the n -th selecting of ONE in $G_1(\mathcal{A}, \mathcal{B})$. TWO, in $G_1(\mathcal{A}, \mathcal{B})$, responds by selecting $M_n = \delta(A_1, M_1, \dots, A_{n-1}, M_{n-1}, A_n)$ since $M_n \in A_n$, then $T(H_1, M_1, \dots, H_{n-1}, M_{n-1}, H_n) = M_n$ and $H_n \cap M_n = \emptyset$. Since T is a winning strategy for TWO in $G(X, I)$, then $\{M_n : n \in N\}$ is not locally finite family. Therefore, $\{M_n : n \in N\} \notin \mathcal{B}$. Hence, δ is a winning strategy for ONE in $G_1(\mathcal{A}, \mathcal{B})$.

Sufficient; let δ be a winning strategy for ONE in $G_1(\mathcal{A}, \mathcal{B})$. We define T to be a winning strategy for TWO in $G(X, I)$. Player ONE in $G(X, I)$ choose an I -compact subset H_1 of X . TWO in $G(X, I)$ looks to the first move for player ONE in $G_1(\mathcal{A}, \mathcal{B})$, say $A_1 = \delta(\emptyset)$, and choose $M_1 \in A_1$ such that $H_1 \cap M_1 = \emptyset$ which is possible by definition of $\mathcal{A}, \mathcal{T}(\mathcal{H}_\infty) = \mathcal{M}_\infty$ which is a legitimate to be the first move for TWO in $G_1(\mathcal{A}, \mathcal{B})$. In the second move, Player ONE in $G(X, I)$ choose an I -compact subset H_2 of X . TWO in $G(X, I)$ looks to the second move for player ONE in $G_1(\mathcal{A}, \mathcal{B})$, say $A_2 = \delta(A_1, M_1)$ and choose $M_2 \in A_2$ such that $H_2 \cap M_2 = \emptyset$ and $T(H_1, M_1, H_2) = M_2$ is a legitimate to be the second move for TWO in $G_1(\mathcal{A}, \mathcal{B})$. In the n -thinning, Player ONE in $G(X, I)$ choose an I -compact subset H_n of X . TWO in $G(X, I)$ looks to the n -th move for player ONE in $G_1(\mathcal{A}, \mathcal{B})$, say $A_n = \delta(A_1, M_1, \dots, A_{n-1}, M_{n-1})$ and choose

$M_n \in A_n$ such that $H_n \cap M_n = \emptyset$ and $T(H_1, M_1, \dots, H_{n-1}, M_{n-1}, H_n) = M_n$ is a legitimate to be the n -th move for TWO in $G_1(\mathcal{A}, \mathcal{B})$. Since δ is a winning strategy for ONE in $G_1(\mathcal{A}, \mathcal{B})$, then $\{M_n : n \in N\}$ is not locally finite family. Hence, δ is a winning strategy for TWO in $G(X, I)$. $\square$

Theorem 4.6. *For any two equivalent ideal topological spaces (X, τ, I) and $(Y, \sigma, f(I))$, the following statements are equivalent.*

- (i) $\text{ONE} \uparrow G(X, I)$.
- (ii) $\text{ONE} \uparrow G(Y, f(I))$.

Proof. (i) $\Rightarrow$ (ii): Let δ be a winning strategy for ONE in $G(X, I)$. We will define δ' to be a winning strategy for ONE in $G(Y, f(I))$. In the first move in $G(Y, f(I))$, ONE looks to the first move for player ONE in $G(X, I)$, say $H_1 = \delta(\emptyset)$ and choose $f(H_1)$ to be the first move for player ONE in $G(Y, f(I))$ which is possible since $f(H_1)$ is an $f(I)$ -compact subset of Y . TWO responds by selecting $f(I)$ -compact subset M_1 of Y such that $f(H_1) \cap M_1 = \emptyset$. It is clear that $f^{-1}(M_1)$ is a legitimate to be the first move for TWO in $G(X, I)$. In the n -th inning in $G(Y, f(I))$ ONE looks to the n -th move for player ONE in $G(X, I)$, say $H_n = \delta(H_1, f^{-1}(M_1), \dots, H_{n-1}, f^{-1}(M_{n-1}))$ and choose $f(H_n)$ to be the n -th move for player ONE in $G(Y, f(I))$ which is possible since $f(H_n)$ is an $f(I)$ -compact subset of Y . TWO responds by selecting $f(I)$ -compact subset M_n of Y such that $f(H_n) \cap M_n = \emptyset$. It is clear that $f^{-1}(M_n)$ is a legitimate n -th move for TWO in $G(X, I)$. Since δ is a winning strategy for ONE in $G(X, I)$, $\{f^{-1}(M_n) : n \in N\}$ is locally compact and $X - \cup \{f^{-1}(M_n) : n \in N\} \in I$. Implies, $\{M_n : n \in N\}$ is locally compact and $X - \cup \{M_n : n \in N\} \in f(I)$. Hence, δ' is a winning strategy for ONE in $G(Y, f(I))$.

In Similar way we can prove (ii) $\Rightarrow$ (i). $\square$

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