ON INTEGRAL INEQUALITIES OF HERMITE-HADAMARD
TYPE FOR LOGARITHMICALLY h -PREINVEX
FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract: In this paper, we consider a new class of convex functions which is called logarithmically h -preinvex functions. We prove several Hermite-Hadamard type inequalities for differentiable logarithmically h -preinvex functions via Fractional Integrals. Some special cases are also discussed. Our results extend and improve the corresponding ones in the literature.

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Key Words: integral inequalities, fractional integrals, hermite-hadamard inequality, logarithmically h -preinvex functions

1. Introduction

The following inequality is well-known in the literature as Hermite-Hadamard inequality. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (1.1)$$

Recently, Hermite-Hadamard type inequality has been the subject of inten-

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sive research. In recent years, several extensions and generalizations have been proposed for classical convexity (see [1 – 25]).

We derive several new Hermite-Hadamard type fractional integral inequalities for logarithmically h -preinvex functions and their variant forms. Results obtained in this paper continue to hold for these special cases. Our results represent significant generalized of the previous results.

2. Preliminares

Definition 1. ([32]) A set K_n is said to be invex set with respect to bifunction $\eta(.,.)$,

$$u + t\eta(v, u) \in K_n, \quad \forall u, v \in K_n, t \in [0, 1]. \quad (2.1)$$

The invex set K_n is also called η -connected set.

Definition 2. ([21]) Let $h : J \rightarrow \mathbb{R}$, where $(0, 1) \subseteq J$ and $h \not\equiv 0$, be an interval in $\mathbb{R}$ and let K be an invex set with respect to $\eta(.,.)$. A nonnegative function $f : K \rightarrow \mathbb{R}$ is called h -preinvex with respect to $\eta(.,.)$, if

$$f(u + t\eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v), \quad \forall u, v \in K, t \in [0, 1]. \quad (2.2)$$

Definition 3. ([14]) Let $K \subset \mathbb{R}$ be an invex set with respect to the bi-function $\eta(.,.)$. then for any $a, b \in K$ and $t \in [0, 1]$, we have

$$\eta(b, b + t\eta(a, b)) = -t\eta(a, b) \text{ and } \eta(b, a + t\eta(a, b)) = (1 - t)\eta(a, b). \quad (2.5)$$

From Definition 3, it follows that

$$\eta(b + t_2\eta(b, a), b + t_1\eta(a, b)) = (t_2 - t_1)\eta(a, b)$$

for every $a, b \in K$ and $t_1, t_2 \in [0, 1]$.

Definition 4. ([24]) Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integral $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha [f(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt \quad x > a \quad (2.6)$$

and

$$J_{b-}^\alpha [f(x)] = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt \quad x < b \quad (2.7)$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ is Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Lemma 1. ([1]) Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta: I \times I \rightarrow \mathbb{R}$ where $\eta(b, a) > 0$. If $f \in L_1[a, a + e^{i\varphi}\eta(b, a)]$ and $\alpha \geq 0$, then

$$\begin{aligned} & [f(a) + f(a + \eta(b, a))] - \frac{\Gamma(\alpha + 1)}{[\eta(b, a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+\eta(b, a))^-}^\alpha f(x) \right\} \\ &= \eta(b, a) \int_0^1 [t^\alpha - (1-t)^\alpha] f(a + t\eta(b, a)) dt. \end{aligned} \quad (2.8)$$

3. Main Results

Theorem 1. Let f be a logarithmically h -preinvex function such that $h(\frac{1}{2}) \neq 0$. Also suppose that Definition 3 holds for η , then, for $\eta(b, a) > 0$, we have

$$\begin{aligned} & \frac{1}{\alpha h(\frac{1}{2})} \ln f\left(\frac{2a + \eta(b, a)}{2}\right) \\ & \leq \frac{\Gamma(\alpha)}{(\eta(b, a))^\alpha} \left\{ J_{a+}^\alpha \ln f(u) + J_{(a+e^{i\varphi}\eta(b, a))^-}^\alpha \ln f(u) \right\} \\ & \leq (\ln f(a) + \ln f(b)) \left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\}. \end{aligned} \quad (3.1)$$

Consequently,

$$\left[f\left(\frac{2a + \eta(b, a)}{2}\right) \right]^{\frac{1}{\alpha h(1/2)}} \leq [f(a)f(b)]^{\frac{1}{\alpha h(1/2)}} \left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\}.$$

Proof. Since f is logarithmically h -preinvex functions, using Definition 3, we have

$$\begin{aligned} f\left(\frac{2a + \eta(b, a)}{2}\right) &= f(a + (1-t)\eta(a, b) + \frac{\eta(a + t\eta(a, b), a + (1-t)\eta(a, b))}{2}) \\ &\leq [f(a + t\eta(a, b))]^{h(1/2)} [f(a + (1-t)\eta(a, b))]^{h(1/2)} \\ &= \{[f(a + t\eta(a, b))] [f(a + (1-t)\eta(a, b))]\}^{h(1/2)}. \end{aligned}$$

Taking the logarithm on both sides of the above inequality yields

$$\ln f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \ln \{[f(a + t\eta(a, b))][f(a + (1 - t)\eta(a, b))]\}^{h(1/2)} \quad (3.2)$$

$$= h\left(\frac{1}{2}\right) \ln \{[f(a + t\eta(a, b))][f(a + (1 - t)\eta(a, b))]\}.$$

Multiplying (3.2) by $t^{\alpha-1}$, $t \in [0, 1]$, and then integrating the resulting inequality with respect to t from 0 to 1, we have

$$\begin{aligned} & \int_0^1 \frac{1}{h(\frac{1}{2})} \left[\ln f\left(\frac{2a + \eta(b, a)}{2}\right) \right] dt \leq \int_0^1 t^{\alpha-1} \ln f(a + t\eta(b, a)) dt \\ & + \int_0^1 t^{\alpha-1} \ln f(a + (1 - t)\eta(b, a)) dt \\ & \frac{1}{\alpha h(\frac{1}{2})} \left[\ln f\left(\frac{2a + \eta(b, a)}{2}\right) \right] \leq \int_0^1 t^{\alpha-1} \ln f(a + t\eta(b, a)) dt \\ & + \int_0^1 t^{\alpha-1} \ln f(a + (1 - t)\eta(b, a)) dt \\ & = \int_a^{a+\eta(b, a)} \left(\frac{u - a}{\eta(b, a)} \right)^{\alpha-1} \ln f(u) \frac{du}{\eta(b, a)} \quad (3.3) \\ & + \int_a^{a+\eta(b, a)} \left(\frac{a + \eta(b, a) - u}{\eta(b, a)} \right)^{\alpha-1} \ln f(u) \frac{du}{\eta(b, a)} \\ & = \frac{1}{(\eta(b, a))^\alpha} \left\{ \int_a^{a+\eta(b, a)} (u - a)^{\alpha-1} \ln f(u) du \right. \\ & + \left. \int_a^{a+\eta(b, a)} (a + \eta(b, a) - u)^{\alpha-1} \ln f(u) du \right\} \\ & = \frac{\Gamma(\alpha)}{(\eta(b, a))^\alpha} \left\{ J_{a^+}^\alpha \ln f(x) + J_{(a+\eta(b, a))^-}^\alpha \ln f(x) \right\}. \end{aligned}$$

Now following inequalities,

$$\ln f(a + t\eta(b, a)) \leq h(1 - t) \ln f(a) + h(t) \ln f(b) \quad (3.4)$$

and

$$\ln f(a + (1 - t)\eta(b, a)) \leq h(t) \ln f(a) + h(1 - t) \ln f(b), \quad (3.5)$$

multiplying (3.4 – 3.5) by $t^{\alpha-1}$, $t \in [0, 1]$, and then integrating the resulting inequality with respect to t from 0 to 1, we have

$$\begin{aligned}
 & \int_0^1 t^{\alpha-1} \ln f(a + t\eta(b, a)) dt \leq \int_0^1 t^{\alpha-1} [h(1-t) \ln f(a) + h(t) \ln f(b)] dt \\
 & \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_a^{a+\eta(b,a)} \left(\frac{u-a}{\eta(b,a)} \right)^{\alpha-1} \ln f(u) \frac{du}{\eta(b,a)} \\
 & \leq \int_0^1 t^{\alpha-1} [h(1-t) \ln f(a) + h(t) \ln f(b)] dt \\
 & \frac{\Gamma(\alpha)}{[\eta(b,a)]^\alpha} (J_{a^+}^\alpha \ln f(u)) \leq \int_0^1 t^{\alpha-1} [h(1-t) \ln f(a) + h(t) \ln f(b)] dt
 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
 & \int_0^1 t^{\alpha-1} \ln f(a + (1-t)\eta(b, a)) dt \leq \int_0^1 t^{\alpha-1} [h(t) \ln f(a) + h(1-t) \ln f(b)] dt \\
 & \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_a^{a+\eta(b,a)} \left(\frac{a+\eta(b,a)-u}{\eta(b,a)} \right)^{\alpha-1} \ln f(u) \frac{du}{\eta(b,a)} \\
 & \leq \int_0^1 t^{\alpha-1} [h(t) \ln f(a) + h(1-t) \ln f(b)] dt \\
 & \frac{\Gamma(\alpha)}{[\eta(b,a)]^\alpha} (J_{(a+\eta(b,a))^-}^\alpha \ln f(u)) \\
 & \leq \int_0^1 t^{\alpha-1} [h(t) \ln f(a) + h(1-t) \ln f(b)] dt.
 \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}
 & \frac{\Gamma(\alpha)}{[\eta(b,a)]^\alpha} \left\{ J_{a^+}^\alpha \ln f(u) + J_{(a+\eta(b,a))^-}^\alpha \ln f(u) \right\} \\
 & \leq \{ \ln f(a) + \ln f(b) \} \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] dt.
 \end{aligned} \tag{3.8}$$

Then, by (3.3) and (3.8), we get the desired inequality (3.1).

Combining (3.6) and (3.7), we have

$$\begin{aligned} & \frac{1}{\alpha h(\frac{1}{2})} \ln f\left(\frac{2a + \eta(b, a)}{2}\right) \\ & \leq (\ln f(a) + \ln f(b)) \left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\} \\ & = (\ln f(a) + \ln f(b)) \left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\} \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \left[f\left(\frac{2a + \eta(b, a)}{2}\right) \right]^{\frac{1}{\alpha h(1/2)}} \\ & \leq \exp^{(\ln f(a) + \ln f(b)) \left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\}} \\ & = \exp^{(\ln f(a) + \ln f(b)) \left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\}} \\ & = (f(a)f(b))^{\left\{ \int_0^1 t^{\alpha-1} h(1-t) dt + \int_0^1 t^{\alpha-1} h(t) dt \right\}}. \end{aligned}$$

Theorem 1 is thus proved. $\square$

Corollary 1. *Let f be a logarithmically s -preinvex function. Also suppose that Definition 3 holds for η , then, for $\eta(b, a) > 0$, we have*

$$\left[\ln f\left(\frac{2a + \eta(b, a)}{2}\right) \right]^{2^{s-1}} \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(u) du \leq \frac{(\ln f(a) + \ln f(b))}{s+1}.$$

Consequently,

$$\left[f\left(\frac{2a + \eta(b, a)}{2}\right) \right]^{2^{s-1}} \leq \exp \left\{ \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(u) du \right\} \leq [f(a)f(b)]^{\left\{ \frac{1}{s+1} \right\}}.$$

Proof. This follows from taking $\alpha = 1$ and $h(t) = t^s$ for $s \in (0, 1]$ in Theorem 1. $\square$

Corollary 2. *Let f be a logarithmically P -preinvex function. Also suppose that Definition 3 holds for η , then, for $\eta(b, a) > 0$, we have*

$$\left[\ln f\left(\frac{2a + \eta(b, a)}{2}\right) \right] \leq \frac{2}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(u) du \leq 2 [\ln f(a) + \ln f(b)].$$

Consequently,

$$\left[f\left(\frac{2a + \eta(b, a)}{2}\right) \right] \leq \exp \left\{ \frac{2}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(u) du \right\} \leq [f(a)f(b)]^2.$$

Proof. This follows from taking $\alpha = 1$ and $h(t) = 1$ in Theorem 1. $\square$

Corollary 3. *Let f be a logarithmically Q -preinvex function. Also suppose that Definition 3 holds for η , then, for $\eta(b, a) > 0$, we have*

$$\left[\frac{1}{4} \ln f\left(\frac{2a + \eta(b, a)}{2}\right) \right] \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(u) du.$$

Consequently,

$$\left[f\left(\frac{2a + \eta(b, a)}{2}\right) \right]^{\frac{1}{4}} \leq \exp \left\{ \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} \ln f(u) du \right\}.$$

Proof. Taking $\alpha = 1$ and $h(t) = \frac{1}{t}$ in Theorem 1, we obtain Corollary 3. $\square$

Remark 1. When $\eta(b, a) = b - a$, the above results reduce to ones for classical logarithmically h -convex functions, logarithmic s -convex functions, logarithmic P -convex functions, and logarithmic Q -convex functions, respectively (see Noor et al., 2013).

Theorem 2. *Let $f : K \rightarrow (0, \infty)$ be a differentiable function such that $f \in L_1[a, a + \eta(b, a)]$. If f, g is logarithmically h -preinvex on K for $q > 1$,*

$h(t) + h(1-t) = 1$ and $\eta(b, a)$, then

$$\left| \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha + 1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} \right| \\ \leq \left(\frac{2(1 - 2^{-\alpha})}{\alpha + 1} \right)^{\frac{1}{p}} \left(\int_0^1 |t^{\alpha} - (1-t)^{\alpha}|^q |f(a)|^{h(1-t)q} |f(b)|^{h(t)q} dt \right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1, the well-known power mean inequality, and the condition that $|f|^q$ is logarithmically h -preinvex gives

$$\left| \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha + 1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} \right| \\ = \left| \int_0^1 [t^{\alpha} - (1-t)^{\alpha}] f(a + t\eta(b, a)) dt \right| \\ \leq \left(\int_0^1 |t^{\alpha} - (1-t)^{\alpha}| dt \right)^{1-1/q} \left(\int_0^1 |[t^{\alpha} - (1-t)^{\alpha}] f(a + t\eta(b, a))|^q dt \right)^{1/q} \\ \leq \left(\frac{2(1 - 2^{-\alpha})}{\alpha + 1} \right)^{1-1/q} \left(\int_0^1 |t^{\alpha} - (1-t)^{\alpha}|^q |f(a)|^{h(1-t)q} |f(b)|^{h(t)q} dt \right)^{\frac{1}{q}}.$$

Theorem 2 is thus proved. $\square$

Remark 2. For different suitable choices of h and $\alpha = 1$, we can obtain corresponding result for logarithmically s -preinvex functions, logarithmically P -preinvex function, and logarithmically preinvex functions.

Corollary 4. Under conditions of Theorem 2, if we take $q = 1$, we have

$$\left| \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha + 1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} \right| \\ \leq \left(\int_0^1 |t^{\alpha} - (1-t)^{\alpha}| |f(a)|^{h(1-t)} |f(b)|^{h(t)} dt \right).$$

Theorem 3. Let $f : K \rightarrow (0, \infty)$ be a differentiable function such that $f \in L[a, a + \eta(b, a)]$. If $|f|^q$ is logarithmically h -preinvex on K for $q > 1$,

$\frac{1}{p} + \frac{1}{q} = 1$ such that and if $h(t) + h(1-t) = 1$, then, for $\eta(b, a)$, we have

$$\left| \frac{\Gamma(\alpha+1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} - \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} \right|^{\frac{1}{q}} \\ \leq \left(\frac{2(1 - 2^{-\alpha p})}{\alpha p + 1} \right)^{\frac{1}{p}} \left(\int_0^1 |f(a)|^{qh(1-t)} |f(b)|^{qh(t)} dt \right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1, we have

$$\left| \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha+1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} \right| \\ = \left| \int_0^1 [t^{\alpha} - (1-t)^{\alpha}] f(a + t\eta(b, a)) dt \right|.$$

From Hölder's inequality, we have

$$\left| \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha+1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} \right|^{\frac{1}{q}} \\ \leq \left(\int_0^1 |t^{\alpha} - (1-t)^{\alpha}|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \\ \leq \left(\frac{2(1 - 2^{-\alpha p})}{\alpha p + 1} \right)^{\frac{1}{p}} \left(\int_0^1 |f(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}}.$$

Here logarithmically h -preinvexity of $|f|^q$, we have following inequality

$$\left| \frac{f(a) + f(a + \eta(b, a))}{\eta(b, a)} - \frac{\Gamma(\alpha+1)}{[\eta(b, a)]^{\alpha+1}} \left\{ J_{a^+}^{\alpha} f(x) + J_{(a+\eta(b, a))^-}^{\alpha} f(x) \right\} \right|^{\frac{1}{q}} \\ \leq \left(\frac{2(1 - 2^{-\alpha p})}{\alpha p + 1} \right)^{\frac{1}{p}} \left(\int_0^1 |f(a)|^{h(1-t)q} |f(b)|^{h(t)q} dt \right)^{\frac{1}{q}}.$$

Theorem 3 is thus proved. $\square$

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