KÖTHE-TOEPLITZ DUALS OF SOME DIFFERENCE SEQUENCE SPACES AND THEIR MATRIX TRANSFORMATIONS OVER NON-ARCHIMEDEAN FIELDS

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Abstract: In this paper we give Köthe-Toeplitz duals of some difference sequence spaces and their inclusion relations using matrix transformations when the sequences, series and the infinite matrices have their entries in a complete non-trivially valued non-archimedean field $K$.

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1. Introduction

Let $p = (p_k)$ denote a sequence of strictly positive real numbers with $1 \leq p_k \leq \sup_{k} p_k < \infty$.

If $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ then the difference sequence spaces of all bounded, convergent and null sequences are defined as follows:

$\Delta \ell_\infty (p) = \left\{ x = (x_k) : \sup_{k} |\Delta x_k|^{p_k} < \infty \right\}$

$\Delta c(p) = \left\{ x = (x_k) : |\Delta x_k - \alpha_k|^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for some } \alpha_k \in K \right\}$ and

$\Delta c_0(p) = \left\{ x = (x_k) : |\Delta x_k|^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty \right\}$

when $p_k = m$, $k = 1, 2, \ldots$, $\Delta \ell_\infty (p) = \ell_\infty (\Delta)$, $\Delta c(p) = c(\Delta)$ and $\Delta c_0(p) = c_0(\Delta)$. 

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2. Köthe-Toeplitz Duals

In this section we give Köthe-Toeplitz duals of some difference sequence spaces.

**Definition 1.** If $E$ is a space of sequences $x = (x_k)$ in $K$ then its Köthe-Toeplitz dual is defined as

$$E^+ = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |x_k y_k| < \infty, \forall y = (y_k) \in E \right\}$$

**Lemma 2.** (see [10]) \( \sup_k |x_k - x_{k+1}|^{p_k} < \infty \) iff:

(i) \( \sup_k k^{-1} |x_k|^{{p_k}} < \infty \) and

(ii) \( \sup_k |x_k - k(k+1)^{-1} x_{k+1}|^{p_k} < \infty \).

**Theorem 3.** Let \( U_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k |a_k| N^{1/p_k} < \infty \right\} \) then for every sequence \( p = (p_k) \) with \( p > 0 \), we have, \( \Delta\ell_\infty^+ (p) = U_1 \).

**Proof.** First to prove: \( U_1 \subset \Delta\ell_\infty^+ (p) \).

Let \( x = (x_k) \in \Delta\ell_\infty (p) \), then for each \( k \in \mathbb{N} \), there exists an integer \( N \) such that

\[ N > \max(1, \sup_k k^{-1} |x_k|^{p_k}) \]  

(1)

Now, for \( a = (a_k) \in U_1 \),

\[
\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} |a_k (x_k - x_{k+1} + x_{k+1})|
\]

\[
= \sum_{k=1}^{\infty} |a_k (x_k - x_{k+1}) + a_k x_{k+1}|
\]

\[
\leq \sum_{k=1}^{\infty} \max \{|a_k||x_k - x_{k+1}|, |a_k||x_{k+1}|\}
\]

\[
\leq \sum_{k=1}^{\infty} |a_k| \max \{|x_k - x_{k+1}|, |x_1 - x_{k+2}|\}
\]
\[ \leq \sum_{k=1}^{\infty} |a_k| \max\{k^{N^1/p_k}, N^{1/p_k}\} \{\text{using (1)}\} \]

\[ \leq \sum_{k=1}^{\infty} k|a_k| N^{1/p_k}, \]

\[ \sum_{k=1}^{\infty} |a_k x_k| < \infty. \]

This implies that, \( a = (a_k) \in \Delta \ell_\infty^+ (p) \). Hence (i) holds.

Now, let us prove the reverse inclusion. That is to prove:

\[ \Delta \ell_\infty^+ (p) \subset U_1 \quad \text{(ii)} \]

Let \( a = (a_k) \in \Delta \ell_\infty^+ (p) \) for every \( x = (x_k) \in \Delta \ell_\infty (p) \) and suppose that \( a \notin U_1 \).

Then there exists an integer \( N > 1 \) such that

\[ \sum_{k=1}^{\infty} k|a_k| N^{1/p_k} \text{ is infinite.} \quad (2) \]

Therefore there exists a strictly increasing sequence \((k_N)\) of positive integers such that \( k = (k_N) \), \( k|a_k| N^{1/p_k} > M \).

We define \( x = (x_k) \) by

\[ x_k = \begin{cases} 0, & k \neq k_N \\ \frac{k^{N^1/p_k}}{M}, & k = k_N \end{cases} \]

Then for every integer \( N > 1 \) and for \( k = k_N \), we have

\[ \sum_{k=1}^{\infty} |a_k x_k| > \infty \{\text{using (2)}\} \]

which is a contradiction to that \( a = (a_k) \in \Delta \ell_\infty^+ (p) \). Hence (ii) holds.

Therefore we have \( \Delta \ell_\infty^+ (p) \subset U_1 \).

That is, the Köthe-Toeplitz dual of \( \Delta \ell_\infty (p) \) is

\[ U_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k|a_k| N^{1/p_k} < \infty \right\} \]

\[ \square \]
Theorem 4. Let
\[ U_2 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k |a_k| N^{-1/p_k} < \infty \right\}, \]
then for every sequence \( p = (p_k) \) with \( p > 0 \), we have \( \Delta c_0^+(p) = U_2 \).

Proof. First to prove: \( U_2 \subset \Delta c_0^+(p) \).

Let \( x = (x_k) \in \Delta c_0(p) \). Then for each \( k \in \mathbb{N} \), there exists an integer \( N \) such that
\[
|\Delta x_k|^{p_k} < \frac{k}{N}, \text{ since } |\Delta x_k|^{p_k} \to 0 \text{ as } k \to \infty. \tag{3}
\]

Now, for \( a = (a_k) \in U_2 \),
\[
\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} |a_k (x_k - x_{k+1}) + a_k x_{k+1}| \\
\leq \sum_{k=1}^{\infty} \max\{|a_k||x_k - x_{k+1}|, |a_k||x_{k+1}|\} \\
\leq \sum_{k=1}^{\infty} |a_k| \max\{|x_k - x_{k+1}|, |x_1 - x_{k+2}|\} \\
\leq \sum_{k=1}^{\infty} |a_k| \max\{||\Delta x_k||, |\Delta x_1||\} \\
\leq \sum_{k=1}^{\infty} |a_k| \max\{kN^{-1/p_k}, N^{-1/p_k}\} \{\text{using (3)}\} \\
\leq \sum_{k=1}^{\infty} k |a_k| N^{-1/p_k}, \\
\sum_{k=1}^{\infty} |a_k x_k| < \infty.
\]

This implies that, \( a \in \Delta c_0^+(p) \).

Hence we have \( U_2 \subset \Delta c_0^+(p) \). The converse part can easily be proved as in the above theorem. \( \square \)
3. Inclusion Theorems

In this section we give inclusion theorems on Matrix transformations between the difference sequence spaces over $K$.

Let $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 1, 2, \ldots$, be an infinite matrix, let $(X,Y)$ denote the set of all matrices $A = (a_{nk})$; $n, k = 1, 2, \ldots$, that transform a sequence $x = (x_k) \in X$ into a sequence $Ax = (A_n(x)) = y = y_n \in Y$ defined by

$$y_n = \sum_{k=1}^{\infty} a_{nk}x_k, \quad n = 1, 2, \ldots.$$  

**Theorem 5.** Let $0 < p_k \leq 1$ for every $k$. Then $A \in (c_0(p), \Delta \ell_\infty (p))$ iff

$$\sup_n \sup_k |a_{nk}|^{p_k} N^{-p_k} < \infty$$

**Proof.** Let $x = (x_k) \in c_0(p)$. Then for each $k \in \mathbb{N}$, there exists an integer $N > 1$ such that

$$|x_k|^{p_k} < \frac{1}{N} \quad (4)$$

Now to prove $Ax \in \Delta \ell_\infty (p)$, we take

$$|\Delta Ax|^{p_k} = |\Delta A_n(x)|^{p_k} = |A_n(x) - A_{n+1}(x)|^{p_k}$$

$$\leq \max \{|A_n(x)|^{p_k}, |A_{n+1}(x)|^{p_k}\}$$

$$\leq \sup_n |A_n(x)|^{p_k}$$

$$\leq \sup_n \sum_{k=1}^{\infty} |a_{nk}x_k|^{p_k}$$

$$\leq \sup_n \max\{|a_{n1}x_1|^{p_k}, |a_{n2}x_2|^{p_k}, \ldots, |a_{nk}x_k|^{p_k}, \ldots\}$$

$$\leq \sup_n \sup_k |a_{nk}|^{p_k} |x_k|^{p_k}$$

$$\leq \sup_n \sup_k |a_{nk}|^{p_k} N^{-p_k} \quad \text{using (4)},$$

$$|\Delta Ax|^{p_k} < \infty.$$  

This implies that $Ax \in \Delta \ell_\infty (p)$.

Therefore, we have $A \in (c_0(p), \Delta \ell_\infty (p))$.

Conversely, suppose that $A \in (c_0(p), \Delta \ell_\infty (p))$, but $\sup_n \left[ \sup_k |a_{nk}|^{p_k} N^{-p_k} \right]$ is infinite.
Then for each $x = (x_k) \in c_0(p)$ as $|x_k|^{p_k} \to 0$ we have

$$|x_k|^{p_k} < \epsilon < 1$$

implies that $\sup_n \left[ \sup_k |a_{nk}|^{p_k} N^{-p_k} |x_k|^{p_k} \right]$ is infinite, i.e.

$$\sup_n \left[ \sup_k |a_{nk} x_k|^{p_k} N^{-p_k} \right]$$

is infinite.

This implies that $Ax \not\in \Delta \ell_{\infty}(p)$ which is a contradiction. This completes the proof of the converse part and hence the theorem. \hfill $\square$

**Theorem 6.** Let $p = (p_k)$ with $1 < p_k \leq \sup_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$. Then $A \in (\ell(p), \Delta c(p))$ iff there exists an integer $N > 1$ such that:

(i) $\Delta a_{nk} \to a_k$, as $n \to \infty$,

(ii) $\sup_k |a_k|^{p_k} N^{1/q_k}$ converges.

**Proof.** Sufficiency. Suppose that (i) & (ii) hold.

Then for each $x = (x_k) \in \ell(p)$, since $\sum_{k=1}^{\infty} |x_k|^{p_k} < \infty$ there exists an integer $N > 1$ such that $(|x_k|^{p_k})^{q_k} < N$.

Let us now prove that $\Delta Ax \in c(p)$. For, we take,

$$|\Delta Ax|^{p_k} = |A_n(x) - A_{n+1}(x)|^{p_k} = \left| \sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k \right|^{p_k}$$

$$\leq \sup_k |a_{nk} - a_{n+1,k}|^{p_k} |x_k|^{p_k}$$

$$\leq \sup_k |\Delta a_{nk}|^{p_k} |x_k|^{p_k}$$

$$\leq \limsup_n \sum_{k=1}^{\infty} |\Delta a_{nk}|^{p_k} |x_k|^{p_k}$$

$$\leq \sup_k |a_k|^{p_k} N^{1/q_k} \quad [\text{by (i)}]$$

$$< \epsilon \quad [\text{by (ii)}].$$

Hence we have $\Delta Ax \in c(p)$, that is $Ax \in \Delta c(p)$. Therefore $A \in (\ell(p), \Delta c(p))$. 

Necessity. Let \( A \in (\ell(p), \Delta c(p)) \) and suppose that \( \sup_k |a_k|^p k N^{1/q_k} \) diverges. Then there exists an integer \( M_1 > 1 \) and for some \( k \in \mathbb{N} \), \( \sup_k |a_k|^p k N^{1/q_k} > M_1 \) and also that \( \Delta a_{n_k} \not\to a_k \), as \( n \to \infty \) for some \( k \in \mathbb{N} \), then there exists an integer \( M_2 > 1 \) such that \( |\Delta a_{n_k} - a_k|^p k > M_2 \).

Now for each \( x = (x_k) \in \ell(p) \), we have that

\[
|\Delta Ax|^p_k = |\Delta A_n(x)|^p_k = \left| \sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k \right|^p_k
\]

\[
= \left| \sum_{k=1}^{\infty} (a_{n,k} - a_{n+1,k}) x_k + a_k x_k - a_k x_k \right|^p_k
\]

\[
\leq \max \{ |\Delta a_{n_k} - a_k|^p |x_k|^p_k, |a_k|^p |x_k|^p_k \}
\]

\[
< N^{1/q_k} \max \{M_1, M_2\}
\]

\[
> MN^{1/q_k} \text{ where } M = \max \{M_1, M_2\}
\]

\[
i.e., |\Delta A_n(x)|^p_k > M_0 > \epsilon \text{ where } M_0 = MN^{1/q_k},
\]

therefore \( \Delta Ax \not\in c(p) \).

That is \( Ax \not\in \Delta c(p) \), which is a contradiction. This completes the proof of the theorem.

\[\Box\]

References


