DISCRETE HARDY-TYPE INEQUALITY WITH KERNEL

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Abstract: Necessary and sufficient conditions are given for the validity of discrete Hardy’s inequality for the sum of two discrete Hardy-type operators with not necessary non-negative coefficients.

AMS Subject Classification: 26D10, 26D15
Key Words: discrete hardy inequality, weighted sequence space and its norm, duality

1. Introduction

Discrete Hardy inequality has been characterized for the discrete Hardy operator in [2,3,9]. Sufficient condition is available in [1] for the validity of the discrete Hardy inequality, for the discrete Hardy-type operator with kernel \( k = \{k_{m,n}\}; \) \( m, n \in \mathbb{Z}_+ \) defined on \( D = \{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+: n \leq m\}, \) \( k_{m,n} \) is non-increasing in \( m \) and non-decreasing in \( n. \) For the discrete Hardy-type operator with general kernel, sufficient conditions for the validity of discrete Hardy inequality is available in [8] which is both necessary and sufficient when the kernel is of product type.

In this note, we consider a different discrete operator

\[
T : \ell^p(v_n) \rightarrow \ell^q(u_n)
\]

defined as

\[
T(a_n) = \phi_{1,m} \sum_{n=1}^{m} \psi_{1,n} k_{m,n} a_n + \phi_{2,m} \sum_{n=m}^{\infty} \psi_{2,n} k_{n,m} a_n
\]

where the kernel involve is discrete Oinarov kernel (not considered in earlier

Received: October 3, 2015
Published: February 15, 2016

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literature) and characterize Hardy inequality. Here \(\{\phi_{i,n}\}, \{\psi_{i,n}\}; i = 1, 2\) and \(\{a_n\}\) are sequences of real numbers not necessarily non-negative. Therefore \(T\) is not necessarily non-negative. Hardy inequality for the operator \(T\) gives better best constant for the Hardy inequality than the same for the single discrete Hardy-type operator. Our study is motivated by this.

Kernel \(k = \{k_{m,n}\}; m, n \in Z_+\) is called discrete Oinarov kernel if (a) \(k_{m,n} \geq 0\) for \(0 < n < m\); (b) \(k_{m,n}\) is increasing in \(m\) and decreasing in \(n\), (c) \(k_{m,n} \approx k_{m,p} + k_{p,n}\) for \(0 < n < p < m\).

Throughout the paper, \(n, m, \alpha, \beta\) are positive integers \((Z_+)\); \(t\) is fixed positive integer; \(p, q\) are real numbers; \(p' = p/(p-1)\) is conjugate to \(p\) and the same is true for \(q\). \(\ell^p(v_n)\) denotes weighted sequence space. \(\{u_n\}, \{v_n\}; n \in Z_+\) are weight sequences.

For complete description of Hardy inequality, we use to refer [5] and monographs [6,7].

## 2. Preliminaries

In the following result, we state the discrete case of [7, Theorem 2.10]:

**Proposition 2.1.** Suppose \(1 < p \leq q < \infty, \{u_n\}, \{v_n\}\) are weight sequences and \(k = \{k_{m,n}\}; m, n \in Z_+\) is discrete Oinarov kernel. The inequality

\[
\left(\sum_{m=1}^{\infty} \left| \sum_{n=1}^{m} k_{m,n} a_n \right|^q u_m \right)^{\frac{1}{q}} \leq C \left(\sum_{m=1}^{\infty} |a_m|^{p} v_m \right)^{\frac{1}{p}},
\]

holds for all non-negative sequence \(\{a_n\} \in \ell^p(v_n)\) and a suitable positive constant \(C\) if and only if \(\text{max}(A, B) < \infty\), where

\[
A = \sup_{m \in Z_+} \left(\sum_{n=m}^{\infty} k_{n,m}^{q} u_n \right)^{\frac{1}{q}} \left(\sum_{n=1}^{m} v_{1-p'}^n \right)^{\frac{1}{p'}},
\]

\[
B = \sup_{m \in Z_+} \left(\sum_{n=m}^{\infty} u_n \right)^{\frac{1}{q}} \left(\sum_{n=1}^{m} k_{m,n}^{p'} v_{1-p'}^n \right)^{\frac{1}{p'}}.
\]

**Proof.** The proof of Proposition 2.1 is similar to continuous case. Continuous case was first proved in [4]. We omit the detail.

By using duality argument and applying suitable substitutions in Proposition (2.1), we can prove the following.
Proposition 2.2. Suppose $1 < p \leq q < \infty$, $\{u_n\}, \{v_n\}$ are weight sequences and $k = \{k_{n,m}\}; n, m \in Z_+$ is discrete Oinarov kernel. The inequality
\[
\left( \sum_{m=1}^{\infty} \left| \sum_{n=m}^{\infty} k_{n,m} a_n \right|^q u_m \right)^{\frac{1}{q}} \leq C \left( \sum_{m=1}^{\infty} |a_m|^p v_m \right)^{\frac{1}{p}},
\]
holds for all non-negative sequence $\{a_n\} \in \ell^p(v_n)$ and a suitable positive constant $C$ if and only if $\max(D, E) < \infty$ where
\[
D = \sup_{m \in Z_+} \left( \sum_{n=1}^{m} k_{m,n}^q u_n \right)^{\frac{1}{q}} \left( \sum_{n=m}^{\infty} v_n^{1-p'} \right)^{\frac{1}{p'}},
\]
\[
E = \sup_{m \in Z_+} \left( \sum_{n=1}^{m} u_n \right)^{\frac{1}{q}} \left( \sum_{n=m}^{\infty} k_{n,m}^p v_n^{1-p'} \right)^{\frac{1}{p'}}.
\]

3. Main Result

Now, we prove our main result.

Theorem 3.1. Suppose $1 < p \leq q < \infty$, $\{\phi_{i,n}\}, \{\psi_{i,n}\}; i = 1, 2$, are sequences of real numbers not necessarily non-negative, $\{u_n\}, \{v_n\}$ are weight sequences and $k = \{k_{m,n}\}; m, n \in Z_+$ is discrete Oinarov kernel. The inequality
\[
\left( \sum_{m=1}^{\infty} \left| k_{m,n} \phi_{1,n} a_n + \phi_{2,m} \sum_{n=m}^{\infty} k_{n,m} \psi_{2,n} a_n \right|^q u_m \right)^{\frac{1}{q}} \leq C \left( \sum_{m=1}^{\infty} |a_m|^p v_m \right)^{\frac{1}{p}},
\]
holds for all sequence $\{a_n\} \in \ell^p(v_n)$ not necessarily non-negative and a suitable positive constant $C$ if and only if $\max(F, G, H, K) < \infty$, where
\[
F = \sup_{m \in Z_+} \left( \sum_{n=m}^{\infty} k_{n,m}^q |\phi_{1,n} u_n|^q \right)^{\frac{1}{q}} \left( \sum_{n=1}^{m} |\phi_{1,n} v_n|^{1-p'} \right)^{\frac{1}{p'}},
\]
\[
G = \sup_{m \in Z_+} \left( \sum_{n=m}^{\infty} |\phi_{1,n} u_n|^q \right)^{\frac{1}{q}} \left( \sum_{n=1}^{m} k_{n,m}^p |\psi_{1,n} v_n|^{1-p'} \right)^{\frac{1}{p'}}.
\]
\[ H = \sup_{m \in \mathbb{Z}^+} \left( \sum_{n=1}^{m} k_{m,n}^q \phi_{2,n} u_n \right)^{\frac{1}{q}} \left( \sum_{n=m}^{\infty} \left| \psi_{2,n} v_n \right|^{\frac{1}{p'}} \right)^{\frac{1}{p'}} , \]

\[ K = \sup_{m \in \mathbb{Z}^+} \left( \sum_{n=1}^{m} \left| \phi_{2,n} \right|^{\frac{1}{q}} \left( \sum_{n=m}^{\infty} \left| k_{n,m}^{p'} \psi_{2,n} v_n \right|^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \right) . \]

**Proof. Sufficiency.** By an application of Proposition (2.1) we find that \( \max(F,G) < \infty \) holds implies the inequality

\[ \left( \sum_{m=1}^{\infty} \left| \phi_{1,m} \sum_{n=1}^{m} k_{m,n} \psi_{1,n} a_n \right|^{q} u_m \right)^{\frac{1}{q}} \leq C \left( \sum_{m=1}^{\infty} \left| a_m \right|^{p} v_m \right)^{\frac{1}{p}} \]

holds for a positive constant \( C \).

Analogously, making similar arguments and using Proposition (2.2), we can prove that \( \max(H,K) < \infty \) holds implies the inequality

\[ \left( \sum_{m=1}^{\infty} \left| \phi_{2,m} \sum_{n=m}^{\infty} k_{n,m} \psi_{2,n} a_n \right|^{q} u_m \right)^{\frac{1}{q}} \leq C \left( \sum_{m=1}^{\infty} \left| a_m \right|^{p} v_m \right)^{\frac{1}{p}} , \]

holds for a positive constant \( C \).

Sufficiency now follows from the inequality

\[ \left( \sum_{m=1}^{\infty} \left| \phi_{1,m} \sum_{n=1}^{m} k_{m,n} \psi_{1,n} a_n \right|^{q} u_m + \phi_{2,m} \sum_{n=m}^{\infty} k_{n,m} \psi_{2,n} a_n \right)^{\frac{1}{q}} \leq \left( \sum_{m=1}^{\infty} \left| a_m \right|^{p} v_m \right)^{\frac{1}{p}} . \]

**Necessity.** For \( \varepsilon > 0, n < t < m \) and \( 1 \leq \alpha < \beta < \infty \) we define modified weight sequence \( \{v_{n_{\varepsilon}}\} \) where

\[ v_{n_{\varepsilon}} = \max \{v_n, k_{t,n}^{p} |\psi_{1,n}|^{p \varepsilon} \} \]

and sequence \( \{a_{t,n}\} \) where

\[ a_{t,n} = \left\{ \begin{array}{ll} \left( k_{t,n} |\psi_{1,n} v_{n_{\varepsilon}}^{1/p} \right)^{p'-1} \text{sgn} \left\{ \psi_{1,n} v_{n_{\varepsilon}}^{1/p} \right\} v_{n_{\varepsilon}}^{-1/p}, & \alpha < n < \beta, \\ 0, & \text{otherwise.} \end{array} \right. \]
For the above choice, the L.H.S. of (3.1) becomes

\[
\left( \sum_{m=1}^{\infty} \left| \phi_{1,m} \sum_{n=1}^{m} k_{m,n} \psi_{1,n} a_{t,n} + \phi_{2,m} \sum_{n=m}^{\infty} k_{n,m} \psi_{2,n} a_{t,n} \right| u_{m} \right)^{\frac{1}{q}}
\]

\[
\geq \left( \sum_{m=\beta}^{\infty} \left| \phi_{1,m} \sum_{n=1}^{m} k_{m,n} \psi_{1,n} a_{t,n} + \phi_{2,m} \sum_{n=m}^{\infty} k_{n,m} \psi_{2,n} a_{t,n} \right| u_{m} \right)^{\frac{1}{q}}
\]

\[
= \left( \sum_{m=\beta}^{\infty} \left| \phi_{1,m} \sum_{n=1}^{m} k_{m,n} \psi_{1,n} a_{t,n} \right| q u_{m} \right)^{1/q} \quad \text{(since } a_{t,n} = 0 \text{ for } n \geq \beta) \]

\[
\geq \left( \sum_{m=\beta}^{\infty} u_{m} |\phi_{1,m}| q \left( \sum_{n=\alpha}^{\beta} k_{m,n} k_{t,n}^{-1} |\psi_{1,n} v_{n}^{-1/p'}| \right)^{1/q} \right)
\]

\[
\text{(Since } (\psi_{1,n} v_{n}^{-1/p}) \text{sgn } \left\{ \psi_{1,n} v_{n}^{-1/p} \right\} = |\psi_{1,n} v_{n}^{-1/p}| \text{)}
\]

\[
> \left( \sum_{m=\beta}^{\infty} u_{m} |\phi_{1,m}| q \left( \sum_{n=\alpha}^{\beta} k_{t,n}^{-1} |\psi_{1,n} v_{n}^{-1/p'}| \right)^{1/q} \right)
\]

\[
\text{(}k_{m,n} \text{ is Oinarov kernel implies } k_{m,n} > k_{t,n} \text{ for } m > t > n) \]

\[
= \left( \sum_{m=\beta}^{\infty} u_{m} |\phi_{1,m}| q \left( \sum_{n=\alpha}^{\beta} k_{t,n}^{-1} |\psi_{1,n} v_{n}^{-1/p'}| \right)^{1/q} \right),
\]

while the R.H.S. of (3.1) can be estimated as

\[
C \left( \sum_{n=1}^{\infty} |a_{t,n}|^{p} v_{n} \right)^{\frac{1}{p}} \leq C \left( \sum_{n=1}^{\infty} |a_{t,n}|^{p} v_{n} \right)^{\frac{1}{p}}
\]

\[
= C \left( \sum_{n=\alpha}^{\beta} k_{t,n}^{-1} |\psi_{1,n} v_{n}^{-1/p'}| \right)^{1/p'}
\]

\[
\leq C \varepsilon^{1-p'} (\beta - \alpha)^{1/p} < \infty.
\]

Consequently the inequality (3.1) yields

\[
\left( \sum_{m=\beta}^{\infty} u_{m} |\phi_{1,m}| q \left( \sum_{n=\alpha}^{\beta} k_{t,n}^{-1} |\psi_{1,n} v_{n}^{-1/p'}| \right)^{1/p'} \right) \leq C < \infty,
\]
which holds for a positive constant $C$ independent of $\alpha$ and $\varepsilon$. For $\alpha \to 1$ and $\varepsilon \to 0$ (via a subsequence), we get $v_{n\varepsilon} \to v_n$ and consequently taking supremum over all integer $\beta \geq 1$, we find that $G$ is finite.

Repeating the above process in the inequality (3.1) with $v_{n\varepsilon}$ and $a_{t,n}$ replaced, respectively, by

$$\tilde{v}_{n\varepsilon} = \max \left\{ k_{n,t}^p |\psi_{2,n}^1|^{p \varepsilon}, v_n \right\}$$

and

$$\tilde{a}_{n,t} = \left\{ \begin{array}{ll}
\{ k_{n,t}^p |\psi_{2,n}^1|^{p \varepsilon - 1/p} \}^{p' - 1} \sgn \{ \psi_{2,n}^1 \} \tilde{v}_{n\varepsilon}^{-1/p}, & \alpha < n < \beta, \\
0, & \text{otherwise},
\end{array} \right.$$  

with $m < t < n$ and $1 \leq \alpha < \beta < \infty$, we find $K < \infty$.

Conjugate operator to $T$ denoted by $T^*$ is defined as

$$T^*(a_n) = \psi_{1,m} \sum_{n=m}^{\infty} k_{n,m} \phi_{1,n} a_n + \psi_{2,m} \sum_{n=1}^{m} k_{m,n} \phi_{2,n} a_n.$$  

By using duality argument it may be shown that the inequality (3.1) and

$$||T^*(a_n)||_{p',v_1^{1-p'}} \leq C ||a_n||_{q',u_1^{1-q'}}$$  

are equivalent, where norm used is that of weighted sequence space. For a new weight sequence $\{u_{n\varepsilon}\}$ where

$$u_{n\varepsilon} = \min \left\{ u_n, k_{t,n}^{-q} |\phi_{2,n}|^{-q \varepsilon - 1} \right\}$$

we find that $||a_n||_{q',u_1^{1-q'}} \leq ||a_n||_{q',u_{n\varepsilon}^{1-q'}}$ holds and therefore, the inequality (3.2) yields

$$||T^*(a_n)||_{p',v_1^{1-p'}} \leq C ||a_n||_{q',u_{n\varepsilon}^{1-q'}}.$$  

For $1 \leq \alpha < \beta < \infty$ and $n < t < m$, define a sequence $\{g_{t,n}\}$ where

$$g_{t,n} = \left\{ \begin{array}{ll}
\{ k_{t,n} |\phi_{2,n} u_{n\varepsilon}^{1/q} \}^{q-1} \sgn \{ \phi_{2,n} u_{n\varepsilon}^{1/q} \} u_{n\varepsilon}^{1/q}, & \alpha < n < \beta, \\
0, & \text{otherwise},
\end{array} \right.$$  

Substituting $a_n = g_{t,n}$ in the inequality (3.3) and using similar argument, we find $H < \infty$. Repeating the above process in the inequality (3.3) with $u_{n\varepsilon}$ and $g_{t,n}$ replaced, respectively, by

$$\tilde{u}_{n\varepsilon} = \min \left\{ u_n, k_{n,t}^{-q} |\phi_{1,n}|^{-q \varepsilon - 1} \right\}$$
and

\[ \tilde{g}_{t,n} = \begin{cases} 
\left\{ k_{n,t} \phi_{1,n} \tilde{u}_{n\varepsilon}^{1/q} \right\}^{q-1} \text{sgn} \left\{ \phi_{1,n} \tilde{u}_{n\varepsilon}^{1/q} \right\} \tilde{u}_{n\varepsilon}^{1/q} , & \alpha < n < \beta, \\
0 , & \text{otherwise},
\end{cases} \]

where \( m < t < n \) and \( 1 \leq \alpha < \beta < \infty \), we find \( F < \infty \). The necessity is now proved. \( \square \)

References


