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DISCRETE HARDY-TYPE INEQUALITY WITH KERNEL

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Abstract: Necessary and sufficient conditions are given for the validity of discrete Hardy's inequality for the sum of two discrete Hardy-type operators with not necessary non-negative coefficients.

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Key Words: discrete hardy inequality, weighted sequence space and its norm, duality

1. Introduction

Discrete Hardy inequality has been characterized for the discrete Hardy operator in [2,3,9]. Sufficient condition is available in [1] for the validity of the discrete Hardy inequality, for the discrete Hardy-type operator with kernel $k = \{k_{m,n}\}$; $m, n \in Z_+$ defined on $D = \{(m,n) \in Z_+ \times Z_+ : n \leq m\}$, $k_{m,n}$ is non-increasing in m and non-decreasing in n. For the discrete Hardy-type operator with general kernel, sufficient conditions for the validity of discrete Hardy inequality is available in [8] which is both necessary and sufficient when the kernel is of product type.

In this note, we consider a different discrete operator

$$T : \ell^p(v_n) \to \ell^q(u_n)$$

defined as

$$T(a_n) = \phi_{1,m} \sum_{n=1}^{m} \psi_{1,n} k_{m,n} a_n + \phi_{2,m} \sum_{n=m}^{\infty} \psi_{2,n} k_{n,m} a_n$$

where the kernel involve is discrete Oinarov kernel (not considered in earlier

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literature) and characterize Hardy inequality. Here $\{\phi_{i,n}\}$, $\{\psi_{i,n}\}$; $i = 1, 2$ and $\{a_n\}$ are sequences of real numbers not necessarily non-negative. Therefore T is not necessarily non-negative. Hardy inequality for the operator T gives better best constant for the Hardy inequality than the same for the single discrete Hardy-type operator. Our study is motivated by this.

Kernel $k = \{k_{m,n}\}; m, n \in Z_+$ is called discrete Oinarov kernel if (a) $k_{m,n} \geq 0$ for $0 < n < m$; (b) $k_{m,n}$ is increasing in m and decreasing in n, (c) $k_{m,n} \approx k_{m,p} + k_{p \cdot n}$ for $0 < n < p < m$.

Throughout the paper, n, m, α, β are positive integers (Z_+); t is fixed positive integer; p, q are real numbers; $p' = p/(p-1)$ is conjugate to p and the same is true for $q \cdot \ell^p(v_n)$ denotes weighted sequence space. $\{u_n\}, \{v_n\}; n \in Z_+$ are weight sequences.

For complete description of Hardy inequality, we use to refer [5] and monographs [6,7].

2. Preliminaries

In the following result, we state the discrete case of [7, Theorem 2.10]:

Proposition 2.1. *Suppose* $1 < p \leq q < \infty, \{u_n\}, \{v_n\}$ *are weight sequences and* $k = \{k_{m,n}\}; m, n \in Z_+$ *is discrete Oinarov kernel. The inequality*

$$\left(\sum_{m=1}^{\infty}\left|\sum_{n=1}^{m} k_{m,n}a_n\right|^q u_m\right)^{\frac{1}{q}} \leq C\left(\sum_{m=1}^{\infty}|a_m|^p v_m\right)^{\frac{1}{p}},$$

holds for all non-negative sequence $\{a_n\} \in \ell^p(v_n)$ *and a suitable positive constant* C *if and only if* $\max(A, B) < \infty$, *where*

$$A = \sup_{m \in Z_+}\left(\sum_{n=m}^{\infty} k_{n,m}^q u_n\right)^{\frac{1}{q}}\left(\sum_{n=1}^{m} v_n^{1-p'}\right)^{\frac{1}{p'}},$$

$$B = \sup_{m \in Z_+}\left(\sum_{n=m}^{\infty} u_n\right)^{\frac{1}{q}}\left(\sum_{n=1}^{m} k_{m,n}^{p'} v_n^{1-p'}\right)^{\frac{1}{p'}}.$$

Proof. The proof of Proposition 2.1 is similar to continuous case. Continuous case was first proved in [4]. We omit the detail.

By using duality argument and applying suitable substitutions in Proposition (2.1), we can prove the following.

Proposition 2.2. *Suppose* $1 < p \le q < \infty, \{u_n\}, \{v_n\}$ *are weight sequences and* $k = \{k_{n,m}\}; n, m \in Z_+$ *is discrete Oinarov kernel. The inequality*

$$\left(\sum_{m=1}^{\infty}\left|\sum_{n=m}^{\infty} k_{n,m}a_n\right|^q u_m\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}|a_m|^p v_m\right)^{\frac{1}{p}},$$

holds for all non-negative sequence $\{a_n\} \in \ell^p(v_n)$ *and a suitable positive constant* C *if and only if* $\max(D, E) < \infty$ *where*

$$D = \sup_{m\in Z_+}\left(\sum_{n=1}^{m} k_{m,n}^q u_n\right)^{\frac{1}{q}}\left(\sum_{n=m}^{\infty} v_n^{1-p'}\right)^{\frac{1}{p'}},$$

$$E = \sup_{m\in Z_+}\left(\sum_{n=1}^{m} u_n\right)^{\frac{1}{q}}\left(\sum_{n=m}^{\infty} k_{n,m}^{p'} v_n^{1-p'}\right)^{\frac{1}{p'}}.$$

3. Main Result

Now, we prove our main result.

Theorem 3.1. *Suppose* $1 < p \le q < \infty$, $\{\phi_{i,n}\}$, $\{\psi_{i,n}\}$; $i = 1, 2$, *are sequences of real numbers not necessarily non-negative,* $\{u_n\}, \{v_n\}$ *are weight sequences and* $k = \{k_{m,n}\}$; $m, n \in Z_+$ *is discrete Oinarov kernel. The inequality*

$$\left(\sum_{m=1}^{\infty}\left|\phi_{1,m}\sum_{n=1}^{m} k_{m,n}\psi_{1,n}a_n + \phi_{2,m}\sum_{n=m}^{\infty} k_{n,m}\psi_{2,n}a_n\right|^q u_m\right)^{\frac{1}{q}} \le C\left(\sum_{m=1}^{\infty}|a_m|^p v_m\right)^{\frac{1}{p}}, \quad (3.1)$$

holds for all sequence $\{a_n\} \in \ell^p(v_n)$ *not necessarily non-negative and a suitable positive constant* C *if and only if* $\max(F, G, H, K) < \infty$, *where*

$$F = \sup_{m\in Z_+}\left(\sum_{n=m}^{\infty} k_{n,m}^q|\phi_{1,n}u_n^{\frac{1}{q}}|^q\right)^{1/q}\left(\sum_{n=1}^{m}|\psi_{1,n}v_n^{-\frac{1}{p}}|^{p'}\right)^{1/p'},$$

$$G = \sup_{m\in Z_+}\left(\sum_{n=m}^{\infty}\left|\phi_{1,n}u_n^{\frac{1}{q}}\right|^q\right)^{\frac{1}{q}}\left(\sum_{n=1}^{m} k_{m,n}^{p'}\left|\psi_{1,n}v_n^{-\frac{1}{p}}\right|^{p'}\right)^{\frac{1}{p'}},$$

$$H = \sup_{m\in Z_+} \left(\sum_{n=1}^{m} k_{m,n}^{q} \left|\phi_{2,n} u_n^{\frac{1}{q}}\right|^q\right)^{\frac{1}{q}} \left(\sum_{n=m}^{\infty} \left|\psi_{2,n} v_n^{-\frac{1}{p}}\right|^{p'}\right)^{\frac{1}{p'}},$$

$$K = \sup_{m\in Z_+} \left(\sum_{n=1}^{m} \left|\phi_{2,n} u_n^{\frac{1}{q}}\right|^q\right)^{\frac{1}{q}} \left(\sum_{n=m}^{\infty} k_{n,m}^{p'} \left|\psi_{2,n} v_n^{-\frac{1}{p}}\right|^{p'}\right)^{\frac{1}{p'}}.$$

Proof. Sufficiency. By an application of Proposition (2.1) we find that $\max(F, G) < \infty$ holds implies the inequality

$$\left(\sum_{m=1}^{\infty} \left|\phi_{1,m} \sum_{n=1}^{m} k_{m,n}\psi_{1,n} a_n\right|^q u_m\right)^{\frac{1}{q}} \le C \left(\sum_{m=1}^{\infty} |a_m|^p v_m\right)^{\frac{1}{p}}$$

holds for a positive constant C.

Analogously, making similar arguments and using Proposition (2.2), we can prove that $\max(H, K) < \infty$ holds implies the inequality

$$\left(\sum_{m=1}^{\infty} \left|\phi_{2,m} \sum_{n=m}^{\infty} k_{n,m}\psi_{2,n} a_n\right|^q u_m\right)^{\frac{1}{q}} \le C \left(\sum_{m=1}^{\infty} |a_m|^p v_m\right)^{\frac{1}{p}},$$

holds for a positive constant C.

Sufficiency now follows from the inequality

$$\left(\sum_{m=1}^{\infty} \left|\phi_{1,m} \sum_{n=1}^{m} k_{m,n}\psi_{1,n} a_n + \phi_{2,m} \sum_{n=m}^{\infty} k_{n,m}\psi_{2,n} a_n\right|^q u_m\right)^{\frac{1}{q}}$$
$$\le \left(\sum_{m=1}^{\infty} \left|\phi_{1,m} \sum_{n=1}^{m} k_{m,n}\psi_{1,n} a_n\right|^q u_m\right)^{\frac{1}{q}} + \left(\sum_{m=1}^{\infty} \left|\phi_{2,m} \sum_{n=m}^{\infty} k_{n,m}\psi_{2,n} a_n\right|^q u_m\right)^{\frac{1}{q}}.$$

Necessity. For $\varepsilon > 0$, $n < t < m$ and $1 \le \alpha < \beta < \infty$ we define modified weight sequence $\{v_{n_\varepsilon}\}$ where

$$v_{n_\varepsilon} = \max\{v_n,\ k_{t,n}^{p} |\psi_{1,n}|^p \varepsilon\}$$

and sequence $\{a_{t,n}\}$ where

$$a_{t,n} = \begin{cases} \left\{k_{t,n} |\psi_{1,n} v_{n_\varepsilon}^{-1/p}\right\}^{p'-1} \operatorname{sgn}\left\{\psi_{1,n} v_{n_\varepsilon}^{-1/p}\right\} v_{n_\varepsilon}^{-1/p}, & \alpha < \ n < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

For the above choice, the L.H.S. of (3.1) becomes

$$\left(\sum_{m=1}^{\infty}\left|\phi_{1,m}\sum_{n=1}^{m}k_{m,n}\psi_{1,n}a_{t,n}+\phi_{2,m}\sum_{n=m}^{\infty}k_{n,m}\psi_{2,n}a_{t,n}\right|^{q}u_m\right)^{\frac{1}{q}}$$

$$\geq\left(\sum_{m=\beta}^{\infty}\left|\phi_{1,m}\sum_{n=1}^{m}k_{m,n}\psi_{1,n}a_{t,n}+\phi_{2,m}\sum_{n=m}^{\infty}k_{n,m}\psi_{2,n}a_{t,n}\right|^{q}u_m\right)^{\frac{1}{q}}$$

$$=\left(\sum_{m=\beta}^{\infty}|\phi_{1,m}\sum_{n=1}^{m}k_{m,n}\psi_{1,n}a_{t,n}|^{q}u_m\right)^{1/q}\qquad(\text{since } a_{t,n}=0 \text{ for } n\geq\beta)$$

$$\geq\left(\sum_{m=\beta}^{\infty}u_m|\phi_{1,m}|^{q}\left(\sum_{n=\alpha}^{\beta}k_{m,n}k_{t,n}^{p'-1}|\psi_{1,n}v_{n_\varepsilon}^{-1/p}|^{p'}\right)^{q}\right)^{1/q}$$

$$(\text{Since } (\psi_{1,n}v_{n_\varepsilon}^{-1/p})\text{sgn}\left\{\psi_{1,n}v_{n_\varepsilon}^{-\frac{1}{p}}\right\}=|\,\psi_{1,n}v_{n_\varepsilon}^{-\frac{1}{p}}\,|)$$

$$>\left(\sum_{m=\beta}^{\infty}u_m|\phi_{1,m}|^{q}\left(\sum_{n=\alpha}^{\beta}k_{t,n}^{p'}|\psi_{1,n}v_{n_\varepsilon}^{-1/p}|^{p'}\right)^{q}\right)^{1/q}$$

($k_{m,n}$ is Oinarov kernal implies $k_{m,n}>k_{t,n}$ for $m>t>n$)

$$=\left(\sum_{m=\beta}^{\infty}u_m\,|\phi_{1,m}|^{q}\right)^{\frac{1}{q}}\left(\sum_{n=\alpha}^{\beta}k_{t,n}^{p'}\left|\psi_{1,n}v_{n_\varepsilon}^{-\frac{1}{p}}\right|^{p'}\right),$$

while the R.H.S. of (3.1) can be estimated as

$$\begin{aligned}C\left(\sum_{n=1}^{\infty}|a_{t,n}|^{p}\,v_n\right)^{\frac{1}{p}}&\leq C\left(\sum_{n=1}^{\infty}|a_{t,n}|^{p}\,v_{n_\varepsilon}\right)^{\frac{1}{p}}\\&=C\left(\sum_{n=\alpha}^{\beta}k_{t,n}^{p'}|\psi_{1,n}v_{n_\varepsilon}^{-\frac{1}{p}}|^{p'}\right)^{1/p'}\\&\leq C\varepsilon^{1-p'}(\beta-\alpha)^{1/p}<\infty.\end{aligned}$$

Consequently the inequality (3.1) yields

$$\left(\sum_{m=\beta}^{\infty}u_m|\phi_{1,m}|^{q}\right)^{1/q}\left(\sum_{n=\alpha}^{\beta}k_{t,n}^{p'}|\psi_{1,n}v_{n_\varepsilon}^{-\frac{1}{p}}|^{p'}\right)^{1/p'}\leq C<\infty,$$

which holds for a positive constant C independent of α and ε. For $\alpha \to 1$ and $\varepsilon \to 0$ (via a subsequence), we get $v_{n_\varepsilon} \to v_n$ and consequently taking supremum over all integer $\beta \geq 1$, we find that G is finite.

Repeating the above process in the inequality (3.1) with v_{n_ε} and $a_{t,n}$ replaced, respectively, by

$$\tilde{v}_{n_\varepsilon} = \max\left\{k_{n,t}^p |\psi_{2,n}|^p \varepsilon, v_n\right\}$$

and

$$\tilde{a}_{n,t} = \begin{cases} \{k_{n,t}|\psi_{2,n}\tilde{v}_{n_\varepsilon}^{-1/p}|\}^{p'-1} \operatorname{sgn}\left\{\psi_{2,n}\tilde{v}_{n_\varepsilon}^{-1/p}\right\} \tilde{v}_{n_\varepsilon}^{-1/p}, & \alpha < n < \beta, \\ 0, & \text{otherwise,} \end{cases}$$

with $m < t < n$ and $1 \leq \alpha < \beta < \infty$, we find $K < \infty$.

Conjugate operator to T denoted by T^* is defined as

$$T^*(a_n) = \psi_{1,m} \sum_{n=m}^{\infty} k_{n,m}\phi_{1,n}a_n + \psi_{2,m} \sum_{n=1}^{m} k_{m,n}\phi_{2,n}a_n.$$

By using duality argument it may be shown that the inequality (3.1) and

$$||T^*(a_n)||_{p',v_n^{1-p'}} \leq C||a_n||_{q',u_n^{1-q'}} \tag{3.2}$$

are equivalent, where norm used is that of weighted sequence space. For a new weight sequence $\{u_{n_\varepsilon}\}$ where

$$u_{n_\varepsilon} = \min\left\{u_n, k_{t,n}^{-q}|\phi_{2,n}|^{-q}\varepsilon^{-1}\right\}$$

we find that $||a_n||_{q',u_n^{1-q'}} \leq ||a_n||_{q',u_{n_\varepsilon}^{1-q'}}$ holds and therefore, the inequality (3.2) yields

$$||T^*(a_n)||_{p',v_n^{1-p'}} \leq C||a_n||_{q',u_{n_\varepsilon}^{1-q'}}. \tag{3.3}$$

For $1 \leq \alpha < \beta < \infty$ and $n < t < m$, define a sequence $\{g_{t,n}\}$ where

$$g_{t,n} = \begin{cases} \left\{k_{t,n}|\phi_{2,n}u_{n_\varepsilon}^{1/q}|\right\}^{q-1} \operatorname{sgn}\left\{\phi_{2,n}u_{n_\varepsilon}^{1/q}\right\} u_{n_\varepsilon}^{1/q}, & \alpha < n < \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting $a_n = g_{t,n}$ in the inequality (3.3) and using similar argument, we find $H < \infty$. Repeating the above process in the inequality (3.3) with u_{n_ε} and $g_{t,n}$ replaced, respectively, by

$$\tilde{u}_{n_\varepsilon} = \min\left\{u_n, k_{n,t}^{-q}|\phi_{1,n}|^{-q}\varepsilon^{-1}\right\}$$

and

$$\tilde{g}_{t,n} = \begin{cases} \left\{k_{n,t}|\phi_{1,n}\tilde{u}_{n_\varepsilon}^{1/q}|\right\}^{q-1} \operatorname{sgn}\left\{\phi_{1,n}\tilde{u}_{n_\varepsilon}^{1/q}\right\}\tilde{u}_{n_\varepsilon}^{1/q}, & \alpha < \ n < \beta, \\ 0, & \text{otherwise}, \end{cases}$$

where $m < t < n$ and $1 \leq \alpha < \beta < \infty$, we find $F < \infty$. The necessity is now proved. $\square$

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