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FAST BLOCK DIAGONALIZATION OF (k, k')-PENTADIAGONAL MATRICES

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Abstract: In this paper, we provide a block diagonalization algorithm of (k, k')-pentadiagonal matrices. The algorithm is a structure-preserving algorithm in that the small diagonal blocks essentially have the same nonzero structure as the original one, and it can be regarded as an extension of the block diagonalization algorithm of k-tridiagonal matrices in [T. Sogabe, M.E.A. El-Mikkawy, Appl. Math. Compute., 218 (2011), 2740-2743].

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1. Introduction

We consider $n \times n$ (k, k')-pentadiagonal matrices $A_n^{(k,k')} \in \mathbb{C}^{n\times n}$ defined as

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$$A_n^{(k,k')} :=$$

$$\begin{array}{c|cccccccccccc|}
 & 1 & 2 & \cdots & k & k{+}1 & k{+}2 & \cdots & k' & k'{+}1 & k'{+}2 & \cdots & n \\
\hline
1 & d_1 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 & a'_1 & 0 & \cdots & 0 \\
2 & 0 & d_2 & & & & a_2 & & & & a'_2 & & \vdots \\
\vdots & \vdots & & \ddots & & & & \ddots & & & & \ddots & 0 \\
k & 0 & & & & & & & & & & & a'_{n-k'} \\
k{+}1 & b_{k+1} & & & & \ddots & & & & \ddots & & & 0 \\
k{+}2 & 0 & b_{k+2} & & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & & & & \ddots & & & & \ddots & 0 \\
k' & 0 & & & & & & & & & & & a_{n-k} \\
k'{+}1 & b'_{k'+1} & & & & \ddots & & & & \ddots & & & 0 \\
k'{+}2 & 0 & b'_{k'+2} & & & & & & & & & & \vdots \\
\vdots & \vdots & & \ddots & & & & \ddots & & & & \ddots & 0 \\
n & 0 & \cdots & 0 & b'_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 & d_n \\
\hline
\end{array}, \tag{1}$$

where $1 \le k \le k' < n$. The matrices arise in a finite difference discretization of partial differential equations (see, e.g., [1, Section 12.1]).

The matrices (1) include three important matrices. First, $A_n^{(1,2)}$ corresponds to an ordinary pentadiagonal matrix. For the recent developments, see, e.g., [2], [5], [6], and [10]. Second, $A_n^{(k,k)}$ corresponds to a k-tridiagonal matrix. For the recent developments, see, e.g., [4], [7], [8], and [9]. Third, if $a'_i = b_j = 0$ for all i and j or $a_i = b'_j = 0$ for all i and j in $A_n^{(k,k')}$, the matrices arise in the discrete hungry Lotka-Volterra system, see, e.g., [3].

The purpose of this paper is to present a block diagonalization algorithm of (k, k')-pentadiagonal matrices. The algorithm can be regarded as an extension of a block diagonalization algorithm of k-tridiagonal matrices in [8].

This paper is organized as follows: in Section 2, we give two lemmas as preliminaries; in Section 3, we present a block diagonalization algorithm of (k, k')-pentadiagonal matrices and describe the nonzero structures of the obtained diagonal blocks; in Section 4, illustrative examples are shown; finally, concluding remarks are made in Section 5.

Preliminaries

In this section, we give two lemmas for proving the theorem in the next section.

Let P and Q be $n \times n$ permutation matrices, and let P' and Q' be $n \times m$ and $n \times \ell$ submatrices of P and Q, respectively. Then, P' and Q' are written by

$$P' = [\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_m}], \quad Q' = [\boldsymbol{e}_{j_1}, \boldsymbol{e}_{j_2}, \ldots, \boldsymbol{e}_{j_\ell}], \tag{2}$$

where $i_p, j_q \in \{1, 2, \ldots, n\}$, $p = 1, 2, \ldots, m$, $q = 1, 2, \ldots, \ell$, and $\boldsymbol{e}_{i_p}$ denotes the n-dimensional i_p-th canonical vector.

We now have the following lemmas:

Lemma 1. *Let $M \in \mathbb{C}^{n\times n}$. Then, multiplying M by $(P')^{\mathrm{T}}$ and Q' in (2) yields an $m \times \ell$ submatrix of $P^{\mathrm{T}}MQ$ where the (p, q) element is represented by*

$$((P')^{\mathrm{T}}MQ')_{p,q} = M_{i_p,j_q},$$

where M_{i_p,j_q} denotes the (i_p, j_q) element of M.

Proof. Easy. □

Lemma 2. *Let $A_n^{(k,k')}$ be an $n \times n$ (k, k')-pentadiagonal matrix, and let $\overline{r}$ be the equivalence class of the set $\mathbb{N}_n := \{i \in \mathbb{N} \,|\, 1 \leq i \leq n\}$ as follows: $\overline{r} := \{j \in \mathbb{N}_n \,|\, j \equiv r \,(\mathrm{mod}\ m)\}$, where $m := \gcd(k, k')$. That is, $\mathbb{N}_n = \bigcup_{r=1}^{m} \overline{r}$. Then, it follows that $(A_n^{(k,k')})_{p,q} = 0$ for $p \in \overline{r_1}$ and $q \in \overline{r_2}$ $(r_1 \neq r_2)$, where $r_1, r_2 \in \{1, 2, \ldots, m\}$.*

Proof. From the definition of $A_n^{(k,k')}$ in (1), the (i, j) element of the matrix is represented by

$$(A_n^{(k,k')})_{i,j} = \begin{cases} d_i & \text{if } \ i = j, \\ a_i & \text{if } \ i = j - k, \\ b_i & \text{if } \ i = j + k, \\ a'_i & \text{if } \ i = j - k', \\ b'_i & \text{if } \ i = j + k', \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

where $1 \leq i \leq n$ and $1 \leq j \leq n$.

Let $\ell \in \{0, -k, k, -k', k'\}$. Then, from m=gcd(k, k'), it follows that $\ell \in m\mathbb{Z}$. Since $q \in \overline{r_2}$, we see that $q \in r_2 + m\mathbb{Z}$. Thus,

$$q + \ell \in r_2 + m\mathbb{Z}. \tag{4}$$

From $p \in \overline{r_1}$, it follows that

$$p \in r_1 + m\mathbb{Z}. \tag{5}$$

$r_1 \neq r_2$ indicates that $(r_1 + m\mathbb{Z}) \cap (r_2 + m\mathbb{Z}) = \emptyset$. Hence, from (4) and (5), we have, for any $p \in \overline{r_1}$ and any $q \in \overline{r_2}$, that

$$p \neq q + \ell, \quad \ell \in \{0, -k, k, -k', k'\}. \tag{6}$$

(3) and (6) lead to $(A_n^{(k,k')})_{p,q} = 0$ for any $p \in \overline{r_1}$ and any $q \in \overline{r_2}$, which concludes the proof. □

2. Main Results

In this section, we propose a block diagonalization algorithm of a (k, k')-pentadiagonal matrix that is an extension of a block diagonalization algorithm of k-tridiagonal matrices in [8].

Using Lemma 2, we can block-diagonalize $A_n^{(k,k')}$ as shown in Theorem 3.

Theorem 3. *Let $A_n^{(k,k')}$ be an $n \times n$ (k, k')-pentadiagonal matrix, let $\mathbb{N}_n$ and $\overline{r}$ be the same notations as in Lemma 2, and let $|\overline{r}|$ be the number of elements in $\overline{r}$. Then, $A_n^{(k,k')}$ is block-diagonalized into a block diagonal matrix with m diagonal blocks by an $n \times n$ permutation matrix P of the form*

$$P := [P_{\overline{1}}, P_{\overline{2}}, \dots, P_{\overline{m}}], \tag{7}$$

where $P_{\overline{r}}$ $(r = 1, 2, \dots, m)$ are $n \times |\overline{r}|$ matrices as follows:

$$P_{\overline{r}} := [\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \dots, \boldsymbol{e}_{i_{|\overline{r}|}}], \tag{8}$$

where $i_p \in \overline{r}$ $(p = 1, 2, \dots, |\overline{r}|)$.

Proof. Substituting (7) into $P^{\mathrm{T}} A_n^{(k,k')} P$ yields

$$P^{\mathrm{T}} A_n^{(k,k')} P = \begin{pmatrix} P_{\overline{1}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{1}} & P_{\overline{1}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{2}} & \cdots & P_{\overline{1}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{m}} \\ P_{\overline{2}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{1}} & P_{\overline{2}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{2}} & \cdots & P_{\overline{2}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{m}} \\ \vdots & \vdots & \ddots & \vdots \\ P_{\overline{m}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{1}} & P_{\overline{m}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{2}} & \cdots & P_{\overline{m}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{m}} \end{pmatrix}.$$

Here, let $P_{\overline{r_1}} := [\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_{|\overline{r_1}|}}]$ and $P_{\overline{r_2}} := [\boldsymbol{e}_{j_1}, \boldsymbol{e}_{j_2}, \ldots, \boldsymbol{e}_{j_{|\overline{r_2}|}}]$ for $i_p \in \overline{r_1}$ and $j_q \in \overline{r_2}$ $(r_1 \neq r_2)$. Then, we have

$$P_{\overline{r_1}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{r_2}} = O_{|\overline{r_1}| \times |\overline{r_2}|}$$

since $(P_{\overline{r_1}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{r_2}})_{p,q} = (A_n^{(k,k')})_{i_p,j_q} = 0$ from Lemmas 1 and 2. Thus, we have

$$P^{\mathrm{T}} A_n^{(k,k')} P = \begin{pmatrix} P_{\overline{1}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{1}} & & \\ & \ddots & \\ & & P_{\overline{m}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{m}} \end{pmatrix}. \tag{9}$$

This conclude the proof. □

Hereafter, for simplicity, (9) is rewritten as

$$P^{\mathrm{T}} A_n^{(k,k')} P = A_{\overline{1}} \oplus A_{\overline{2}} \oplus \cdots \oplus A_{\overline{m}},$$

where $A_{\overline{r}} := P_{\overline{r}}^{\mathrm{T}} A_n^{(k,k')} P_{\overline{r}}$ and "$\oplus$" denotes direct sum.

A block diagonalization algorithm in Theorem 3 is summarized in Algorithm 1.

Algorithm 1

Step 1: Generate $P_{\overline{1}}, P_{\overline{2}}, \ldots, P_{\overline{m}}$ using (8).

Step 2: Generate a permutation matrix P using (7).

Step 3: $P^{\mathrm{T}} A_n^{(k,k')} P = A_{\overline{1}} \oplus A_{\overline{2}} \oplus \cdots \oplus A_{\overline{m}}$.

Here, we have the following two notes: if $k = k'$, then Algorithm 1 reduces to the algorithm in [8]; $A_n^{(k,k')}$ is block-diagonalized regardless of order of canonical vectors in (8).

The following proposition may lead to a further block-diagonalization of $A_n^{(k,k')}$.

Proposition 4. *Let $A_n^{(k,k')}$ be an $n \times n$ (k,k')-pentadiagonal matrix, where n and k satisfy that $\mathbb{N}_{n,k} (:= \{i \in \mathbb{N}_n \mid n-k < i \leq k\}) \neq \emptyset$. Let $\mathbb{N}_n$, $\overline{r}$, and $|\overline{r}|$ be the same notations as in Theorem 3, and let $|\mathbb{N}_{n,k}|$ be the number of elements* *in* $\mathbb{N}_{n,k}$*. Then,* $A_n^{(k,k')}$ *is block-diagonalized into a block diagonal matrix with* $(m+|\mathbb{N}_{n,k}|)$ *diagonal blocks by an* $n\times n$ *permutation matrix* P' *of the form*

$$P' := [P'_{\overline{1}}, P'_{\overline{2}}, \ldots, P'_{\overline{m}}], \tag{10}$$

where $P'_{\overline{r}}$ $(r=1,2,\ldots,m)$ *are* $n\times|\overline{r}|$ *matrices as follows:*

$$P'_{\overline{r}} := [\hat{P}_{\overline{r}^*}, \hat{P}_{\overline{r}'}], \tag{11}$$

$$\hat{P}_{\overline{r}^*} := [\boldsymbol{e}_{i_1}, \boldsymbol{e}_{i_2}, \ldots, \boldsymbol{e}_{i_{|\overline{r}^*|}}],\ \hat{P}_{\overline{r}'} := [\boldsymbol{e}_{j_1}, \boldsymbol{e}_{j_2}, \ldots, \boldsymbol{e}_{j_{|\overline{r}'|}}], \tag{12}$$

where $j_q \in \overline{r}' := \{i\in\mathbb{N}_{n,k} | i\in\overline{r}\}$ $(q=1,2,\ldots,|\overline{r}'|)$ *and* $i_p\in\overline{r}^* := \overline{r}\setminus\overline{r}'$ $(p=1,2,\ldots,|\overline{r}^*|)$.

Proof. Since P in Theorem 3 includes P' in (10), we have

$$(P')^{\mathrm{T}} A_n^{(k,k')} P' = A'_{\overline{1}} \oplus A'_{\overline{2}} \oplus \cdots \oplus A'_{\overline{m}},$$

where $A'_{\overline{r}} := (P'_{\overline{r}})^{\mathrm{T}} A_n^{(k,k')} P'_{\overline{r}}$ $(r=1,2,\ldots,m)$. Substituting (11) into the definition of $A'_{\overline{r}}$ yields

$$\begin{aligned} A'_{\overline{r}} &= (P'_{\overline{r}})^{\mathrm{T}} A_n^{(k,k')} P'_{\overline{r}} \\ &= [\hat{P}_{\overline{r}^*}, \hat{P}_{\overline{r}'}]^{\mathrm{T}} A_n^{(k,k')} [\hat{P}_{\overline{r}^*}, \hat{P}_{\overline{r}'}] \\ &= \begin{pmatrix} \hat{P}^{\mathrm{T}}_{\overline{r}^*} A_n^{(k,k')} \hat{P}_{\overline{r}^*} & \hat{P}^{\mathrm{T}}_{\overline{r}^*} A_n^{(k,k')} \hat{P}_{\overline{r}'} \\ \hat{P}^{\mathrm{T}}_{\overline{r}'} A_n^{(k,k')} \hat{P}_{\overline{r}^*} & \hat{P}^{\mathrm{T}}_{\overline{r}'} A_n^{(k,k')} \hat{P}_{\overline{r}'} \end{pmatrix}. \end{aligned} \tag{13}$$

Here, let $i\in\mathbb{N}_n$ and $j\in\mathbb{N}_{n,k}$. Then, since $j+k>n$, $j+k'>n$, $j-k<1$, and $j-k'<1$, we have

$$i\neq j\pm k, \quad i\neq j\pm k'.$$

Thus, using (3), we obtain

$$i\in\mathbb{N}_n \text{ and } j\in\mathbb{N}_{n,k} \Rightarrow (A_n^{(k,k')})_{i,j} = d_i\delta_{i,j}, \tag{14}$$

where $\delta_{i,j}$ denotes the Kronecker's delta. Similarly, let $i\in\mathbb{N}_{n,k}$ and $j\in\mathbb{N}_n$. Then, since $i+k>n$, $i+k'>n$, $i-k<1$, and $i-k'<1$, we have

$$j\neq i\pm k, \quad j\neq i\pm k'.$$

Thus, using (3), we obtain

$$i\in\mathbb{N}_{n,k} \text{ and } j\in\mathbb{N}_n \Rightarrow (A_n^{(k,k')})_{i,j} = d_i\delta_{i,j}. \tag{15}$$

From Lemma 1 and (12), it follows that

$$(\hat{P}_{\overline{r}^*}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}'})_{p,q} = (A_n^{(k,k')})_{i_p,j_q}, \tag{16}$$
$$(\hat{P}_{\overline{r}'}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}^*})_{q,p} = (A_n^{(k,k')})_{j_q,i_p}. \tag{17}$$

Since $\overline{r}' \cap \overline{r}^* = \emptyset$, we obtain $i_p \neq j_q$. Thus, from (14) and (16), it follows that $(\hat{P}_{\overline{r}^*}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}'})_{p,q} = 0$. Similarly, it follows from (15) and (17) that we have $(\hat{P}_{\overline{r}'}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}^*})_{q,p} = 0$. Therefore,

$$\hat{P}_{\overline{r}^*}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}'} = O_{|\overline{r}^*|\times|\overline{r}'|},\ \hat{P}_{\overline{r}'}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}^*} = O_{|\overline{r}'|\times|\overline{r}^*|}. \tag{18}$$

Substituting (18) into (13) yields

$$A'_{\overline{r}} = \hat{P}_{\overline{r}^*}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}^*} \oplus \hat{P}_{\overline{r}'}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}'}. \tag{19}$$

From Lemma 1 and (12), it follows that

$$(\hat{P}_{\overline{r}'}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}'})_{q,q'} = (A_n^{(k,k')})_{j_q,j_{q'}}$$

where $j_q, j_{q'} \in \overline{r}'$. From $\overline{r}' \subseteq \mathbb{N}_{n,k} \subseteq \mathbb{N}_n$, we see that j_q and $j_{q'}$ are written by $j_q \in \mathbb{N}_n$ and $j_{q'} \in \mathbb{N}_{n,k}$. Thus, using (14), we have

$$(A_n^{(k,k')})_{j_q,j_{q'}} = d_{j_q} \delta_{j_q,j_{q'}}.$$

Hence,

$$\hat{P}_{\overline{r}'}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}'} = d_{j_1} \oplus d_{j_2} \oplus \cdots \oplus d_{j_{|\overline{r}'|}}. \tag{20}$$

Substituting (20) into (19) yields

$$A'_{\overline{r}} = \hat{P}_{\overline{r}^*}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}^*} \oplus (d_{j_1} \oplus d_{j_2} \oplus \cdots \oplus d_{j_{|\overline{r}'|}}).$$

Namely, $A'_{\overline{r}}$ is further block-diagonalized into a block diagonal matrix with one diagonal block of the size $|\overline{r}^*| \times |\overline{r}^*|$ and $|\overline{r}'|$ diagonal blocks of the size 1×1 by $P'_{\overline{r}}$ in (11).

Therefore, using P' in (10), $A_n^{(k,k')}$ is block-diagonalized into a block diagonal matrix with m diagonal blocks of the form $\hat{P}_{\overline{r}^*}^{\mathrm{T}} A_n^{(k,k')} \hat{P}_{\overline{r}^*}$ and $|\mathbb{N}_{n,k}|$ diagonal blocks of the size 1×1. □

A block diagonalization algorithm in Proposition 4 is summarized in Algorithm 2.

Algorithm 2

Step 1: Generate $P'_{\overline{r}} := [\hat{P}_{\overline{r}^*}, \hat{P}_{\overline{r}'}]$ for $r = 1, 2, \ldots, m$ using (11).

Step 2: Generate a permutation matrix P' using (10).

Step 3: $(P')^{\mathrm{T}} A_n^{(k,k')} P' = A'_{\overline{1}} \oplus A'_{\overline{2}} \oplus \cdots \oplus A'_{\overline{m}}$.

Next, the nonzero structures of diagonal blocks obtained from Algorithms 1 is shown in Proposition 5.

Proposition 5. *Let $A_{\overline{r}}$ be diagonal blocks obtained from Step 3 of Algorithm 1, where $r = 1, 2, \ldots, m$. Let $i_1 < i_2 < \cdots < i_{|\overline{r}|}$ in (8), and let $\tilde{k} = k/m$ and $\tilde{k}' = k'/m$. Then, $A_{\overline{r}}$ has the following nonzero structure:*

$$(A_{\overline{r}})_{p,q} = \begin{cases} d_{i_p} & \text{if } \ p = q, \\ a_{i_p} & \text{if } \ p = q - \tilde{k}, \\ b_{i_p} & \text{if } \ p = q + \tilde{k}, \\ a'_{i_p} & \text{if } \ p = q - \tilde{k}', \\ b'_{i_p} & \text{if } \ p = q + \tilde{k}', \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Proof. Using Lemmas 1 and 2, the (p, q) element of $A_{\overline{r}}$ is represented by

$$(A_{\overline{r}})_{p,q} = (A_n^{(k,k')})_{i_p,i_q} = \begin{cases} d_{i_p} & \text{if } \ i_p = i_q, \\ a_{i_p} & \text{if } \ i_p = i_q - k, \\ b_{i_p} & \text{if } \ i_p = i_q + k, \\ a'_{i_p} & \text{if } \ i_p = i_q - k', \\ b'_{i_p} & \text{if } \ i_p = i_q + k', \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

where $i_p, i_q \in \overline{r}$. Here, it follows from the definition of $\overline{r}$, $k = m\tilde{k}$, $k' = m\tilde{k}'$, and the assumption that we obtain

$$i_q \pm k = i_{q \pm \tilde{k}}, \quad i_q \pm k' = i_{q \pm \tilde{k}'}. \tag{23}$$

Substituting (23) into (22) yields

$$(A_{\overline{r}})_{p,q} = \begin{cases} d_{i_p} & \text{if } \ i_p = i_q, \\ a_{i_p} & \text{if } \ i_p = i_{q-\tilde{k}}, \\ b_{i_p} & \text{if } \ i_p = i_{q+\tilde{k}}, \\ a'_{i_p} & \text{if } \ i_p = i_{q-\tilde{k}'}, \\ b'_{i_p} & \text{if } \ i_p = i_{q+\tilde{k}}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we obtain (21). This concludes the proof. □

We see from (3) and (21) that $A_{\overline{r}}$ inherits essentially the same structure from the original matrix (1). Thus, Algorithm 1 is a structure-preserving algorithm.

3. Illustrative Examples

In this section, illustrative examples of Algorithms 1 and 2 are provided.

Example 6. Let $A_{10}^{(3,6)}$ be a $(3,6)$-pentadiagonal matrix as follows:

$$A_{10}^{(3,6)} := \begin{pmatrix} d_1 & 0 & 0 & a_1 & 0 & 0 & a'_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & a_2 & 0 & 0 & a'_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & a_3 & 0 & 0 & a'_3 & 0 \\ b_4 & 0 & 0 & d_4 & 0 & 0 & a_4 & 0 & 0 & a'_4 \\ 0 & b_5 & 0 & 0 & d_5 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & b_6 & 0 & 0 & d_6 & 0 & 0 & a_6 & 0 \\ b'_7 & 0 & 0 & b_7 & 0 & 0 & d_7 & 0 & 0 & a_7 \\ 0 & b'_8 & 0 & 0 & b_8 & 0 & 0 & d_8 & 0 & 0 \\ 0 & 0 & b'_9 & 0 & 0 & b_9 & 0 & 0 & d_9 & 0 \\ 0 & 0 & 0 & b'_{10} & 0 & 0 & b_{10} & 0 & 0 & d_{10} \end{pmatrix}.$$

The result of Algorithm 1 applied to $A_{10}^{(3,6)}$ is shown next. From $m = 3$, $\mathbb{N}_{10}$ is divided into the following three equivalence classes: $\overline{1} = \{1, 4, 7, 10\}$; $\overline{2} = \{2, 5, 8\}$; $\overline{3} = \{3, 6, 9\}$. Step 1 yields $P_{\overline{1}} = [\boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_7, \boldsymbol{e}_{10}]$, $P_{\overline{2}} = [\boldsymbol{e}_2, \boldsymbol{e}_5, \boldsymbol{e}_8]$, and $P_{\overline{3}} = [\boldsymbol{e}_3, \boldsymbol{e}_6, \boldsymbol{e}_9]$. Step 2 yields $P = [P_{\overline{1}}, P_{\overline{2}}, P_{\overline{3}}]$. In Step 3, we have $P^{\mathrm{T}} A_{10}^{(3,6)} P = A_{\overline{1}} \oplus A_{\overline{2}} \oplus A_{\overline{3}}$, where

$$A_{\overline{1}} = \begin{pmatrix} d_1 & a_1 & a'_1 & 0 \\ b_4 & d_4 & a_4 & a'_4 \\ b'_7 & b_7 & d_7 & a_7 \\ 0 & b'_{10} & b_{10} & d_{10} \end{pmatrix}, \quad A_{\overline{2}} = \begin{pmatrix} d_2 & a_2 & a'_2 \\ b_5 & d_5 & a_5 \\ b'_8 & b_8 & d_8 \end{pmatrix},$$

$$A_{\overline{3}} = \begin{pmatrix} d_3 & a_3 & a'_3 \\ b_6 & d_6 & a_6 \\ b'_9 & b_9 & d_9 \end{pmatrix}.$$

Example 7. Let $A_7^{(4,6)}$ be a $(4,6)$-pentadiagonal matrix as follows:

$$A_7^{(4,6)} := \begin{pmatrix} d_1 & 0 & 0 & 0 & a_1 & 0 & a'_1 \\ 0 & d_2 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & d_4 & 0 & 0 & 0 \\ b_5 & 0 & 0 & 0 & d_5 & 0 & 0 \\ 0 & b_6 & 0 & 0 & 0 & d_6 & 0 \\ b'_7 & 0 & b_7 & 0 & 0 & 0 & d_7 \end{pmatrix}.$$

From $m = 2$, $\mathbb{N}_7$ is divided into the following two equivalence classes: $\overline{1} = \{1, 3, 5, 7\}$; $\overline{2} = \{2, 4, 6\}$. Here, Algorithm 2 can be applied since $\mathbb{N}_{7,4} = \{4\} \neq \emptyset$.

First, the result of Algorithm 1 applied to $A_7^{(4,6)}$ is shown next. Steps 1 and 2 yield $P = [P_{\overline{1}}, P_{\overline{2}}]$, where $P_{\overline{1}} = [\boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_5, \boldsymbol{e}_7]$ and $P_{\overline{2}} = [\boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_6]$. In Step 3, we have $P^{\mathrm{T}} A_7^{(4,6)} P = A_{\overline{1}} \oplus A_{\overline{2}}$, where

$$A_{\overline{1}} = \begin{pmatrix} d_1 & 0 & a_1 & a'_1 \\ 0 & d_3 & 0 & a_3 \\ b_5 & 0 & d_5 & 0 \\ b'_7 & b_7 & 0 & d_7 \end{pmatrix}, \quad A_{\overline{2}} = \begin{pmatrix} d_2 & 0 & a_2 \\ 0 & d_4 & 0 \\ b_6 & 0 & d_6 \end{pmatrix}.$$

Second, the result of Algorithm 2 applied to $A_7^{(4,6)}$ is shown next. Before Step 1, $\overline{2}' = \{4\}$ and $\overline{2}^* = \{2, 6\}$ are generated. Steps 1 and 2 yield $P' = [P'_{\overline{1}}, P'_{\overline{2}}]$, where $P'_{\overline{1}} = P_{\overline{1}}$ and $P'_{\overline{2}} = [\hat{P}_{\overline{2}^*}, \hat{P}_{\overline{2}'}] = \big[[\boldsymbol{e}_2, \boldsymbol{e}_6], [\boldsymbol{e}_4]\big]$. In Step 3, we have $(P')^{\mathrm{T}} A_7^{(4,6)} P' = A'_{\overline{1}} \oplus A'_{\overline{2}}$, where

$$A'_{\overline{1}} = A_{\overline{1}}, \quad A'_{\overline{2}} = \hat{P}_{\overline{2}^*} A_7^{(4,6)} \hat{P}_{\overline{2}^*}^{\mathrm{T}} \oplus d_4 = \begin{pmatrix} d_2 & a_2 \\ b_6 & d_6 \end{pmatrix} \oplus d_4.$$

In this case, $A_{\overline{2}}$ is further block-diagonalized into $A'_{\overline{2}}$ by Algorithm 2.

4. Concluding Remarks

In this paper, we proposed a block diagonalization algorithm of (k, k')-pentadiagonal matrices that is an extension of the block diagonalization algorithm of

k-tridiagonal matrices in [8]. As for the obtained diagonal blocks, we showed that the nonzero structures of the diagonal blocks are essentially the same as that of the original matrix.

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References

[1] R.L. Burden, J.D. Faires, *Numerical Analysis* (*ninth edition*), Brooks/Cole, USA (2005).

[2] Y. Chen, A new algorithm for computing the determinant and the inverse of a pentadiagonal Toeplitz matrix, *Engineering-London*, **5**, No. 5A (2013), 25-28.

[3] A. Fukuda, E. Ishiwata, Y. Yamamoto, M. Iwasaki, Y. Nakamura, Integrable discrete hungry systems and their related matrix eigenvalues, *Ann. Mat. Pur. Appl.*, **192**, No. 3 (2013), 423-445.

[4] J. Jia, T. Sogabe, M.E.A. El-Mikkawy, Inversion of k-tridiagonal matrices with Toeplitz structure, *Comput. Math. Appl.*, **65**, No. 1 (2013), 116-125.

[5] S.S. Askar, A.A. Karawia, On solving pentadiagonal linear systems via transformations, *Math. Probl. Eng.*, **2015**, ID 232456 (2015), 9 pages.

[6] E. Kilic, M. El-Mikkawy, A computational algorithm for special nth-order pentadiagonal Toeplitz determinants, *Appl. Math. Comput.*, **199**, No. 2 (2008), 820-822.

[7] E. Kırklar, F. Yılmaz, A note on k-tridiagonal k-Toeplitz matrices, *Ala. J. Math.*, **39**, (2015), 1-4.

[8] T. Sogabe, M.E.A. El-Mikkawy, Fast block diagonalization of k-tridiagonal matrices, *Appl. Math. Comput.*, **218**, No. 6 (2011), 2740-2743.

[9] A. Yalçiner, The LU factorization and determinants of the k-tridiagonal matrices, *Asian-European J. Math.*, **4**, No. 1 (2011), 187-197.

[10] X. Zhao, T. Huang, On the inverse of a general pentadiagonal matrix, *Appl. Math. Comput.*, **202**, No. 2 (2008), 639-646.