HAUSDROFF PROPERTY OF CARTESIAN AND
TENSOR PRODUCT OF GRAPHS

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\textbf{Abstract:} A simple graph $G$ is said to be Hausdroff if for any two distinct vertices $u$ and $v$ of $G$, one of the following conditions hold:

1. Both $u$ and $v$ are isolated
2. Either $u$ or $v$ is isolated
3. There exist two nonadjacent edges $e_1$ and $e_2$ of $G$ such that $e_1$ is incident with $u$ and $e_2$ is incident with $v$.

In this paper we derive sufficient conditions for cartesian and tensor products of two graphs to be Hausdroff.

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\section{1. Introduction}

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of $G$ by $V(G)$, the set of edges of $G$ by $E(G)$ and the minimum degree of $G$ by $\delta(G)$.
The degree [4] of a vertex \( v \) in a graph \( G \), denoted by \( \text{deg} v \), is the number of edges incident with \( v \). A pendant vertex [6] in a graph \( G \) is a vertex of degree one. A vertex \( v \) is isolated [2] if \( \text{deg} v = 0 \). By an empty graph [5] we mean a graph with no edges. Two vertices \( u \) and \( v \) of \( G \) are adjacent[8], if \( uv \) is an edge of \( G \). A simple graph is said to be complete[7] if every pair of distinct vertices of \( G \) are adjacent in \( G \). A complete graph of \( n \) vertices is denoted by \( K_n \). A connected graph that has no cut vertices is called a block [3]. A block of \( G \) containing exactly one cut vertex of \( G \) is called an end-block [3] of \( G \). The Cartesian product[9] \( G \square H \) of two graphs \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \) is the graph with vertex set \( V(G) \times V(H) \) where the vertex \( (u_1, v_1) \) is adjacent to the vertex \( (u_2, v_2) \) whenever \( u_1u_2 \in E(G) \) and \( v_1 = v_2 \), or \( u_1 = u_2 \) and \( v_1v_2 \in E(H) \). The Tensor product (or direct product)[1] \( G \times H \) of two graphs \( G \) and \( H \) is the graph with the vertex set \( V(G) \times V(H) \), two vertices \( (u_1, v_1) \) and \( (u_2, v_2) \) being adjacent in \( G \times H \) if, and only if, \( u_1u_2 \in E(G) \) and \( v_1v_2 \in E(H) \). A graph \( G \) is said to be Hausdorff [10] if for any two distinct vertices \( u \) and \( v \) of \( G \), one of the following three conditions hold: (1) Both \( u \) and \( v \) are isolated (2) Either \( u \) or \( v \) is isolated (3) There exist two nonadjacent edges \( e_1 \) and \( e_2 \) of \( G \) such that \( e_1 \) is incident with \( u \) and \( e_2 \) is incident with \( v \). From the definition of a Hausdorff graph we have if \( G \) is a graph with \( \delta(G) = 1 \), then it cannot be Hausdorff. In particular \( K_2 \) is not Hausdorff. Also if \( G \) is Hausdorff, then any supergraph of \( G \) is Hausdorff.

**Theorem 1.** [10] Let \( G = (V(G), E(G)) \) be a graph with \( \delta(G) \geq 3 \) then, \( G \) is Hausdorff.

### 2. Cartesian Product

From the definition of cartesian product of graphs we have:

**Proposition 2.** The cartesian product \( K_n \square K_n \) is Hausdorff for every \( n \).

**Proposition 3.** The cartesian product \( K_n \square K_m \) (\( n \neq m \)) is Hausdorff if, and only if, either \( n = 1 \) and \( m \geq 4 \) or \( n \geq 2 \) and \( m \geq 2 \).

**Corollary 4.** The cartesian product \( C_n \square C_m \) is Hausdorff \( \forall n, m \).

**Proposition 5.** The cartesian product \( P_n \square P_m \) is Hausdorff \( \forall n, m \).

**Example 6.**

Example 6 shows that the cartesian product of two non-Hausdorff graphs can be Hausdorff.
Theorem 7. Let $G_1$ and $G_2$ be two graphs with no isolated vertices. Then $G_1 \Box G_2$ is Hausdroff.

Proof. Let $(u_i, v_j)$ and $(u_r, v_s)$ be two distinct vertices of $G_1 \Box G_2$.

Case 1. $u_i = u_r$

Then the vertices $v_j$ and $v_s$ of $G_2$ are distinct. Since $G_1$ is a graph with no isolated vertices, $G_1$ contains a vertex $v_p$ such that $u_i$ and $v_p$ are adjacent in $G_1$. Then $(u_i, v_j)(u_p, v_j)$ and $(u_i, v_s)(u_p, v_s)$ are two nonadjacent edges of $G_1 \Box G_2$.

Case 2. $u_i \neq u_r$

In this case, either $v_j = v_s$ or $v_j \neq v_s$. Suppose $v_j = v_s$. Since $G_1 \Box G_2 = G_2 \Box G_1$, the result follows as in Case 1. So we need only to consider the case $v_j \neq v_s$. In this case if $u_i$ is adjacent to $u_r$, then $(u_i, v_j)(u_r, v_j)$ and $(u_r, v_s)(u_i, v_s)$ are two nonadjacent edges of $G_1 \Box G_2$ incident with $(u_i, v_j)$ and $(u_r, v_s)$ respectively. If $u_i$ is not adjacent to $u_r$, since $G_1$ is free from isolated vertices, there exist vertices $u_p$ and $u_q$ distinct from $u_i$ and $u_r$ such that $u_i$ is adjacent to $u_p$ and $u_r$ is adjacent to $u_q$ in $G_1$. ( $u_p$ may be equal to $u_q$ ) Then $(u_i, v_j)(u_p, v_j)$ and $(u_r, v_s)(u_q, v_s)$ are two nonadjacent edges of $G_1 \Box G_2$ incident with $(u_i, v_j)$ and $(u_r, v_s)$ respectively.

Hence the theorem.

Remark 8. Theorem 7 need not be true if both the graphs contain isolated vertices. (see. Figure [2]).

If $G_1$ contains an isolated vertex $u$ and $G_2$ contains a pendant edge $vw$ then $(u, v)(u, w)$ is a pendant edge of the cartesian product $G_1 \Box G_2$ of $G_1$ and $G_2$. Thus in such cases $G_1 \Box G_2$ can never be Hausdroff. We state this result as a proposition as follows:
Proposition 9. The cartesian product $G_1 \square G_2$ of two graphs $G_1$ and $G_2$ is not Hausdroff if $\delta(G_1) = 0$ and $\delta(G_2) = 1$.

The question then arise is that what happens to the cartesian product when we increase the minimum degree of the graph $G_2$. Unfortunately, the result remains failed in certain cases. For example, consider the graphs $G_1$ and $G_2$ and their cartesian product in Figure 3. There $\delta(G_1) = 0$ and $\delta(G_2) = 2$. The cartesian product $G_1 \square G_2$ of $G_1$ and $G_2$ contains a triangle, hence it cannot be Hausdroff.

Example 10.

But one can overcome this difficult situation by giving some restrictions to the graph $G_2$.

Theorem 11. Let $G_1$ be any graph and $G_2$ be a graph with no triangle as end-block. If $\delta(G_2) = 2$, then $G_1 \square G_2$ is Hausdroff.
Proof. Let \((u_i, v_j)\) and \((u_r, v_s)\) be two distinct vertices of \(G_1 \Box G_2\).

Case 1. \(u_i = u_r\)

Suppose \(v_j\) and \(v_s\) are adjacent vertices of \(G_2\). Since \(\delta(G_2) = 2\), there exists a vertex \(v_p\) distinct from \(v_s\) such that \(v_j\) is adjacent to \(v_p\). If \(v_s\) and \(v_p\) are not adjacent then \(v_s\) must be adjacent to some vertex \(v_q\) of \(G_2\) distinct from \(v_j\) and \(v_p\). Then \((u_i, v_j)(u_i, v_p)\) and \((u_i, v_s)(u_i, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\) incident with \((u_i, v_j)\) and \((u_r, v_s)\) respectively.

If \(v_s\) and \(v_p\) are adjacent then the subgraph induced by the vertices \(v_j, v_s, v_p\) is a triangle in \(G_2\). Since \(G_2\) contains no triangle as end-block either \(v_s\) is adjacent to a vertex \(v_q\) distinct from \(v_j\) and \(v_p\) or \(v_j\) is adjacent to a vertex \(v_t\) distinct from \(v_s\) and \(v_p\). In the first case \((u_i, v_j)(u_i, v_p)\) and \((u_i, v_s)(u_i, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\). In the second case \((u_i, v_j)(u_i, v_t)\) and \((u_i, v_s)(u_i, v_p)\) are two nonadjacent edges of \(G_1 \Box G_2\).

Suppose \(v_j\) and \(v_s\) are not adjacent in \(G_2\). In this case since \(\delta(G_2) = 2\), we can choose two distinct vertices \(v_p\) and \(v_q\) of \(G_2\) such that \(v_p\) is adjacent to \(v_j\) and \(v_q\) is adjacent to \(v_s\), then \((u_i, v_j)(u_i, v_p)\) and \((u_i, v_s)(u_i, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\) incident with \((u_i, v_j)\) and \((u_r, v_s)\) respectively.

Case 2. \(u_i \neq u_r\)

Suppose \(v_j = v_s\), since \(\delta(G_2) = 2\), there exist two distinct vertices \(v_p\) and \(v_q\) such that \(v_j\) is adjacent to both \(v_p\) and \(v_q\). Then \((u_i, v_j)(u_i, v_p)\) and \((u_r, v_s)(u_r, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\).

Suppose \(v_j \neq v_s\), then by proceeding as in the proof of Case 1 we get two nonadjacent edges incident with the vertices \((u_i, v_j)\) and \((u_r, v_s)\). First of all we consider the case \(v_j\) and \(v_s\) are adjacent vertices of \(G_2\). Since \(\delta(G_2) = 2\), there exists a vertex \(v_p\) distinct from \(v_s\) such that \(v_p\) is adjacent to \(v_j\). If \(v_s\) and \(v_p\) are not adjacent then \(v_s\) must be adjacent to some vertex \(v_q\) of \(G_2\) distinct from \(v_j\) and \(v_p\). Which implies \((u_i, v_j)(u_i, v_p)\) and \((u_i, v_s)(u_i, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\) incident with \((u_i, v_j)\) and \((u_r, v_s)\) respectively. If \(v_s\) and \(v_p\) are adjacent then \(v_j, v_s, v_p\) form a triangle. Since \(G_2\) contains no triangle as end-block either \(v_s\) is adjacent to a vertex \(v_q\) distinct from \(v_j\) and \(v_p\) or \(v_j\) is adjacent to a vertex \(v_t\) distinct from \(v_s\) and \(v_p\). In the first case \((u_i, v_j)(u_i, v_p)\) and \((u_r, v_s)(u_r, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\). In the second case \((u_i, v_j)(u_i, v_t)\) and \((u_r, v_s)(u_r, v_p)\) are two nonadjacent edges of \(G_1 \Box G_2\).

Suppose \(v_j\) and \(v_s\) are not adjacent in \(G_2\). In this case, since \(\delta(G_2) = 2\), we can choose two distinct vertices \(v_p\) and \(v_q\) of \(G_2\) such that \(v_j\) is adjacent to \(v_p\) and \(v_s\) is adjacent to \(v_q\). Then, \((u_i, v_j)(u_i, v_p)\) and \((u_r, v_s)(u_r, v_q)\) are two nonadjacent edges of \(G_1 \Box G_2\).

Thus in all the cases we have proved that for any two distinct vertices
(\(u_i, v_j\)), (\(u_r, v_s\)) of \(G_1 \square G_2\) there exist two nonadjacent edges \(e_1\) and \(e_2\) of \(G_1 \square G_2\) such that \(e_1\) is incident with \((u_i, v_j)\) and \(e_2\) is incident with \((u_r, v_s)\). Hence \(G_1 \square G_2\) is Hausdorff.

**Theorem 12.** Let \(G_1\) be any graph and \(G_2\) be a graph with \(\delta(G_2) \geq 3\), then \(G_1 \square G_2\) is Hausdorff.

**Proof.** Let \((u, v) \in V(G_1 \square G_2)\), then \(u \in V(G_1)\) and \(v \in V(G_2)\). Since \(\delta(G_2) \geq 3\), \(v\) is adjacent to at least three vertices, say \(v_1, v_2, v_3\) of \(G_2\). Then the vertex \((u, v)\) is adjacent to the vertices \((u, v_1), (u, v_2)\) and \((u, v_3)\) of \(G_1 \square G_2\). Therefore, \(\deg(u, v) \geq 3\). Since \((u, v)\) is an arbitrary vertex of \(G_1 \square G_2\), \(\delta(G_1 \square G_2) \geq 3\). Hence by Theorem 1, \(G_1 \square G_2\) is Hausdorff.

**Proposition 13.** Let \(G_1\) be an empty graph and \(G_2\) be a Hausdorff graph then \(G_1 \square G_2\) is Hausdorff.

**Proof.** Let \(\{u_1, u_2, \ldots, u_n\}\) and \(\{v_1, v_2, \ldots, v_m\}\) be the vertex sets of \(G_1\) and \(G_2\) respectively. For \(i = 1, 2, \ldots, n\), let \(H_i = \{(u_i, v_j); j = 1, 2, \ldots, m\}\). Then \(G_1 \square G_2 = \bigcup_{i=1}^n H_i\). Note that for every \(i\), \(H_i\) is isomorphic to \(G_2\) and hence Hausdorff. Therefore, the graph \(G_1 \square G_2\), being the union of Hausdorff graphs, is Hausdorff.

**Remark 14.** Since \(G_1 \square G_2 = G_2 \square G_1\), Propositions 9, 13 and Theorems 7,11,12 are still true even if we interchange the roles of \(G_1\) and \(G_2\).

**Theorem 15.** Cartesian product of two Hausdorff graphs is Hausdorff.

**Proof.** Let \(G_1\) and \(G_2\) be two Hausdorff graphs. Then, for \(i = 1, 2\), \(G_i = K_i \cup H_i\), where \(V(K_i)\) is the set all isolated vertices of \(G_i\) and \(V(H_i)\) is the set of all non-isolated vertices of \(G_i\). Hence \(G_1 \square G_2 = (K_1 \square K_2) \cup (K_1 \square H_2) \cup (H_1 \square K_2) \cup (H_1 \square H_2)\). By Proposition 13, \((K_1 \square K_2), (K_1 \square H_2)\) and \((H_1 \square K_2)\) are Hausdorff. By Theorem 7, \(H_1 \square H_2\) is Hausdorff. Therefore, \(G_1 \square G_2\) is Hausdorff.
Another interesting graph product that we can consider is that of the tensor product. Let us start with the tensor product of $K_2$ and $K_3$. Though both $K_2$ and $K_3$ are non-Hausdorff their tensor product seems to be Hausdorff.

Note that the graph $K_2$ is free from isolated vertices and the graph $K_3$ has minimum degree 2. Lemma 16 shows that this result is true in general. That is, if $\delta(G_1) \geq 1$ and $\delta(G_2) = 0$, then $G_1 \times G_2$ is Hausdorff.

**Lemma 16.** Let $G_1$ be a graph with no isolated vertices and $G_2$ be a graph with $\delta(G_2) = 2$. Then $G_1 \times G_2$ is Hausdorff.

**Proof.** Let $V(G_1) = \{u_i, i = 1, 2, \ldots m\}$ and $V(G_2) = \{v_j, j = 1, 2, \ldots n\}$. Then, $V(G_1 \times G_2) = \{(u_i, v_j); i = 1, 2, \ldots m, j = 1, 2, \ldots n\}$. Consider two distinct vertices $(u_i, v_j)$ and $(u_r, v_s)$ of $G_1 \times G_2$. Since $G_1$ contains no isolated vertices, the vertex $u_i$ is adjacent to at least one vertex of $G_1$.

**Case 1.** $u_i$ and $u_r$ are adjacent vertices of $G_1$.

Since $\delta(G_2) = 2$, there exists a vertex $v_p$ distinct from $v_s$ such that $v_p$ is adjacent to $v_j$. Similarly there exists a vertex $v_q$ distinct from $v_j$ such that $v_q$ is adjacent to $v_s$. Then $(u_i, v_j)(u_r, v_p)$ and $(u_r, v_s)(u_i, v_q)$ are two nonadjacent edges of $G_1 \times G_2$ incident with $(u_i, v_j)$ and $(u_r, v_s)$ respectively.

**Case 2.** $u_i$ and $u_r$ are nonadjacent vertices of $G_1$.

Choose vertices $u_p$ and $u_q$ of $G_1$, which are adjacent to the vertices $u_i$ and $u_r$ respectively. If $v_j$ and $v_s$ are adjacent in $G_2$, then, the edges $(u_i, v_j)(u_p, v_s)$ and $(u_r, v_s)(u_q, v_j)$ are two nonadjacent edges of $G_1 \times G_2$ incident with $(u_i, v_j)$ and $(u_r, v_s)$ respectively. Otherwise, since $\delta(G_2) = 2$, we can choose two distinct vertices $v_p$ and $v_q$ of $G_2$ such that $v_p$ is adjacent to $v_j$ and $v_q$ is adjacent to $v_s$. Then, the edges $(u_i, v_j)(u_p, v_p)$ and $(u_r, v_s)(u_q, v_q)$ are two nonadjacent edges of $G_1 \times G_2$ incident with $(u_i, v_j)$ and $(u_r, v_s)$ respectively.
Theorem 17 shows that the restriction $\delta(G_2) = 2$ on the second graph $G_2$ can be withdrawn.

**Theorem 17.** Let $G_1$ be a graph with no isolated vertices and $G_2$ be a graph with $\delta(G_2) \geq 2$. Then, the tensor product $G_1 \times G_2$ of $G_1$ and $G_2$ is Hausdroff.

**Proof.** If $\delta(G_2) = 2$, then the proof follows from Lemma 16. Now suppose $\delta(G_2) \geq 3$. Let $(u, v)$ be a vertex of $G_1 \times G_2$. Since $G_1$ is a graph with no isolated vertices, the vertex $u$ is adjacent to at least one vertex say $w$ of $G_1$. Since $\delta(G_2) \geq 3$, the vertex $v$ is adjacent to at least three vertices say $v_1, v_2, v_3$ of $G_2$. Then the vertex $(u, v)$ is adjacent to the vertices $(w, v_1), (w, v_2), (w, v_3)$ of $G_1 \times G_2$. Therefore, $\deg(u, v) \geq 3$. Since $(u, v)$ is arbitrary, $\delta(G_1 \times G_2) \geq 3$. Hence by Theorem 1, $G_1 \times G_2$ is Hausdroff.

**Lemma 18.** If one of $G_1$ and $G_2$ be empty graphs then, $G_1 \times G_2$ is Hausdroff.

**Proof.** Let one of $G_1$ and $G_2$ be empty graphs then, $G_1 \times G_2$ is an empty graph. This implies $G_1 \times G_2$ is Hausdroff.

**Theorem 19.** Let $G_1$ be any graph and $G_2$ be a graph with $\delta(G_2) \geq 2$, then $G_1 \times G_2$ is Hausdroff.

**Proof.** We can write $G_1 = K \cup H$, where $V(K)$ is the set all isolated vertices of $G_1$ and $V(H)$ is the set all non-isolated vertices of $G_1$. Then $G_1 \times G_2 = (K \times G_2) \cup (H \times G_2)$. Since $K$ is empty by Lemma 18, $K \times G_1$ is Hausdroff. By Theorem 17, $H \times G_2$ is Hausdroff. Therefore, the graph $G_1 \times G_2$, being the union of Hausdroff graphs, is Hausdroff.

**Corollary 20.** For every $n, m \geq 3$, $C_n \times C_m$ is Hausdroff.

**Theorem 21.** Let $G_1$ and $G_2$ be two graphs such that both $G_1$ and $G_2$ contain at least one pendant vertex. Then $G_1 \times G_2$ can never be Hausdroff.

**Proof.** Let $u$ be a pendant vertex with pendant edge $ux$ in $G_1$ and let $v$ be a pendant vertex with $uy$ as pendant edge in $G_2$. Then $(u, v)(x, y)$ is a pendant edge in $G_1 \times G_2$. Therefore, $G_1 \times G_2$ is not Hausdroff.

**Theorem 22.** Tensor product of any two Hausdroff graphs is Hausdroff.
Proof. Let $G_1$ and $G_2$ be the given Hausdroff graphs. For $i = 1, 2$, we write $G_i = K_i \cup H_i$, where $V(K_i)$ is the set all isolated vertices of $G_i$ and $V(H_i)$ is the set all non-isolated vertices of $G_i$. Then $G_1 \times G_2 = (K_1 \times K_2) \cup (K_1 \times H_2) \cup (H_1 \times K_2) \cup (H_1 \times H_2)$. Since both $K_1$ and $K_2$ are empty graphs by Lemma 18, $K_1 \times K_2$, $K_1 \times H_2$, $H_1 \times K_2$ are Hausdroff. By Theorem 19, $H_1 \times H_2$ is Hausdroff. Therefore, the graph $G_1 \times G_2$, being the finite union of Hausdroff graphs, is Hausdroff.

4. Conclusion

In this paper we have discussed conditions under which Cartesian product of two graphs is Hausdroff. It is identified that Cartesian product of two Hausdroff Graphs is Hausdroff. Conditions under which Tensor product of two graphs become Hausdroff have been formulated. There are many unsolved problems in this area which are yet to be settled.

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References
