IDEALS IN $P(r, m)$ $\Gamma$-SEMINEAR-RINGS

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Abstract: In this paper, we discuss in detail the behaviour of ideals of a $P(r, m)$ $\Gamma$-seminear-ring. We have shown that in a $P(1, 2)(P(2, 1))$ $\Gamma$-seminear-ring, every left ideal (right ideal) of $R$ is also an ideal. We also obtain the notions of prime ideal, completely prime ideal and primary ideal coincide in a $P(r, m)$ $\Gamma$-seminear-ring which admits mate functions.

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1. Introduction

The concept of seminear-rings was introduced by B. V. Rootsealaar in 1962 [14]. It is known that seminear-rings are common generalization of nearrings and semirings. Right seminear-rings are algebraic systems $(R, +, \cdot)$ with two binary associative operations, a zero 0 with $x + 0 = 0 + x = x$ and $x0 = 0x = 0$ for any $x \in R$ and one distributive law $(x + y)z = xz + yz$ for all $x, y, z \in R$. If we replace the above distributive law by $x(y + z) = xy + xz$, then $R$ is called a left seminear-ring. Throughout this paper $R$ stands for a right seminear-
ring \((R, +, .)\). The notion of \(\Gamma\) - seminear-rings were first introduced by Sajeepianskool [11] as a generalization of \(\Gamma\) - near-rings and \(\Gamma\) - semirings and then \(\Gamma\) - rings. In this paper we first define \(P(r, m)\) \(\Gamma\) - seminear-rings and we discuss in detail the behaviour of ideals of a \(P(r, m)\) \(\Gamma\) - seminear-ring.

2. Preliminaries

In this section we list some basic definitions and results from the theory of \(\Gamma\) - seminear-rings that are used in the development of the paper.

Definition 1. \([11]\) Let \(R\) be an additive semigroup and \(\Gamma\) a nonempty set. Then \(R\) is called a right \(\Gamma\) - seminear-ring if there exists a mapping \(R \times \Gamma \times R \rightarrow R\) satisfying the following conditions:

(i) \((a + b)\gamma c = a\gamma c + b\gamma c\)

(ii) \((a\gamma b)\beta c = a\gamma (b\beta c)\) for all \(a, b, c \in R\) and \(\gamma, \beta \in \Gamma\)

Definition 2. \([11]\) Let \(R\) be a \(\Gamma\) - seminear-ring under the mapping \(f : R \times \Gamma \times R \rightarrow R\). a subsemigroup \(A\) of \(R\) is called a sub \(\Gamma\) - seminear-ring of \(R\) if \(A\) is a \(\Gamma\) - seminear-ring under the restriction of \(f\) to \(A \times \Gamma \times A\).

Definition 3. \([11]\) A non-empty subset \(I\) of a \(\Gamma\) - seminear-ring \(R\) is called a left (right) ideal if

(i) for all \(x, y \in I\), \(x + y \in I\) and

(ii) for all \(x \in I\), \(r \in R\) and \(\gamma \in \Gamma\), \(r\gamma x(x\gamma r) \in I\),

\(I\) is said to be an ideal of \(R\) it is both a left and a right ideal.

Definition 4. \([1]\) An ideal \(I\) of \(\Gamma\) - seminear-ring \(R\) is called

(i) a Prime ideal if \(\Lambda \Gamma B \subseteq I \Rightarrow A \subseteq I\) or \(B \subseteq I\) holds for all ideals \(A, B\) of \(R\).

(ii) a completely prime ideal if for \(a, b \in R\), \(\gamma \in \Gamma\), \(a\gamma b \in I \Rightarrow a \in I\) or \(b \in I\).

(iii) a completely semiprime ideal if for \(x \in R\), \(x^2 \in I\) implies \(x \in I\).

(iv) a primary ideal if \(a\gamma b\beta c \in I\) and if the product of any two of \(a, b, c\) not in \(I\), \(\gamma, \beta \in \Gamma\), then the \(k^{th}\) power of the third element is in \(I\).
(v) a semiprime ideal if \( I^2 \subseteq P \Rightarrow I \subseteq P \) for all ideals \( I \) of \( R \).

**Definition 5.** [11] A \( \Gamma \)-seminear-ring \( R \) is called
(i) a prime \( \Gamma \)-seminear-ring if \( \{0\} \) is a \( \Gamma \)-prime ideal.
(ii) a semiprime \( \Gamma \)-seminear-ring if \( \{0\} \) is a \( \Gamma \)-semiprime ideal.

**Definition 6.** [1] A \( \Gamma \)-seminear-ring \( R \) is called left (right) normal if \( a \in R \gamma (a \gamma R) \) for each \( a \in R, \gamma \in \Gamma \). \( R \) is normal if it is both left and right normal.

**Definition 7.** [13] A map \( f \) from \( R \) into \( R \) is called a mate function for \( R \) if \( x = x \gamma f(x) \gamma x \) for all \( x \) in \( R \), \( \gamma \in \Gamma \) (\( f(x) \) is called a mate of \( x \)).

**Definition 8.** A \( \Gamma \)-seminear-ring \( R \) is called an integral \( \Gamma \)-seminear-ring if \( R \) has no non-zero divisors. Obviously every \( \Gamma \)-seminear-field is an integral \( \Gamma \)-seminear-ring.

**Definition 9.** For \( A \subseteq R \), we define the radical \( \sqrt{A} \) of \( A \) to be \( \{a \in R/a^k \in A \) for some positive integer \( k \} \). Obviously \( A \subseteq \sqrt{A} \).

**Definition 10.** [5] A left ideal \( A \) of \( R \) is called essential if \( A \cap B = \{0\} \), where \( B \) is any left ideal of \( R \), implies \( B = \{0\} \).

**Definition 11.** An ideal \( I \) of \( R \) is called a strictly prime ideal if for left ideals \( A,B \) of \( R \), \( A \gamma B \subseteq I \) implies \( A \subseteq I \) or \( B \subseteq I \).

### 3. \( P(r,m) \) \( \Gamma \)-Semeinar-Rings

In this section we give the precise definition of a \( P(r,m) \) \( \Gamma \)-seminear-ring and illustrate this concept with suitable examples.

**Definition 12.** Let \( r,m \) be two positive integers. We say that \( R \) is a \( P(r,m) \) \( \Gamma \)-seminearring if \( x^r \gamma R = R \gamma x^m \) for all \( x \) in \( R \) and \( \gamma \in \Gamma \).

**Example 13.** (a) Let \( R = \{0,a,b,c,d\} \). We define the semigroup operations “+” and “\( \gamma \)” in \( R \) as follows.
Then \((R, +, \Gamma)\) is a \(P(r, m) \Gamma\) - seminear-ring for all positive integers \(r\) and \(m\), \(\gamma \in \Gamma\).

(b) The direct product of any two \(\Gamma\) - seminear fields is a \(P(r, m) \Gamma\) - seminear-ring for all positive integers \(r\) and \(m\).

(c) The Boolean \(P(1, 1) \Gamma\) - seminear-ring is a \(P(r, m) \Gamma\) - seminear-ring for all positive integers \(r\) and \(m\).

**Proposition 14.** If \(R\) has a mate function \(f\) then \(R\) is a left (right) normal \(\Gamma\) - seminear-ring.

**Proof.** Since \(R\) has a mate function \(f\) for all \(x \in R, \gamma \in \Gamma\), \(x = x\gamma f(x)\gamma x \in R\gamma x(x\gamma R)\). Obviously then \(R\) is a left (right) normal \(\Gamma\) - seminear-ring.

**Proposition 15.** In a \(P(1, 2) \Gamma\) - seminear-ring, \(E \subseteq C(R)\)

**Proof.** Since 0 \(\in E\), it is non-empty. Let \(e \in E\), As \(R\) is \(P(1, 2)\), \(e\gamma R = R\gamma e^2\) \(\Rightarrow e\gamma R = R\gamma e \Rightarrow e\gamma R\gamma e = e\gamma (R\gamma e) = e\gamma (e\gamma R) = e^2\gamma R = e\gamma R\). Hence \(e\gamma R = e\gamma R\gamma e = R\gamma e\). For \(x \in R, \gamma \in \Gamma\) there exist \(u, v \in R\) such that \(x\gamma e = e\gamma u\gamma e\) and \(e\gamma x = e\gamma v\gamma e\). These imply \(e\gamma x\gamma e = e\gamma (x\gamma e) = e\gamma (e\gamma u\gamma e) = e\gamma u\gamma e = x\gamma e\) and \(e\gamma x\gamma e = (e\gamma x)\gamma e = (e\gamma u\gamma e)\gamma e = e\gamma x\). Thus \(e\gamma x = e\gamma x\gamma e = x\gamma e\) for all \(x \in R, \gamma \in \Gamma\). Therefore \(E \subseteq C(R)\).

**Proposition 16.** Let \(R\) be a \(P(1, 2) \Gamma\) - seminear-ring. Then every left ideal of \(R\) is an ideal.

**Proof.** If \(A\) is a left ideal of \(R\) then \(R\Gamma A \subseteq A\). Let \(a \in A\) and \(y \in R\). We have \(a\gamma y \in a\gamma R = R\gamma a^2 \Rightarrow a\gamma y = y\gamma a^2 = (y\gamma a)\gamma a\) (for some \(y\gamma a\) in \(R\)) \(\in R\gamma a\). This forces \(a\gamma y \in R\Gamma A \subseteq A \Rightarrow A\Gamma R \subseteq A\). Hence \(A\) is an ideal.

**Remark 17.** We observe that as in Proposition 16, every right ideal of \(R\)
is also an ideal in a $P(2, 1) \Gamma$ - seminear-ring.

**Theorem 18.** Let $R$ admit a mate function $f$. Then $R$ is a $P(r, m) \Gamma$ - seminear-ring for all positive integers $r$ and $m$ if and only if $R$ is a $P(1, 2) \Gamma$ - seminear-ring.

**Proof.** If part: Since $R$ is a $P(1, 2) \Gamma$ - seminear-ring $\Rightarrow E \subseteq C(R)$ (By proposition 15) Let $r, m$ be any two positive integers. Let $a \in x^r \gamma R$. Therefore $a = x^r \gamma y$ for some $y$ in $R$. Now $a = (x^r \gamma f(x) \gamma x)^r \gamma y = x^r \gamma (f(x) \gamma x)^r \gamma y$ (since $f(x) \gamma x \in E \subseteq C(R)) = x^r \gamma f(x) \gamma x \gamma y = x^r \gamma y f(x) \gamma x$ (since $E \subseteq C(R)) = x^r \gamma y f(x) \gamma x = x^r \gamma y (f(x)) m x^m$ (since $E \subseteq C(R)) = (x^r \gamma y (f(x)) m) x^m \in R \gamma x^m$. Therefore $x^r \gamma R \subseteq R \gamma x^m$. In a similar fashion we get $R \gamma x^m \subseteq x^r \gamma R$. Hence $x^r \gamma R = R \gamma x^m$ and $R$ is a $P(r, m) \Gamma$ - seminear-ring. The converse is obvious - it follows by taking $r = 1$ and $m = 2$.

**Theorem 19.** Let $R$ be a $P(r, m) \Gamma$ - seminear-ring with a mate function $f$ and let $A$ and $B$ be any two left ideals of $R$. Then we have the following:

(i) $\sqrt{A} = A$,

(ii) $A \cap B = A \Gamma B$,

(iii) $A^2 = A$,

(iv) If $A \subseteq B$ then $A \Gamma B = A$,

(v) $A \cap R \Gamma B = A \Gamma B$,

(vi) $A$ is a $P(r, m)$ Sub-$\Gamma$ - seminear-ring.

**Proof.** We first observe that in view of Theorem 18 we need only to consider the special case when $r = 1$ and $m = 2$. Thus we take $R$ to be a $P(1, 2) \Gamma$ - seminear-ring with a mate function. (i.e) $R$ is a right normal (By Proposition 14).

(i) Let $x \in \sqrt{A}$. Then there exists some positive integer $k$ such that $x^k \in A$. Since $R$ is an right normal $\Gamma$ - seminear-ring $x \in x^r \gamma R = R \gamma x^2 \Rightarrow x = y \gamma x^2$ for some $y \in R \Rightarrow x = y \gamma x^2 = y \gamma (y \gamma x^2) \gamma x = y^2 \gamma x^3 = \cdots = y^{k-1} \gamma x^k \in R \Gamma A \subseteq A$. (i.e) $x \in A, \gamma \in \Gamma$. Therefore $\sqrt{A} \subseteq A$. But obviously $A \subseteq \sqrt{A}$ and (i) follows.
(ii) By proposition 16 both \( A \) and \( B \) are ideals and consequently
\[
A \Gamma B \subseteq A \cap B. \tag{1}
\]

To prove the reverse inclusion we note that for any \( x \in A \cap B \), \( x = x \gamma f(x) \gamma x = (x \gamma f(x)) \gamma x \in (A \Gamma R) \Gamma B \subseteq A \Gamma B \Rightarrow x \in A \Gamma B \). Therefore
\[
A \cap B \subseteq A \Gamma B. \tag{2}
\]

From (1) and (2) we get \( A \cap B = A \Gamma B \).

(iii) Taking \( B = A \) in (ii) we get \( A \Gamma A = A \cap A = A^2 = A \).

(iv) If \( A \subseteq B \Rightarrow A \cap B = A \) and (ii) gives \( A = A \Gamma B \).

(v) We have \( A \cap R \Gamma B \subseteq A \cap B \) (since \( R \Gamma B \subseteq B \)). Therefore
\[
A \cap R \Gamma B \subseteq A \Gamma B \tag{3}
\]
(using (ii)).

Also \( A \Gamma B = A \cap B = A \) and \( A \Gamma B \subseteq R \Gamma B \). Therefore
\[
A \Gamma B \subseteq A \cap R \Gamma B. \tag{4}
\]

From (3) and (4) we get \( A \Gamma B = A \cap R \Gamma B \).

(vi) Let \( a \in A \). As \( a \gamma A \subseteq a \gamma R = R \gamma a^2 \), there exists \( y \in R \), for every \( x \in A \), such that \( a \gamma x = y \gamma a^2 \). Now \( a \gamma x = y \gamma a \gamma a = y (a \gamma f(a) \gamma a) \gamma a = (y \gamma a \gamma f(a)) \gamma a^2 = a' \gamma a^2 \). where \( a' = y \gamma a \gamma f(a) \in (R \Gamma A) \Gamma R \subseteq A \). Therefore
\[
a \gamma A \subseteq A \gamma a^2. \tag{5}
\]

Conversely if \( z \in A \) then \( z \gamma a^2 \in A \gamma a^2 \subseteq R \gamma a^2 = a \gamma R \Rightarrow \) there exists \( w \in R \) such that \( z \gamma a^2 = a \gamma w = a \gamma f(a) \gamma a \gamma w = a \gamma (f(a) \gamma a \gamma w) = a \gamma z' \) where \( z' = f(a) \gamma a \gamma w \in R \Gamma A \Gamma R \subseteq A \). Therefore
\[
A \gamma a^2 = a \gamma A. \tag{6}
\]

From (5) and (6) we get
\[
a \gamma A = A \gamma a^2 \tag{7}
\]
for all \( a \in A, \gamma \in \Gamma \). From (6) and (7), \( A \) is a \( P(r, m) \) Sub \( \Gamma \) - seminearring.
**Theorem 20.** If $R$ is a $P(r, m)$ $\Gamma$-seminear-ring with a mate function $f$ then $R$ has the following properties

(i) $R$ is a semiprime $\Gamma$-seminear-ring

(ii) $R\gamma x\gamma R\gamma y = R\gamma x \cap R\gamma y = R\gamma x\gamma y$ for all $x, y \in R, \gamma \in \Gamma$.

**Proof.** In view of the Theorem 18 we can take $R$ as a $P(1, 2)$ $\Gamma$-seminear-ring with a mate function $f$.

(i) Let $A$ be a left ideal of $R$. Then it is clear from Proposition 16, $A$ is an ideal of $R$. Let $I$ be any ideal of $R$ such that $I^2 \subseteq A$. If $x \in I$ then $x = x\gamma f(x)\gamma x \in II(R\Gamma I) \subseteq I^2 \subseteq A \Rightarrow x \in A$. Thus $I \subseteq A$. Therefore $A$ is a $\Gamma$-semiprime ideal. In particular $\{0\}$ is a $\Gamma$-semiprime ideal and therefore $R$ is a semiprime $\Gamma$ seminear-ring.

(ii) As $R\gamma x$ and $R\gamma y$ are left ideals of $R$, it follows from the Theorem 19(ii) that $R\gamma x \cap R\gamma y = (R\gamma x)\gamma (R\gamma y)$. Also $R\gamma x = R\gamma x \cap R = R\gamma x\gamma R$. Hence $R\gamma x\gamma y = R\gamma x\gamma R\gamma y = R\gamma x \cap R\gamma y$ and (ii) follows.

**Theorem 21.** Let $R$ be a $P(r, m)$ $\Gamma$-seminear-ring with a mate function $f$ and let $P$ be a ideal of $R$. Then the following are equivalent

(i) $P$ is a prime ideal

(ii) $P$ is a completely prime ideal

(iii) $P$ is a primary ideal

**Proof.** (i) $\Rightarrow$ (ii). Let $a\gamma b \in P$. By Theorem 20(ii), $R\gamma a\gamma R\gamma b = R\gamma a\gamma b \subseteq R\Gamma P \subseteq P$. Since $R\gamma a$ and $R\gamma b$ are ideals in $R$ (by Proposition 16) and also $P$ is prime, $R\gamma a\gamma R\gamma b \subseteq P \Rightarrow R\gamma a \subseteq P$ or $R\gamma b \subseteq P$.

Suppose $R\gamma a \subseteq P$. Then $a = (a\gamma f(a))\gamma a \in P$ and $R\gamma b \subseteq P \Rightarrow b = (b\gamma f(b))\gamma b \in P$. Hence $P$ is a completely prime ideal.

Proof of (ii) $\Rightarrow$ (i) obvious.

(ii) $\Rightarrow$ (iii) : Theorem 20(ii) guarantees that for all $\gamma \in \Gamma, x, y \in R, R\gamma x\gamma y = R\gamma x \cap R\gamma y$. As $R\gamma x \cap R\gamma y = R\gamma y \cap R\gamma x$, we see that $R\gamma x\gamma y = R\gamma y\gamma x$ for all $x, y \in R$. In a similar fashion it follows that for all $a, b, c \in R$.
\[ R \gamma a \gamma b \gamma c = R \gamma b \gamma c \gamma a = R \gamma c \gamma a \gamma b = R \gamma a \gamma c \gamma b = R \gamma b \gamma a \gamma c = R \gamma c \gamma b \gamma a. \]

Suppose \( a \gamma b \gamma c \in P \) and \( a \gamma b \notin P \). Since \( R \) is a \( P(r, m) \) \( \Gamma \) - seminear-ring with a mate function, it is a normal \( \Gamma \) - seminear-ring. Therefore \( a \gamma b \gamma c \in R \gamma a \gamma b \gamma c \subseteq R \Gamma P \subseteq P \) and therefore \( (a \gamma b) \gamma c \in P \Rightarrow c \in P \) (as \( P \) is a completely prime ideal and since \( a \gamma b \notin P \)). Again suppose \( a \gamma b \gamma c \in P \) and \( a \gamma c \notin P \). To get the desired result we proceed as follows. Consider \( a \gamma c \gamma b \in R \gamma a \gamma c \gamma b = R \gamma a \gamma b \gamma c \subseteq R \Gamma P \subseteq P \). Thus \( a \gamma c \gamma b = (a \gamma c) \gamma b \in P \). If \( a \gamma c \notin P \) then \( b \in P \) as before. Continuing in the same way, it follows that if \( a \gamma b \gamma c \in P \) and if the product of any two of \( a, b, c \) does not fall in \( P \) then the third falls in \( P \). Hence \( P \) is a primary ideal.

\((iii) \Rightarrow (ii)\): Let \( a \gamma b \in P \) and \( a \notin P \). First we observe that \( f(a) \gamma a \notin P \). For, if \( f(a) \gamma a \in P \Rightarrow a = a \gamma (f(a) \gamma a) \in R \Gamma P \subseteq P \) which is a contradiction. Also \( f(a) \gamma a \gamma b \in R \Gamma P \subseteq P \). Thus \( f(a) \gamma a \gamma b \in P \) and \( f(a) \gamma a \notin P \). As \( P \) is a primary ideal of \( R \), \( b^k \in P \Rightarrow b \) for some positive integer \( k \). Now \( b^k \in P \Rightarrow b \in \sqrt{P} \) and \( \sqrt{P} = P \) by Theorem 19(i). Thus \( b \in P \) and (ii) follows.

**Theorem 22.** Let \( R \) be a \( P(r, m) \) \( \Gamma \) - seminear-ring with mate functions. If \( R \) is prime then \( R \) has no non-zero divisors.

**Proof.** Let \( x, y \in R \) such that \( x \gamma y = 0 \). Clearly \( R \gamma x \) and \( R \gamma y \) are ideals of \( R \) and by Theorem 20(ii) \( R \gamma x R \gamma y = R \gamma x \gamma y = R \gamma 0 = \{0\} \). Since \( R \) is prime we have either \( R \gamma x = \{0\} \) or \( R \gamma y = \{0\} \). If \( f \) is a mate function for \( R \) then we have \( x = x \gamma f(x) \gamma x \in R \gamma x \) and \( y = y \gamma f(y) \gamma y \in R \gamma y \). Therefore \( x = 0 \) or \( y = 0 \). Hence \( R \) has no non-zero divisors.

**Proposition 23.** Let \( R \) be a \( P(r, m) \) \( \Gamma \) - seminear-ring admitting mate functions. If \( R \) has no non-zero divisors, then every ideal of \( R \) is essential.

**Proof.** Let \( A \neq 0 \) be an ideal of \( R \). Suppose there exists an ideal \( B \) of \( R \) such that \( A \Gamma B = \{0\} \). Theorem 19(ii) demands that \( A \Gamma B = \{0\} \). Since \( R \) has no non-zero divisors, we get \( B = \{0\} \) and the result follows.

**References**


