International Journal of Pure and Applied Mathematics
Volume 106 No. 2 2016, 533-541
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v106i2.16
ijpam.eu

IDEALS IN $P(r, m)$ Γ-SEMINEAR-RINGS

R. Perumal[1 §], P. Chinnaraj[2]

[1]Department of Mathematics
SRM University
Kattankulathur, 603203, Tamilnadu, INDIA

[2]Department of Mathematics
PSG Institute of Technology and Applied Research
Coimbatore, 641603, Tamilnadu, INDIA

Abstract: In this paper, we discuss in detail the behaviour of ideals of a $P(r, m)$ Γ - seminear-ring. We have shown that in a $P(1, 2)(P(2, 1))$ Γ - seminear-ring, every left ideal (right ideal) of R is also an ideal. We also obtain the notions of prime ideal, completely prime ideal and primary ideal coincide in a $P(r, m)$ Γ - seminear-ring which admits mate functions.

AMS Subject Classification: 16Y60

Key Words: left (right) ideal, prime ideal, completely prime ideal, primary ideal, $P(r, m)$ Γ - seminear-ring

1. Introduction

The concept of seminear-rings was introduced by B. V. Rootselaar in 1962 [14]. It is known that seminear-rings are common generalization of nearrings and semirings. Right seminear-rings are algebraic systems $(R, +, .)$ with two binary associative operations, a zero 0 with $x + 0 = 0 + x = x$ and $x0 = 0x = 0$ for any $x \in R$ and one distributive law $(x + y)z = xz + yz$ for all $x, y, z \in R$. If we replace the above distributive law by $x(y + z) = xy + xz$, then R is called a left seminear-ring. Throughout this paper R stands for a right seminear-

Received: September 9, 2015
Published: February 15, 2016

§Correspondence author

ring $(R, +, .)$. The notion of Γ - seminear-rings were first introduced by Sajee pianskool [11] as a generalization of Γ - near-rings and Γ - semirings and then Γ - rings. In this paper we first define $P(r, m)$ Γ - seminear-rings and we discuss in detail the behaviour of ideals of a $P(r, m)$ Γ - seminear-ring.

2. Preliminaries

In this section we list some basic definitions and results from the theory of Γ - seminear-rings that are used in the development of the paper.

Definition 1. *[11] Let R be an additive semigroup and Γ a nonempty set. Then R is called a right Γ - seminear-ring if there exists a mapping $R \times \Gamma \times R \to R$ satisfying the following conditions:*

(i) $(a + b)\gamma c = a\gamma c + b\gamma c$

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c)$ *for all* $a, b, c \in R$ *and* $\gamma, \beta \in \Gamma$

Definition 2. *[11] Let R be a Γ - seminear-ring under the mapping $f : R \times \Gamma \times R \to R$. a subsemigroup A of R is called a sub Γ - seminear-ring of R if A is a Γ - seminear-ring under the restriction of f to $A \times \Gamma \times A$.*

Definition 3. *[11] A non-empty subset I of a Γ - seminear-ring R is called a left (right) ideal if*

(i) for all $x, y \in I$, $x + y \in I$ and

(ii) for all $x \in I$,$r \in R$ and $\gamma \in \Gamma$, $r\gamma x(x\gamma r) \in I$,
I is said to be an ideal of R it is both a left and a right ideal.

Definition 4. *[1] An ideal I of Γ - seminear-ring R is called*

(i) a Prime ideal if $A\Gamma B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$ holds for all ideals A, B of R.

(ii) a completely prime ideal if for $a, b \in R$, $\gamma \in \Gamma$, $a\gamma b \in I \Rightarrow a \in I$ or $b \in I$.

(iii) a completely semiprime ideal if for $x \in R$, $x^2 \in I$ implies $x \in I$.

(iv) a primary ideal if $a\gamma b\beta c \in I$ and if the product of any two of a, b, c not in I, $\gamma, \beta \in \Gamma$, then the k^{th} power of the third element is in I.

(v) a semiprime ideal if $I^2 \subseteq P \Rightarrow I \subseteq P$ for all ideals I of R.

Definition 5. *[11] A Γ - seminear-ring R is called*

(i) a prime Γ - seminear-ring if $\{0\}$ is a Γ - prime ideal.

(ii) a semiprime Γ - seminear-ring if $\{0\}$ is a Γ - semiprime ideal.

Definition 6. *[1] A Γ - seminear-ring R is called left (right) normal if $a \in R\gamma a(a\gamma R)$ for each $a \in R, \gamma \in \Gamma$. R is normal if it is both left and right normal.*

Definition 7. *[13] A map "f" from R into R is called a mate function for "R" if $x = x\gamma f(x)\gamma x$ for all x in R , $\gamma \in \Gamma$ ($f(x)$ is called a mate of x).*

Definition 8. *A Γ - seminear-ring R is called an integral Γ - seminear-ring if R has no non-zero divisors. Obviously every Γ - seminear-field is an integral Γ - seminear-ring.*

Definition 9. *For $A \subseteq R$, we define the radical $\sqrt{A}$ of A to be $\{a \in R/a^k \in A$ for some positive integer $k\}$. Obviously $A \subseteq \sqrt{A}$.*

Definition 10. *[5] A left ideal A of R is called essential if $A \cap B = \{0\}$, where B is any left ideal of R, implies $B = \{0\}$.*

Definition 11. *An ideal I of R is called a strictly prime ideal if for left ideals A,B of R, $A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.*

3. $P(r,m)$ Γ - Seminear-Rings

In this section we give the precise definition of a $P(r,m)$ Γ - seminear-ring and illustrate this concept with suitable examples.

Definition 12. *Let r, m be two positive integers. We say that R is a $P(r,m)$ Γ - seminearring if $x^r\gamma R = R\gamma x^m$ for all x in R and $\gamma \in \Gamma$.*

Example 13. (a) Let $R = \{0, a, b, c, d\}$. We define the semigroup operations "$+$" and "γ" in R as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	c
d	d	a	b	c	d

γ	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
d	0	a	b	c	d

Then $(R, +, \Gamma)$ is a $P(r, m)$ Γ - seminear-ring for all positive integers r and m, $\gamma \in \Gamma$.

(b) The direct product of any two Γ - seminear fields is a $P(r, m)$ Γ - seminear-ring for all positive integers r and m.

(c) The Boolean $P(1, 1)$ Γ - seminear-ring is a $P(r, m)$ Γ - seminear-ring for all positive integers r and m.

Proposition 14. *If R has a mate function f then R is a left (right) normal Γ - seminear-ring.*

Proof. Since R has a mate function f for all $x \in R$, $\gamma \in \Gamma$, $x = x\gamma f(x)\gamma x \in R\gamma x(x\gamma R)$. Obviously then R is a left (right) normal Γ - seminear-ring.

Proposition 15. *In a $P(1, 2)$ Γ - seminear-ring, $E \subseteq C(R)$*

Proof. Since $0 \in E$, it is non-empty. Let $e \in E$, As R is $P(1, 2)$, $e\gamma R = R\gamma e^2 \Rightarrow e\gamma R = R\gamma e \Rightarrow e\gamma R\gamma e = e\gamma(R\gamma e) = e\gamma(e\gamma R) = e^2\gamma R = e\gamma R$. Hence $e\gamma R = e\gamma R\gamma e = R\gamma e$. For $x \in R$, $\gamma \in \Gamma$ there exist $u, v \in R$ such that $x\gamma e = e\gamma u\gamma e$ and $e\gamma x = e\gamma v\gamma e$. These imply $e\gamma x\gamma e = e\gamma(x\gamma e) = e\gamma(e\gamma u\gamma e) = e\gamma u\gamma e = x\gamma e$ and $e\gamma x\gamma e = (e\gamma x)\gamma e = (e\gamma v\gamma e)\gamma e = e\gamma x$. Thus $e\gamma x = e\gamma x\gamma e = x\gamma e$ for all $x \in R, \gamma \in \Gamma$. Therefore $E \subseteq C(R)$.

Proposition 16. *Let R be a $P(1, 2)$ Γ - seminear-ring. Then every left ideal of R is an ideal.*

Proof. If A is a left ideal of R then $R\Gamma A \subseteq A$. Let $a \in A$ and $y \in R$. We have $a\gamma y \in a\gamma R = R\gamma a^2 \Rightarrow a\gamma y = y'\gamma a^2 = (y'\gamma a)\gamma a$ (for some y' in R)$\in R\gamma a$. This forces $a\gamma y \in R\Gamma A \subseteq A \Rightarrow A\Gamma R \subseteq A$. Hence A is an ideal.

Remark 17. We observe that as in Proposition 16, every right ideal of R

is also an ideal in a $P(2,1)$ Γ - seminear-ring.

Theorem 18. *Let R admit a mate function f. Then R is a $P(r,m)$ Γ - seminear-ring for all positive integers r and m if and only if R is a $P(1,2)$ Γ - seminear-ring.*

Proof. If part: Since R is a $P(1,2)$ Γ - seminear-ring $\Rightarrow E \subseteq C(R)$ (By proposition 15) Let r, m be any two positive integers. Let $a \in x^r\gamma R$. Therefore $a = x^r\gamma y$ for some y in R. Now $a = (x\gamma f(x)\gamma x)^r\gamma y = x^r\gamma(f(x)\gamma x)^r\gamma y$ (since $f(x)\gamma x \in E \subseteq C(R)) = x^r\gamma(f(x)\gamma x)\gamma y = x^r\gamma y\gamma f(x)\gamma x$ (since $E \subseteq C(R)) =$ $x^r\gamma y\gamma(f(x)\gamma x)^m$ (since $f(x)\gamma x \in E) = x^r\gamma y\gamma(f(x))^m x^m$ (since $E \subseteq C(R)) =$ $(x^r\gamma y\gamma(f(x))^m)\gamma x^m \in R\gamma x^m$. Therefore $x^r\gamma R \subseteq R\gamma x^m$. In a similar fashion we get $R\gamma x^m \subseteq x^r\gamma R$. Hence $x^r\gamma R = R\gamma x^m$ and R is a $P(r,m)$ Γ - seminear-ring. The converse is obvious - it follows by taking $r = 1$ and $m = 2$.

Theorem 19. *Let R be a $P(r,m)$ Γ - seminear-ring with a mate function f and let A and B be any two left ideals of R. Then we have the following:*

(i) $\sqrt{A} = A$,

(ii) $A \bigcap B = A\Gamma B$,

(iii) $A^2 = A$,

(iv) If $A \subseteq B$ *then* $A\Gamma B = A$,

(v) $A \bigcap R\Gamma B = A\Gamma B$,

(vi) A *is a* $P(r,m)$ *Sub-*Γ *- seminear-ring.*

Proof. We first observe that in view of Theorem 18 we need only to consider the special case when $r = 1$ and $m = 2$. Thus we take R to be a $P(1,2)$ Γ - seminear-ring with a mate function. (i.e) R is a right normal (By Proposition 14).

(i) Let $x \in \sqrt{A}$. Then there exists some positive integer k such that $x^k \in A$. Since R is an right normal Γ - seminear-ring $x \in x\gamma R = R\gamma x^2 \Rightarrow x = y\gamma x^2$ for some $y \in R \Rightarrow x = y\gamma x\gamma x = y\gamma(y\gamma x^2)\gamma x = y^2\gamma x^3 = \cdots = y^{k-1}\gamma x^k \in$ $R\Gamma A \subseteq A$. (i.e) $x \in A, \gamma \in \Gamma$. Therefore $\sqrt{A} \subseteq A$. But obviously $A \subseteq \sqrt{A}$ and (i) follows.

(ii) By proposition 16 both A and B are ideals and consequently

$$A\Gamma B \subseteq A \cap B. \tag{1}$$

To prove the reverse inclusion we note that for any $x \in A\bigcap B, x = x\gamma f(x)\gamma x = (x\gamma f(x))\gamma x \in (A\Gamma R)\Gamma B \subseteq A\Gamma B \Rightarrow x \in A\Gamma B$. Therfore

$$A\bigcap B \subseteq A\Gamma B. \tag{2}$$

From (1) and (2) we get $A\bigcap B = A\Gamma B$.

(iii) Taking $B = A$ in (ii) we get $A\Gamma A = A\bigcap A \Rightarrow A^2 = A$.

(iv) If $A \subseteq B \Rightarrow A\bigcap B = A$ and (ii) gives $A = A\Gamma B$.

(v) We have $A\bigcap R\Gamma B \subseteq A\bigcap B$(since $R\Gamma B \subseteq B$). Therefore

$$A\bigcap R\Gamma B \subseteq A\Gamma B \tag{3}$$

(using(ii)).
Also $A\Gamma B = A\bigcap B = A$ and $A\Gamma B \subseteq R\Gamma B$. Therfore

$$A\Gamma B \subseteq A\bigcap R\Gamma B. \tag{4}$$

From (3) and (4) we get $A\Gamma B = A\bigcap R\Gamma B$.

(vi) Let $a \in A$. As $a\gamma A \subseteq a\gamma R = R\gamma a^2$, there exists $y \in R$, for every $x \in A$, such that $a\gamma x = y\gamma a^2$. Now $a\gamma x = y\gamma a\gamma a = y\gamma(a\gamma f(a)\gamma a)\gamma a = (y\gamma a\gamma f(a))\gamma a^2 = a'\gamma a^2$. where $a' = y\gamma a\gamma f(a) \in (R\Gamma A)\Gamma R \subseteq A$. Therfore

$$a\gamma A \subseteq A\gamma a^2. \tag{5}$$

Conversely if $z \in A$ then $z\gamma a^2 \in A\gamma a^2 \subseteq R\gamma a^2 = a\gamma R \Rightarrow$ there exists $w \in R$ such that $z\gamma a^2 = a\gamma w = a\gamma f(a)\gamma a\gamma w = a\gamma(f(a)\gamma a\gamma w) = a\gamma z'$ where $z' = f(a)\gamma a\gamma w \in R\Gamma A\Gamma R \subseteq A$. Therefore

$$A\gamma a^2 = a\gamma A. \tag{6}$$

From (5) and (6) we get

$$a\gamma A = A\gamma a^2 \tag{7}$$

for all $a \in A, \gamma \in \Gamma$. From (6) and (7), A is a $P(r,m)$ Sub Γ - seminear-ring.

Theorem 20. *If R is a $P(r, m)$ Γ - seminearring with a mate function f then R has the following properties*

(i) R is a semiprime Γ - seminear-ring

(ii) $R\gamma x\gamma R\gamma y = R\gamma x \bigcap R\gamma y = R\gamma x\gamma y$ for all $x, y \in R, \gamma \in \Gamma$.

Proof. In view of the Theorem 18 we can take R as a $P(1, 2)$ Γ - seminear-ring with a mate function f.

(i) Let A be a left ideal of R. Then it is clear from Proposition 16, A is an ideal of R. Let I be any ideal of R such that $I^2 \subseteq A$. If $x \in I$ then $x = x\gamma f(x)\gamma x \in I\Gamma(R\Gamma I) \subseteq I^2 \subseteq A \Rightarrow x \in A$. Thus $I \subseteq A$. Therefore A is a Γ-semiprime ideal. In particular $\{0\}$ is a Γ - semiprime ideal and therefore R is a semiprime Γ seminear-ring.

(ii) As $R\gamma x$ and $R\gamma y$ are left ideals of R, it follows from the Theorem 19(ii) that $R\gamma x \bigcap R\gamma y = (R\gamma x)\gamma(R\gamma y)$. Also $R\gamma x = R\gamma x \bigcap R = R\gamma x\gamma R$. Hence $R\gamma x\gamma y = R\gamma x\gamma R\gamma y = R\gamma x \bigcap R\gamma y$ and (ii) follows.

Theorem 21. *Let R be a $P(r, m)$ Γ - seminear-ring with a mate function f and let P be a ideal of R. Then the following are equivalent*

(i) P is a prime ideal

(ii) P is a completely prime ideal

(iii) P is a primary ideal

Proof. $(i) \Rightarrow (ii)$. Let $a\gamma b \in P$. By Theorem 20(ii), $R\gamma a\gamma R\gamma b = R\gamma a\gamma b \subseteq R\Gamma P \subseteq P$. Since $R\gamma a$ and $R\gamma b$ are ideals in R (by Proposition 16) and also P is prime, $R\gamma a\gamma R\gamma b \subseteq P \Rightarrow R\gamma a \subseteq P$ or $R\gamma b \subseteq P$.

Suppose $R\gamma a \subseteq P$. Then $a = (a\gamma f(a))\gamma a \in P$ and $R\gamma b \subseteq P \Rightarrow b = (b\gamma f(b))\gamma b \in P$. Hence P is a completely prime ideal.

Proof of $(ii) \Rightarrow (i)$ obvious.

$(ii) \Rightarrow (iii)$: Theorem 20(ii) gurantees that for all $\gamma \in \Gamma$, $x, y \in R$, $R\gamma x\gamma y = R\gamma x \bigcap R\gamma y$. As $R\gamma x \bigcap R\gamma y = R\gamma y \bigcap R\gamma x$, we see that $R\gamma x\gamma y = R\gamma y\gamma x$ for all $x, y \in R$. In a smilar fashion it follows that for all $a, b, c \in R$

$$R\gamma a\gamma b\gamma c = R\gamma b\gamma c\gamma a = R\gamma c\gamma a\gamma b = R\gamma a\gamma c\gamma b = R\gamma b\gamma a\gamma c = R\gamma c\gamma b\gamma a.$$

Suppose $a\gamma b\gamma c \in P$ and $a\gamma b \notin P$. Since R is a $P(r,m)$ Γ - seminear-ring with a mate function, it is a normal Γ - seminear-ring. Therefore $a\gamma b\gamma c \in R\gamma a\gamma b\gamma c \subseteq R\Gamma P \subseteq P$ and therefore $(a\gamma b)\gamma c \in P \Rightarrow c \in P$ (as P is a completely prime ideal and since $a\gamma b \notin P$). Again suppose $a\gamma b\gamma c \in P$ and $a\gamma c \notin P$. To get the desired result we proceed as follows. Consider $a\gamma c\gamma b \in R\gamma a\gamma c\gamma b = R\gamma a\gamma b\gamma c \subseteq R\Gamma P \subseteq P$. Thus $a\gamma c\gamma b = (a\gamma c)\gamma b \in P$. If $a\gamma c \notin P$ then $b \in P$ as before. Continuing in the same way, it follows that if $a\gamma b\gamma c \in P$ and if the product of any two of a, b, c does not fall in P then the third falls in P. Hence P is a primary ideal.

$(iii) \Rightarrow (ii)$: Let $a\gamma b \in P$ and $a \notin P$. First we observe that $f(a)\gamma a \notin P$. For, if $f(a)\gamma a \in P \Rightarrow a = a\gamma(f(a)\gamma a) \in R\Gamma P \subseteq P$ which is a contradiction. Also $f(a)\gamma a\gamma b \in R\Gamma P \subseteq P$. Thus $f(a)\gamma a\gamma b \in P$ and $f(a)\gamma a \notin P$. As P is a primary ideal of R, $b^k \in P \Rightarrow b$ for some positive integer k. Now $b^k \in P \Rightarrow b \in \sqrt{P}$ and $\sqrt{P} = P$ by Theorem 19 (i). Thus $b \in P$ and (ii) follows.

Theorem 22. *Let R be a $P(r,m)$ Γ - seminear-ring with mate functions. If R is prime then R has no non-zero divisors.*

Proof. Let $x, y \in R$ such that $x\gamma y = 0$. Clearly $R\gamma x$ and $R\gamma y$ are ideals of R and by Theorem 20(ii) $R\gamma x\gamma R\gamma y = R\gamma x\gamma y = R\gamma 0 = \{0\}$. Since R is prime we have either $R\gamma x = \{0\}$ or $R\gamma y = \{0\}$. If f is a mate function for R then we have $x = x\gamma f(x)\gamma x \in R\gamma x$ and $y = y\gamma f(y)\gamma y \in R\gamma y$. Therefore $x = 0$ or $y = 0$. Hence R has no non-zero divisors.

Proposition 23. *Let R be a $P(r,m)$ Γ - seminear-ring admitting mate functions. If R has no non-zero divisors, then every ideal of R is essential.*

Proof. Let $A \neq 0$ be an ideal of R. Suppose there exists an ideal B of R such that $A \bigcap B = \{0\}$. Theorem 19(ii) demands that $A\Gamma B = \{0\}$. Since R has no non-zero divisors, we get $B = \{0\}$ and the result follows.

References

[1] Ahsan. J., Seminear-rings characterized by their S-ideals. I, *proc. japan. acad.*, **71A** (1995), 101-103.

[2] Ahsan. J., Seminear-rings characterized by their S-ideals. II, *proc.japan.acad.*, **71A** (1995), 111-113.

[3] Balakrishnan R. and Perumal R., Left Duo Seminear-rings, *Scientia Magna.*, **8(3)** (2012), 115 -120.

[4] Balakrishnan. R and Suryanarayanan. S., P(r,m) Near-rings, *Bull. Malaysian Math. Soc. (Second Series).*, **23** (2000), 117-130.

[5] Oswald.A., Near-rings in which every N-subgroup is principal, *Proc. London Math. Soc.*, **(3)28** (1974), 67-88.

[6] Perumal R., Balakrishnan R. and Uma S., Some Special Seminear-ring Structures, *Ultra Scientist of Physical Sciences.*, **23(2)** (2011),427 - 436.

[7] Perumal R., Balakrishnan R. and Uma S., Some Special Seminear-ring Structures II, *Ultra Scientist of Physical Sciences.*, **24(1)** (2012),91 - 98.

[8] Perumal R. and Balakrishnan R., Left Bipotent Seminear-rings , *International Journal of Algebra.*, **6(26)** (2012),1289 -1295.

[9] Perumal R. and Chinnaraj P., Medial Left Bipotent Seminear-rings., *Springer Proceedings in Mathematics and Statistics.*, **139** (2015), 451-457.

[10] Pilz Günter., *Near Rings*, North-Holland, Amsterdam, second edition, (1983).

[11] Sajee Pianskool., Simple $\Gamma-$Seminearrings, *Journal of Mathematics research.*, **(1)2** (2009), 124-129.

[12] Shabir. M. and Ahamed. I., Weakly regular seminearrings, *International Electronic Journal of Algebra.*, **2** (2007), 114-126 .

[13] Suryanarayanan. S and Ganesan. N., Stable and Pseudostable near-rings, *Indian J. Pure and Appl. Math.*, **19** (December, 1988), 1206-1216.

[14] Van Hoorn. W.G. and Van Rootselaar. B., Fundamental notions in the theory of seminear-rings., *compositio mathematic.*, **18** (1967), 65-78.

[15] Weinert. H.J., Seminear-rings, seminearfield and their semigroup theoretical background, *Semigroup Forum.*, **24**(1982), 231- 254.