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ON 2-ABSORBING δ-PRIMARY GAMMA-IDEAL OF GAMMA RING

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Abstract: In this paper, the notion of 2-absorbing δ-primary Γ-ideal of Γ-ring is introduced which unify 2-absorbing Γ-ideal and 2-absorbing δ-primary Γ-ideal , and several properties are investigated. Here δ is a mapping that assigns to each Γ-ideal J a Γ-ideal $\delta(J)$ of the same Γ-ring such that:(1) $(\forall I \in \mathbf{J}(M))(I \subseteq \delta(I))$,(2) $(\forall I, J \in \mathbf{J}(M)))(I \subseteq J \Rightarrow \delta(I) \subseteq \delta(J))$, where $\mathbf{J}(M)$ is the set of Γ-ideal of Γ-ring M.

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Key Words: Γ-ideal, intersection preserving, global Γ-ideal expansion, Γ-ring homomorphism, 2-absorbing δ-primary

1. Introduction

The notion of a Γ-ring is introduced by Nobusawa in[6], as more general than a ring. In [1] Barnes weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Barnes[1], Kyuno [4] and Luh[5] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Prime ideals and primary ideals are two of the most important structures in ring theory. In [8] Zhao investigated the possibil-

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ity of a unified approach to studying such two ideals, and introduced the notion of δ-primary ideals for a mapping δ that assigns to each ideal I an ideal $\delta(I)$ of the same ring. Such δ-primary ideals unify the prime and primary ideals under one frame. In [2] and [3], The concepts of prime ideal and primary ideal was extended to the context of 2-absorbing ideal and 2-absorbing primary ideal. In [7] Jun and all are trying to apply the Zhao's idea in ring theory to a Γ-ring. In this paper, we introduced the notion of 2-absorbing δ-primary Γ-ideal of Γ-ring which unify 2-absorbing Γ-ideal and 2-absorbing δ-primary Γ-ideal. A number of results concerning 2-absorbing δ-primary Γ-ideals of Γ-ring are given.

2. Preliminaries

Recall that if M and Γ be two Abelian groups and for all $x, y \in M$ and all $\alpha, \beta \in \Gamma$ the conditions: 1. $x\alpha y \in M$;
2. $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$, $x\alpha(y+z) = x\alpha y + x\alpha z$;
3. $(x\alpha y)\beta z = x\alpha(y\beta z)$;
are satisfied, then we call M a Γ-ring. By a right (resp. left) Γ-ideal of a Γ-ring M we mean an additive subgroup U of M such that $U\,\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and a left Γ-ideal, then we say that U is a Γ-ideal of M. A Γ-ideal I of M is said to be prime if for any ideals U and V of M, $U\Gamma V \subseteq I$ implies $U \subseteq I$ or $V \subseteq I$. We note from [1] that a proper Γ-ideal I of M is prime if a $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$ for all $a, b \in M$. A mapping $\sigma : M \to M'$ of Γ-rings is called a Γ-ring homomorphism if it satisfies:
1. $\sigma(a+b) = \sigma(a) + \sigma(b)$ for all $a, b \in M$;
2.$\sigma(a\gamma b) = \sigma(a)\gamma\sigma(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$
For more detail see for example [1] and [7].

Definition 2.1. [7] A Γ-ideal I of M is said to be primary if it satisfies:

$$(\forall a, b \in M)(\forall \gamma \in \Gamma)(a\gamma b \in I, a \notin I \Rightarrow b \in \sqrt{I})$$

Denote by $\mathbf{J}(M)$ the set of all Γ-ideal of M

Definition 2.2. [7] An expansion of Γ-ideal in M is defined to be a function $\delta : \mathbf{J}(M) \to \mathbf{J}(M)$ such that:

1. $(\forall I \in \mathbf{J}(M))(I \subseteq \delta(I))$;
2. $(\forall I, J \in \mathbf{J}(M)))(I \subseteq J \Rightarrow \delta(I) \subseteq \delta(J))$.

Example 1. (1) The identity function $Id : \mathbf{J}(M) \to \mathbf{J}(M)$ is a Γ-ideal expansion of M. (2) The constant function $c : \mathbf{J}(M) \to \mathbf{J}(M)$, $I \mapsto M$, is a

Γ-ideal expansion of M. (3)The radical of Γ-ideal, $\sqrt{} : \mathbf{J}(M) \to \mathbf{J}(M), I \mapsto \sqrt{I}$ is a Γ-ideal expansion of M

Definition 2.3. [7] Given a Γ-ideal expansion δ of M, a Γ-ideal $I \in \mathbf{J}(M)$ is said to be δ-primary if it satisfies:

$$(\forall a, b \in M)(\forall \gamma \in \Gamma)(a\gamma b \in I, a \notin I \Rightarrow b \in \delta(I))$$

Note that the following notions are defined in commutative ring M.

Definition 2.4. [2] A proper ideal I of M is called a 2-absorbing ideal of M if whenever $a, b, c \in M$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

Definition 2.5. [3] A proper ideal I of M is said to be a 2-absorbing primary ideal of M if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$

we can give the following definitions, they are analogue of definitions 2.4 and 2.5 in the Context of Γ-ring.

Definition 2.6. A proper Γ-ideal I of Γ-ring M is called a 2-absorbing Γ-ideal of M if whenever $a, b, c \in M$, $\alpha, \beta \in \Gamma$ and $a\alpha b\beta c \in I$, then $a\alpha b \in I$ or $a\beta c \in I$ or $b\beta c \in I$.

Definition 2.7. A proper Γ-ideal I of Γ-ring M is called a 2-absorbing primary Γ-ideal of M if whenever $a, b, c \in M$, $\alpha, \beta \in \Gamma$ and $a\alpha b\beta c \in I$, then $a\alpha b \in I$ or $a\beta c \in \sqrt{I}$ or $b\beta c \in \sqrt{I}$.

Remark 1. every 2-absorbing Γ-ideal of Γ-ring M is 2-absorbing primary Γ-ideal of Γ-ring M.

3. 2-Absorbing δ-Primary Γ-Ideal

In this section, we investigate 2-absorbing δ-primary Γ-ideal of Γ-ring M which unify 2-absorbing Γ-ideal and 2-absorbing primary Γ-ideal of M. In what follows let M denote a Γ-ring.

Definition 3.1. Given a Γ-ideal expansion δ of M, a Γ-ideal $I \in \mathbf{J}(M)$ is said to be 2-absorbing δ-primary if it satisfies:

$$(\forall a, b, c \in M)(\forall \alpha, \beta \in \Gamma)(a\alpha b\beta c \in I \Rightarrow a\alpha b \in I \;\; or \;\; a\beta c \in \delta(I) \;\; or \;\; c\beta b \in \delta(I))$$

Example 2. (1) If $\delta(I) = Id(I)$, 2-absorbing δ-primary Γ-ideal is just 2-absorbing Γ-ideal as defined in definition 2.6.

(2) If $\delta(I) = \sqrt{I}$, 2-absorbing δ-primary Γ-ideal is just 2-absorbing primary Γ-ideal as defined in definition 2.7.

In the following we will give a list of results, they are an extension of some results in [7].

Theorem 3.2. *Let δ and γ be Γ-ideal expansions of M. If $\delta(I) \subseteq \gamma(I)$ for all $I \in \mathbf{J}(M)$, then every 2-absorbing δ-primary Γ-ideal is also 2-absorbing γ-primary.*

Proof. Let I be an 2-absorbing δ-primary Γ-ideal of M. Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ be such that $a\alpha b\beta c \in I$. Then $a\alpha b \in I$ or $a\beta c \in \delta(I)$ or $b\beta c \in \delta(I)$. Since $I \subseteq \delta(I) \subseteq \gamma(I)$ by assumption. Hence I is a 2-absorbing γ-primary Γ-ideal of M. □

Theorem 3.3. *Let δ be a Γ-ideal expansion of M. For any subset S of M, denote by $\mathbf{J}_\delta(S)$ the intersection of all 2-absorbing δ-primary Γ-ideals of M containing S. Then the function $h : \mathbf{J}(M) \to \mathbf{J}(M)$ given by $h(I) = \mathbf{J}_\delta(I)$ for all $I \in \mathbf{J}(M)$ is a Γ-ideal expansion of M.*

Proof. Proof. $I \subseteq \mathbf{J}_\delta(I) = h(I)$ for all $I \in \mathbf{J}(M)$. Let $I, J \in \mathbf{J}(M)$ be such that $I \subseteq J$. Then

$$h(I) = \mathbf{J}_\delta(I) = \bigcap \in \{H \in \mathbf{J}_(M) | I \subseteq H \ \ and \ \ H \ \ is \ \ 2{-}absorbing \ \ \delta{-}primary\}$$

$$\subseteq \bigcap \{H \in \mathbf{J}(M) | J \subseteq H \ \ and \ \ H \ \ is \ \ 2{-}absorbing \ \ \delta{-}primary\} = \mathbf{J}_\delta(J) = h(J).$$

□

Theorem 3.4. *Let δ be a Γ-ideal expansion of M. If $\{J_i | i \in \Lambda\}$ is a directed collection of 2-absorbing δ-primary Γ-ideals of M, where Λ is an index set, then the Γ-ideal $J := \bigcup_{i\in\Lambda} J_i$ is 2-absorbing δ-primary.*

Proof. Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ be such that $a\alpha b\beta c \in J$. Then there exists J_i such that $a\alpha b\beta c \in J_i$. Since J_i is 2-absorbing δ-primary and $J_i \subseteq J$, it follows that $a\alpha b \in J_i$ or $a\beta c \in \delta(J_i)$ or $b\beta c \in \delta(J_i)$. Since $J_i \subseteq \delta(J_i) \subseteq \delta(J)$, $a\alpha b \in J$ or $a\beta c \in \delta(J)$ or $b\beta c \in \delta(J)$, so that J is 2-absorbing δ-primary. □

Recall that A Γ-ideal expansion δ is said to be intersection preserving if it satisfies:

$$(\forall I, J \in \mathbf{J}(M))(\delta(I \cap J) = \delta(I) \cap \delta(J))$$

A Γ-ideal expansion δ is said to be global if for each Γ-ring homomorphism $\sigma : M \to M^{'}$ of Γ-rings, the following holds:

$$(\forall I \in \mathbf{J}(M^{'})(\delta(\sigma^{-1}(I)) = \sigma^{-1}(\delta(I)))$$

Note that the Γ-ideal expansion Id of M in Example 1(1) is both intersection preserving and global.

Example 3. [7, Theorem 8] For each $I \in \mathbf{J}(M)$, let

$$\mathbf{B} := \bigcap\{J|I \subseteq J \ \ and \ \ J \ \ is \ \ a \ \ prime \ \ \Gamma - ideal \ \ of \ \ M\}$$

Then a function $\delta : \mathbf{J}(M) \to \mathbf{J}(M)$ given by $\delta(I) = \mathbf{B}(I)$ for all $I \in \mathbf{J}(M)$ is an intersection preserving Γ-ideal expansion of M.

Theorem 3.5. *Let δ be a Γ-ideal expansion of M which is intersection preserving. If $I_1, I_2, ..., I_n$ are 2-absorbing δ-primary Γ-ideals of M and $J = \delta(I_k)$ for all $k = 1, 2, ..., n$, then $I := \bigcap_{k=1}^{n} I_k$ is an 2-absorbing δ-primary Γ-ideal of M.*

Proof. Obviously, $I := \bigcap_{k=1}^{n} I_k$ is a Γ-ideal of M. Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ be such that $a\alpha b\beta c \in I$ and $a\alpha b \notin I$. Then $a\alpha b \notin I_k$ for some $k \in \{1, 2, ..., n\}$. But $a\alpha b\beta c \in I \subseteq I_k$ and I_k is 2-absorbing δ-primary, which imply that $a\beta c \in \delta(I_k)$ or $b\beta c \in \delta(I_k)$ Since δ is intersection preserving ,we have

$$\delta(I) = \delta(\bigcap_{k=1}^{n} I_k) = \bigcap_{k=1}^{n} \delta(I_k) = J = \delta(I_k)$$

and so $a\beta c \in \delta(I)$ or $b\beta c \in \delta(I)$ Therefore I is an 2-absorbing δ-primary Γ-ideal of M. □

Recall that, $\sigma : M \to M^{'}$ be a Γ-ring homomorphism of Γ-rings. Note that if J is a Γ-ideal of $M^{'}$, then $\sigma^{-1}(J)$ is a Γ-ideal of M, and that if σ is surjective and I is a Γ-ideal of M, then $\sigma(I)$ is a Γ-ideal of $M^{'}$.

Theorem 3.6. *Let δ be a Γ-ideal expansion which is global and let $\sigma : M \to M^{'}$ be a Γ-ring homomorphism of Γ-rings. If J is an 2-absorbing δ-primary Γ-ideal of $M^{'}$, then $\sigma^{-1}(J)$ is an 2-absorbing δ-primary Γ-ideal of M.*

Proof. Let $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ be such that $a\alpha b\beta c \in \sigma^{-1}(J)$. Then $\sigma(a)\alpha\sigma(b)\beta\sigma(c) \in J$, which imply that $\sigma(a)\alpha\sigma(b) \in J$ or $\sigma(a)\beta\sigma(c) \in \delta(J)$ or $\sigma(b)\beta\sigma(c) \in \delta(J)$. Since δ is global, it follows that $a\alpha b \in \sigma^{-1}(J))$ or $a\beta b \in \sigma^{-1}(\delta(J)) = \delta(\sigma^{-1}(J))$ or $b\beta c \in \delta(\sigma^{-1}(J))$ hance $\sigma^{-1}(J)$ is 2-absorbing δ-primary. □

Recall that, if $\sigma : M \to M^{'}$ is a Γ-ring homomorphism of Γ-rings, then $\sigma^{-1}(\sigma(I)) = I$ for any $I \in \mathbf{J}(M)$ that contains $\ker(\sigma)$.

Theorem 3.7. *Let $\sigma : M \to M^{'}$ be a surjective Γ-ring homomorphism of Γ-rings and let I be a Γ-ideal of M that contains $\ker(\sigma)$. Then I is 2 -absorbing δ-primary if and only if $\sigma(I)$ is an 2 -absorbing δ-primary Γ-ideal of $M^{'}$, where δ is a global Γ-ideal expansion.*

Proof. If σ(I) is an 2 -absorbing δ-primary Γ-ideal of $M^{'}$, then I is 2 -absorbing δ-primary by $I = \sigma^{-1}(\sigma(I))$ and Theorem 3.6. Suppose that I is 2 -absorbing δ-primary. Let $x, y, z \in M^{'}$ and $\alpha, \beta \in \Gamma$ be such that $x\alpha y\beta z \in \sigma(I)$. Since σ is surjective, we have $\sigma(a) = x$, $\sigma(b) = y$ and $\sigma(c) = z$ for some $a, b, c \in M$. Then $\sigma(a\alpha b\beta c) = \sigma(a)\alpha\sigma(b)\beta\sigma(c) = x\alpha y\beta z \in \sigma(I)$, which imply that $a\alpha b\beta c \in \sigma^{-1}(\sigma(I)) = I$. Since I is 2 -absorbing δ-primary, it follows that $a\alpha b \in I$ or $a\beta c \in \delta(I)$ or $b\beta c \in \delta(I)$ so that $x\alpha y \in \sigma(I)$ or $x\beta z \in \sigma(\delta(I))$ or $y\beta z \in \sigma(\delta(I))$. Using the fact that δ is global , we have

$$\delta(I) = \delta(\sigma^{-1}(\sigma(I))) = \sigma^{-1}(\delta(\sigma(I)))$$

and so $\sigma(\delta(I)) = \sigma(\sigma^{-1}(\delta(\sigma(I)))) = \delta(\sigma(I))$ since σ is surjective. Therefore $\sigma(I)$ is 2 -absorbing δ-primary. □

The following result, it is an extension of [3, lemma 2.18] .

Lemma 3.8. *Let δ be a Γ-ideal expansion of M and I be a 2-absorbing δ-primary Γ-ideal of M and suppose that $a\alpha b\beta J \subseteq I$ for some elements $a, b \in M$ and some ideal J of M and $\alpha, \beta \in \Gamma$. If $a\alpha b \notin I$, then $a\beta J \subseteq \delta(I)$ or $b\beta J \subseteq \delta(I)$.*

Proof. Suppose that $a\beta J \nsubseteq \delta(I)$ and $b\beta J \nsubseteq \delta(I)$. Then $a\beta j_1 \notin \delta(I)$ and $b\beta j_2 \notin \delta(I)$ for some $j_1, j_2 \in J$. Since $a\alpha b\beta j_1 \in I$ and $a\alpha b \notin I$ and $a\beta j_1 \notin \delta(I)$, we have $b\beta j_1 \in \delta(I)$. Since $a\alpha b\beta j_2 \in I$ and $a\alpha b \notin I$ and $b\beta j_2 \notin \delta(I)$, we have $a\beta j_2 \in \delta(I)$. Now, since $a\alpha b\beta(j_1 + j_2) \in I$ and $a\alpha b \notin I$, we have $a\beta(j_1 + j_2) \in \delta(I)$ or $b\beta(j_1 + j_2) \in \delta(I)$. Suppose that $a\beta(j_1 + j_2) = a\beta j_1 + a\beta j_2 \in \delta(I)$. Since $a\beta j_2 \in \delta(I)$, we have $a\beta j_1 \in \delta(I)$, a contradiction. Suppose that $b\beta(j_1 + j_2) = b\beta j_1 + b\beta j_2 \in \delta(I)$. Since $b\beta j_1 \in \delta(I)$, we have $b\beta j_2 \in \delta(I)$, a contradiction again. Thus $a\beta J \subseteq \delta(I)$ or $b\beta J \subseteq \delta(I)$. □

The following result, it is an extension of [3, theorem 2.19].

Theorem 3.9. *Let δ be a Γ-ideal expansion of M and I be a proper Γ-ideal of M. Then I is a 2-absorbing δ-primary Γ-ideal if and only if whenever $I_1\Gamma I_2\Gamma I_3 \subseteq I$ for some Γ-ideals I_1, I_2, I_3 of M, then $I_1\Gamma I_2 \subseteq I$ or $I_2\Gamma I_3 \subseteq \delta(I)$ or $I_1\Gamma I_3 \subseteq \delta(I)$.*

Proof. Suppose that whenever $I_1\Gamma I_2\Gamma I_3 \subseteq I$ for some Γ-ideals I_1, I_2, I_3 of M, then $I_1\Gamma I_2 \subseteq I$ or $I_2\Gamma I_3 \subseteq \delta(I)$ or $I_1\Gamma I_3 \subseteq \delta(I)$. Then clearly I is a 2-absorbing δ-primary Γ-ideal of M by definition. Conversely, suppose that I is a 2-absorbing δ-primary Γ-ideal of M and $I_1\Gamma I_2\Gamma I_3 \subseteq I$ for some Γ-ideals I_1, I_2, I_3 of M, such that $I_1\Gamma I_2 \nsubseteq I$. We show that $I_1\Gamma I_3 \subseteq \delta(I)$ or $I_2\Gamma I_3 \subseteq \delta(I)$. Suppose that neither $I_1\Gamma I_3 \subseteq \delta(I)$ nor $I_2\Gamma I_3 \subseteq \delta(I)$. Then there are $q_1 \in I_1$ and $q_2 \in I_2$ such that neither $q_1\beta I_3 \subseteq \delta(I)$ nor $q_2\beta I_3 \subseteq \delta(I)$. Since $q_1\alpha q_2\beta I_3 \subseteq I$ and neither $q_1\beta I_3 \subseteq \delta(I)$ nor $q_2\beta I_3 \subseteq \delta(I)$, we have $q_1\alpha q_2 \in I$ by Lemma 3.8. Since $I_1\Gamma I_2 \nsubseteq I$, we have $a\alpha b \nsubseteq I$ for some $a \in I_1, b \in I_2$. Since $a\alpha b\beta I_3 \subseteq I$ and $a\alpha b \notin I$, we have $a\beta I_3 \subseteq \delta(I)$ or $b\beta I_3 \subseteq \delta(I)$ by Lemma 3.8 . We consider three cases.

Case one: Suppose that $a\beta I_3 \subseteq \delta(I)$, but $b\beta I_3 \nsubseteq \delta(I)$. Since $q_1\alpha b\beta I_3 \subseteq I$ and neither $b\beta I_3 \subseteq \delta(I)$ nor $q_1\beta I_3 \subseteq \delta(I)$, we conclude that $q_1\alpha b \in I$ by Lemma 3.8. Since $(a+q_1)\alpha b\beta I_3 \subseteq I$ and $a\beta I_3 \subseteq \delta(I)$, but $q_1\beta I_3 \nsubseteq \delta(I)$, we conclude that $(a+q_1)\beta I_3 \nsubseteq \delta(I)$. Since neither $b\beta I_3 \subseteq \delta(I)$ nor $(a+q_1)\beta I_3 \subseteq \delta(I)$, we conclude that $(a+q_1)\alpha b \in I$ by Lemma 3.8. Since $(a+q_1)\alpha b = a\alpha b + q_1\alpha b \in I$ and $q_1\alpha b \in I$, we conclude that $a\alpha b \in I$, a contradiction.

Case two: Suppose that $b\beta I_3 \subseteq \delta(I)$, but $a\beta I_3 \nsubseteq \delta(I)$. Since $a\alpha q_2\beta I_3 \subseteq I$ and neither $a\beta I_3 \subseteq \delta(I)$ nor $q_2\beta I_3 \subseteq \delta(I)$, we conclude that $a\alpha q_2 \in I$. Since $a\alpha(b+q_2)\beta I_3 \subseteq I$ and $b\beta I_3 \subseteq \delta(I)$, but $q_2\beta I_3 \nsubseteq \delta(I)$, we conclude that $(b+q_2)\beta I_3 \nsubseteq \delta(I)$. Since neither $a\beta I_3 \subseteq \delta(I)$ nor $(b+q_2)\beta I_3 \subseteq \delta(I)$, we conclude that $a\alpha(b+q_2) \in I$ by Lemma 3.8. Since $a\alpha(b+q_2) = a\alpha b + a\alpha q_2 \in I$ and $a\alpha q_2 \in I$, we conclude that $a\alpha b \in I$, a contradiction.

Case three: Suppose that $a\beta I_3 \subseteq \delta(I)$ and $b\beta I_3 \subseteq \delta(I)$. Since $b\beta I_3 \subseteq \delta(I)$ and $q_2\beta I_3 \nsubseteq \delta(I)$, we conclude that $(b+q_2)\beta I_3 \nsubseteq \delta(I)$. Since $q_1\alpha(b+q_2)\beta I_3 \subseteq I$ and neither $q_1\beta I_3 \subseteq \delta(I)$ nor $(b+q_2)\beta I_3 \subseteq \delta(I)$, we conclude that $q_1\beta(b+q_2) = q_1\beta b + q_1\beta q_2 \in I$ by Lemma 3.8. Since $q_1\beta q_2 \in I$ and $q_1\alpha(b+q_1)\beta q_2 \in I$, we conclude that $q_1\beta b \in I$. Since $a\beta I_3 \subseteq \delta(I)$ and $q_1\beta I_3 \nsubseteq \delta(I)$, we conclude that $(a+q_1)\beta I_3 \nsubseteq \delta(I)$. Since $(a+q_1)\alpha q_2\beta I_3 \subseteq I$ and neither $q_2\beta I_3 \subseteq \delta(I)$ nor $(a+q_1)\beta I_3 \subseteq \delta(I)$, we conclude that $(a+q_1)\alpha q_2 = a\alpha q_2 + q_1\alpha q_2 \in I$ by Lemma 3.8. Since $q_1\alpha q_2 \in I$ and $a\alpha q_2 + q_1\alpha q_2 \in I$, we conclude that $a\alpha q_2 \in I$. Now, since $(a+q_1)\alpha(b+q_2)\beta I_3 \subseteq I$ and neither $(a+q_1)\beta I_3 \subseteq \delta(I)$ nor $(b+q_2)\beta I_3 \subseteq \delta(I)$, we conclude that $(a+q_1)\alpha(b+q_2) = a\alpha b + a\alpha q_2 + q_1\alpha b + q_1\alpha q_2 \in I$ by Lemma 3.8. Since $a\alpha q_2, q_1\alpha b, q_1\alpha q_2 \in I$, we have $a\alpha q_2 + q_1\alpha b + q_1\alpha q_2 \in I$. Since $a\alpha b + a\alpha q_2 + q_1\alpha b + q_1\alpha q_2 \in I$ and $a\alpha q_2 + b\alpha q_1 + q_1\alpha q_2 \in I$, we conclude that $a\alpha b \in I$, a contradiction. Hence $I_1\Gamma I_3 \subseteq \delta(I)$ or $I_2\Gamma I_3 \subseteq \delta(I)$. □

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References

[1] W.E. Barnes, On the Γ-rings of Nobusawa, Pacific J. Math., **18** (1966), 411-422.

[2] A. Badawi, On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc. , **75**, No.3 (2007), 417-429.

[3] A. Badawi, U. Tekir, and E. Yetkin, On 2-absorbing primary ideals in commutative rings, Bull. Korean Math. Soc. , **51**, No.4 (2014), 1163-1173.

[4] S. Kyuno, On prime gamma rings, Pacific J. Math., **75**, No. 1 (1978),185-190.

[5] Luh, On the theory of simple Γ-rings, Michigan Math. J., **16** (1969),65-75.

[6] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math., **1** (1964), 81-89.

[7] Y. B. Jun and M. A. ztrk, On gamma-ideal expansions of gamma-rings, international journal of pure applied mathematics, **23**, No.2 (2005), 243-249.

[8] D.Zhao, δ-primary ideals of commutative rings, Kyungpook Math. J., **41** (2001), 17-22.