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A NOTE ON THE GUTMAN INDEX OF JACO GRAPHS

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Abstract: The concept of the *Gutman index,* denoted $Gut(G)$ was introduced for a connected undirected graph G. In this note we apply the concept to the underlying graphs of the family of Jaco graphs, (*directed graphs by definition*), and decribe a recursive formula for the *Gutman index* $Gut(J^*_{n+1}(x))$. We also determine the *Gutman index* for the trivial *edge-joint* between Jaco graphs.

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1. Introduction

For general reference to notation and concepts of graph theory see [2]. Unless mentioned otherwise, a graph $G = G(V, E)$ on $\nu(G)$ vertices (order of G) with $\epsilon(G)$ edges (size of G) will be a finite undirected and connected simple graph. The degree of a vertex in G is denoted $d_G(v)$ and if the context of G is clear the degree is denoted $d(v)$ for brevity. Also in a directed graph $G^{\rightarrow}$ the degree is $d_{G^{\rightarrow}}(v) = d^+_{G^{\rightarrow}}(v) + d^-_{G^{\rightarrow}}(v)$ or for brevity, $d(v) = d^+(v) + d^-(v)$ if G is clear.

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The concept of the Gutman index $Gut(G)$ of a connected undirected graph G was introduced in 1994 by Gutman [4]. It is defined to be $Gut(G) = \sum\limits_{\{v,u\}\subseteq V(G)} d_G(v)d_G(u)d_G(v,u)$, where $d_G(v)$ and $d_G(u)$ are the degree of v and u in G respectively, and $d_G(v,u)$ is the distance between v and u in G. Clearly, if the vertices of G of order n are randomly labeled $v_1, v_2, v_3, ..., v_n$ the definition states that $Gut(G) = \sum\limits_{\ell=1}^{n-1}\sum\limits_{j=\ell+1}^{n} d_G(v_\ell)d_G(v_j)d_G(v_\ell, v_j)$. Worthy results are reported in Andova et al. [1] and Dankelmann et al. [3].

2. The Gutman Index of the Underlying Graph of a Jaco Graph

Despite earlier definitions in respect of the family of Jaco graphs [5, 6], the definitions found in [7] serve as the unifying definitions. For ease of reference some of the important definitions are repeated here.

Definition 2.1. [7] Let $f(x) = mx + c; x \in \mathbb{N}$, $m, c \in \mathbb{N}_0$. The family of infinite linear Jaco graphs denoted by $\{J_\infty(f(x)) : f(x) = mx + c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is defined by $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$, $A(J_\infty(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

Definition 2.2. [7] The family of finite linear Jaco graphs denoted by $\{J_n(f(x)) : f(x) = mx + c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is defined by $V(J_n(f(x))) = \{v_i : i \in \mathbb{N}, i \leq n\}$, $A(J_n(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

The reader is referred to [7] for the definition of the *prime Jaconian vertex* and the *Hope graph*. The graph has four fundamental properties which are:

(i) $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$ and,

(ii) if v_j is the head of an arc then the tail is always a vertex v_i, $i < j$ and,

(iii) if v_k, for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j and finally,

(iv) the degree of vertex k is $d(v_k) = f(k)$.

The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) $v_t, t > n$. Hence, trivially $d(v_i) \leq i$ for $i \in \mathbb{N}$. For $m = 0$ and $c \geq 0$ two special classes of disconnected linear Jaco graphs exist. For $c = 0$ the Jaco graph $J_n(0)$ is a null graph (*edgeless graph*) on n vertices. For $c > 0$, the Jaco graph $J_n(c) = \bigcup\limits_{\lfloor\frac{n}{c+1}\rfloor-copies} K^{\rightarrow}_{c+1} \bigcup K^{\rightarrow}_{n-(c+1)\cdot\lfloor\frac{n}{c+1}\rfloor}$.

since the Gutman index is defined for connected graphs the bound $m \geq 1$ will apply.

In this note we only consider the case $m = 1$, $c = 0$. The generalisation for $f(x) = mx + c$ in general remains open. Denote the underlying Jaco graph by $J_n^*(f(x))$. A recursive formula of the Gutman index $Gut(J_{n+1}^*(x))$ in terms of $Gut(J_n^*(x))$ is given in the next theorem.

Theorem 2.1. *For the underlying graph $J_n^*(x)$ of a finitie Jaco Graph $J_n(x), n \in \mathbb{N}, n \geq 2$ with Jaconian vertex v_i we have that recursively:*

$$Gut(J_{n+1}^*(x)) = Gut(J_n^*(x)) + \sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k, v_t)$$
$$+ \sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n}(d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q)) + (n-i)(\sum_{k=1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k, v_n)$$
$$+ \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t)) + (n-i-1) + i(n-i).$$

Proof. Consider the underlying Jaco graph, $J_n^*(x), n \in \mathbb{N}, n \geq 2$ with prime Jaconian vertex v_i. Now consider $J_{n+1}^*(x)$. From the definition of a Jaco graph the extension from $J_n^*(x)$ to $J_{n+1}^*(x)$ adds the vertex v_{n+1} and the edges $v_{i+1}v_{n+1}, v_{i+2}v_{n+1}, ..., v_nv_{n+1}$.

Step 1: Consider any ordered pair of vertices $(v_k, v_q)_{k<q}, 1 \leq k \leq i-1$, and $k+1 \leq q \leq i$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_k)d_{J_{n+1}^*(1)}(v_q)d_{J_{n+1}^*(x)}(v_k, v_q) = d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_q)d_{J_n^*(x)}(v_k, v_q).$$

By applying this step $\forall v_k, 1 \leq k \leq i-1$, and $\forall v_q, k+1 \leq q \leq i$ with $k < q$ we obtain:

$$\sum_{k=1}^{i-1}\sum_{q=k+1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_q)d_{J_n^*(x)}(v_k, v_q).$$

Step 2: Consider any vertex $v_k, 1 \leq k \leq i$ and any other vertex $v_t, i+1 \leq t \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_k)d_{J_{n+1}^*(x)}(v_t)d_{J_{n+1}^*(x)}(v_k, v_t) = d_{J_n^*(x)}(v_k)(d_{J_n^*(x)}(v_t)+1)d_{J_n^*(x)}(v_k, v_t)$$
$$= d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_k, v_t) + d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k, v_t).$$

By applying this step $\forall v_k, 1 \leq k \leq i$ and $\forall v_t, i+1 \leq t \leq n$, we obtain:

$$\sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_k,v_t) + \sum_{k=1}^{i}\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k,v_t).$$

Step 3: Consider any two distinct vertices $v_t, v_q, i+1 \leq t \leq n-1$, and $t+1 \leq q \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$\begin{aligned} &d_{J_{n+1}^*(x)}(v_t)d_{J_{n+1}^*(x)}(v_q)d_{J_{n+1}^*(x)}(v_t,v_q) \\ &\quad = (d_{J_n^*(x)}(v_t)+1)(d_{J_n^*(x)}(v_q)+1)d_{J_n^*(x)}(v_t,v_q) \\ &\quad\quad = d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_q) + d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q) + 1. \end{aligned}$$

By applying this step $\forall v_t, i+1 \leq t \leq n-1$ and $\forall v_q, t+1 \leq q \leq n$, we obtain:

$$\begin{aligned} &\sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n} d_{J_n^*(x)}(v_t)d_{J_n^*(x)}(v_q) \\ &\quad + \sum_{t=i+1}^{n-1}\sum_{q=t+1}^{n} (d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q)) + (n-i-1). \end{aligned}$$

Step 4: Consider any vertex $v_k, 1 \leq k \leq i$ and the vertex v_{n+1}. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_k)d_{J_{n+1}^*(x)}(v_{n+1})d_{J_{n+1}^*(x)}(v_k,v_{n+1}) = d_{J_n^*(x)}(v_k)(n-i)(d_{J_n^*(x)}(v_k,v_n)+1).$$

By applying this step $\forall v_k, 1 \leq k \leq i$ we obtain:

$$\begin{aligned} &\sum_{k=1}^{i} d_{J_n^*(x)}(v_k)(n-i)(d_{J_n^*(x)}(v_k,v_n)+1) \\ &\quad = (n-i)\sum_{k=1}^{i} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_k,v_n) + i(n-i). \end{aligned}$$

Step 5: Consider any vertex $v_t, i+1 \leq t \leq n$ and the vertex v_{n+1}. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n+1}^*(x)}(v_t)d_{J_{n+1}^*(x)}(v_{n+1})d_{J_{n+1}^*(x)}(v_t,v_{n+1}) = d_{J_n^*(x)}(v_t)(n-i)d_{J_n^*(x)}(v_t,v_n).$$

By applying this step $\forall v_t, i+1 \leq t \leq n$ we obtain:

$$\sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t)(n-i) = (n-i) \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t).$$

Final Summation Step: Adding Steps 1 to 5 and noting that:

$$Gut(J_n^*(x)) = \sum_{k=1}^{i-1} \sum_{q=k+1}^{i} d_{J_n^*(x)}(v_k) d_{J_n^*(x)}(v_q) d_{J_n^*(x)}(v_k, v_q) + \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k) d_{J_n^*(x)}(v_t) d_{J_n^*(x)}(v_k, v_t) + \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} d_{J_n^*(x)}(v_t) d_{J_n^*(x)}(v_q),$$

provides the result:

$$Gut(J_{n+1}^*(x)) = Gut(J_n^*(x)) + \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_k) d_{J_n^*(x)}(v_k, v_t) + \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} (d_{J_n^*(x)}(v_t) + d_{J_n^*(x)}(v_q)) + (n-i)(\sum_{k=1}^{i} d_{J_n^*(x)}(v_k) d_{J_n^*(x)}(v_k, v_n) + \sum_{t=i+1}^{n} d_{J_n^*(x)}(v_t)) + (n-i-1) + i(n-i).$$

□

3. The Gutman Index of the Edge-Joint between $J_n^*(x), n \in \mathbb{N}$ and $J_m^*(x), m \in \mathbb{N}$

The concept of an *edge-joint* between two simple undirected graphs G and H is defined below.

Definition 3.1. The edge-joint of two simple undirected graphs G and H is the graph obtained by linking the edge vu, $v \in V(G), u \in V(H)$ and denoted, $G \rightsquigarrow_{vu} H$.

Note. $G \rightsquigarrow_{vu} H = G \cup H + vu$, $v \in V(G), u \in V(H)$.

The next theorem provides $Gut(J_n^*(x) \rightsquigarrow_{v_1u_1} J_m^*(x))$ in terms of $Gut(J_n^*(x))$ and $Gut(J_m^*(x))$. The edge-joint $J_n^*(x) \rightsquigarrow_{v_1u_1} J_m^*(x)$ is called *trivial*. Edge-joints $J_n^*(x) \rightsquigarrow_{v_iu_j} J_m^*(x)$, $i \neq 1$ or $j \neq 1$ are called *non-trivial*. For families (classes) of graphs such as paths P_n, cycles C_n, complete graphs K_n, Jaco graphs $J_n(f(x))$, etc, the notation is abbreviated as $P_n \rightsquigarrow_{vu} P_m = P_{n,m}^{\rightsquigarrow_{vu}}$ and $J_n^*(f(x)) \rightsquigarrow_{v_iu_j} J_m^*(f(x)) = J_{n,m}^{\rightsquigarrow_{v_iu_j}}$, etc.

Theorem 3.1. *For the underlying graphs $J_n^*(x)$ and $J_m^*(x)$ of the finitie Jaco Graphs $J_n(x), J_m(x)$, $n, m \in \mathbb{N}$ and $n \geq m \geq 2$:*

$$\begin{aligned}Gut(J_n^*(x) \rightsquigarrow_{v_1u_1} J_m^*(x)) &= Gut(J_{n,m}^{\rightsquigarrow_{v_1u_1}}) = Gut(J_n^*(x)) + Gut(J_m^*(x))\\ &+ \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell) d_{J_n^*(x)}(v_1, v_\ell) + \sum_{s=2}^{m} d_{J_m^*(x)}(u_s) d_{J_m^*(x)}(u_1, u_s)\\ &+ \sum_{t=2}^{m} (d_{J_n^*(x)}(v_1) + 1) d_{J_m^*(x)}(u_t)(d_{J_m^*(x)}(u_1, u_t) + 1)\\ &+ \sum_{k=2}^{n}\sum_{t=2}^{m} d_{J_n^*(x)}(v_k) d_{J_m^*(x)}(u_t)(d_{J_n^*(x)}(v_1, v_k) + d_{J_m^*(x)}(u_1, u_t) + 1) + 4.\end{aligned}$$

Proof. Consider the underlying Jaco graphs, $J_n^*(x), J_m^*(x)$, with $n, m \in \mathbb{N}$ and $n \geq m \geq 2$ with $J_m(x)$ having prime Jaconian vertex u_i. Also label the vertices of $J_n^*(x)$ and $J_m^*(x)$; $v_1, v_2, v_3, ..., v_n$ and $u_1, u_2, u_3, ..., u_m$, respectively. Consider $J_{n,m}^{\rightsquigarrow_{v_1u_1}} = J_n^*(x) \cup J_m^*(x) + v_1u_1$. Without loss of generality apply the piecewise definition:

$$\begin{aligned}Gut(J_{n,m}^{\rightsquigarrow_{v_1u_1}}) &= \sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n} d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_k) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_\ell) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_k, v_\ell)\\ &+ \sum_{t=1}^{m-1}\sum_{s=t+1}^{m} d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(u_t) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(u_s) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(u_t, u_s)\\ &+ \sum_{k=1}^{n}\sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_k) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(u_t) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_k, u_t) + d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_1)\\ &\qquad d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(u_1) d_{J_{n,m}^{\rightsquigarrow_{v_1u_1}}}(v_1, u_1).\end{aligned}$$

Step 1(a): Consider vertex v_1 and vertex $v_\ell, 2 \leq \ell \leq n$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n,m}^{\leadsto v_1u_1}}(v_1)d_{J_{n,m}^{\leadsto v_1u_1}}(v_\ell)d_{J_{n,m}^{\leadsto v_1u_1}}(v_1,v_\ell) = (d_{J_n^*(x)}(v_1)+1)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell)$$
$$= d_{J_n^*(x)}(v_1)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell) + d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell).$$

By applying this step $\forall v_\ell, 2 \leq \ell \leq n$ we obtain:

$$\sum_{\ell=2}^{n} d_{J_n^*(x)}(v_1)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell).$$

Step 1(b): For all ordered pairs of vertices $(v_k, v_\ell)_{k<\ell}$ with $2 \leq k \leq n-1$ and $3 \leq \ell \leq n$ we have that:

$$\sum_{k=2}^{n-1}\sum_{\ell=k+1}^{n} d_{J_{n,m}^{\leadsto v_1u_1}}(v_k)d_{J_{n,m}^{\leadsto v_1u_1}}(v_\ell)d_{J_{n,m}^{\leadsto v_1u_1}}(v_k,v_\ell)$$
$$= \sum_{k=2}^{n-1}\sum_{\ell=k+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_k,v_\ell).$$

By applying this step $\forall (v_k, v_\ell)_{k<\ell}$, $1 \leq k \leq n-1$ and $2 \leq \ell \leq n$, we obtain:

$$\sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n} d_{J_n^*(x)}(v_k)d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_k,v_\ell) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell)$$
$$= Gut(J_n^*(x)) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell).$$

Step 2: Similar to Step 1 we have that:

$$\sum_{t=1}^{m-1}\sum_{s=t+1}^{m} d_{J_{n,m}^{\leadsto v_1u_1}}(u_t)d_{J_{n,m}^{\leadsto v_1u_1}}(u_s)d_{J_{n,m}^{\leadsto v_1u_1}}(u_t,u_s)$$
$$= \sum_{t=1}^{m-1}\sum_{s=t+1}^{m} d_{J_m^*(x)}(u_t)d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_t,u_s) + \sum_{s=2}^{m} d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_1,u_s)$$
$$= Gut(J_m^*(x)) + \sum_{s=2}^{m} d_{J_m^*(x)}(u_s)d_{J_m^*(x)}(u_1,u_s).$$

Step 3: To conclude this step we will provide the next partial summation as a piecewise summation, to be:

$$\sum_{k=1}^{n}\sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t)$$
$$= \sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_t)$$
$$+ \sum_{k=2}^{n}\sum_{t=2}^{m} d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t).$$

Step 3(a): Consider vertex v_1 and vertex $u_t, 2 \leq t \leq m$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_t)$$
$$= (d_{J_n^*(x)}(v_1)+1)d_{J_m^*(x)}(u_t)(d_{J_m^*(x)}(u_1,u_t)+1).$$

By applying this step $\forall u_t, 2 \leq t \leq m$ we obtain:

$$\sum_{t=2}^{m}(d_{J_n^*(x)}(v_1)+1)d_{J_m^*(x)}(u_t)(d_{J_m^*(x)}(u_1,u_t)+1).$$

Step 3(b): Consider vertex $v_k, 2 \leq k \leq n$ and vertex $u_t, 2 \leq t \leq m$. By applying the definition of the Gutman index to this pair of vertices we have the term:

$$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_t)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_k,u_t)$$
$$= d_{J_n^*(x)}(v_k)d_{J_m^*(x)}(u_t)(d_{J_n^*(x)}(v_1,v_k)+d_{J_m^*(x)}(u_1,u_t)+1).$$

By applying the step $\forall v_k, 2 \leq k \leq n$ and $\forall u_t, 2 \leq t \leq m$, we obtain:

$$\sum_{k+2}^{n}\sum_{t=2}^{m} d_{J_n^*(x)}(v_k)d_{J_m^*(x)}(u_t)(d_{J_n^*(x)}(v_1,v_k)+d_{J_m^*(x)}(u_1,u_t)+1).$$

Step 4: It is easy to see that:

$$d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(u_1)d_{J_{n,m}^{\rightsquigarrow v_1u_1}}(v_1,u_1) = 4.$$

Final Summation Step: Adding Steps 1 to 4 provides the result:

$$Gut(J_{n,m}^{\rightsquigarrow v_1u_1}) = Gut(J_n^*(x)) + Gut(J_m^*(x)) + \sum_{\ell=2}^{n} d_{J_n^*(x)}(v_\ell)d_{J_n^*(x)}(v_1,v_\ell)$$

$$+\sum_{s=2}^{m} d_{J^*_m(x)}(u_s)d_{J^*_m(x)}(u_1,u_s)+\sum_{t=2}^{m}(d_{J^*_n(x)}(v_1)+1)d_{J^*_m(x)}(u_t)(d_{J^*_m(x)}(u_1,u_t)+1)$$
$$+\sum_{k=2}^{n}\sum_{t=2}^{m} d_{J^*_n(x)}(v_k)d_{J^*_m(x)}(u_t)(d_{J^*_n(x)}(v_1,v_k)+d_{J^*_m(x)}(u_1,u_t)+1)+4.$$

□

4. Conclusion

For the simple case $f(x) = x$ the calculation of the Gutman index for Jaco graph and the edge-joint between them is immediately complicated. Finding a result similar to Theorem 3.1 for $J^*_n(x) \rightsquigarrow_{v_i u_j} J^*_m(x)$, $i \neq 1$ or $j \neq 1$ *(non-trivial edge-joints)* remains open. The single most important challege is to find a closed formula for the number of edges in $J_n(x)$. Such closed formula will enable finding a closed formula for distances between given vertices and a simplied formula for many invariants of Jaco graphs might result from such finding. Hence, important open questions remain such as: *Is there a closed formula for the number of edges of $J_n(x)$, $n \in \mathbb{N}$? Is there a closed formula for the cardinality of the Jaconian set $\mathbb{J}(J_n(x))$ of $J_n(x), n \in \mathbb{N}$? Is there a closed formula for $d_{J^*_n(x)}(v_1, v_n)$ in $J^*_n(x), n \in \mathbb{N}$?*. Refer to [7] for further reading.

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