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FIXED POINT POINTS OF RATIONAL TYPE CONTRACTIONS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we prove common fixed point results for mappings satisfying some contractive condition in multiplicative metric spaces. We also give examples in support of our results.

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1. Introduction and Preliminaries

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [2] introduced the concept of multiplicative metric spaces as follows:

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Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping d together with X, that is, (X, d) is a multiplicative metric space.

Example 1.2. ([6]) Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d^* : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $|\cdot|^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1; \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}^n_+, d^*)$ is a multiplicative metric space.

Example 1.3. ([7]) Let $d : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined as $d(x, y) = a^{|x-y|}$ for all $x, y \in \mathbb{R}$, where $a > 1$. Then d is a multiplicative metric and $(\mathbb{R}, d)$ is a multiplicative metric space. We may call it usual multiplicative metric spaces.

Remark 1.4. We note that the Example 1.2 is valid for positive real numbers and Example 1.3 is valid for all real numbers.

Example 1.5. ([7]) Let (X, d) be a metric space. Define a mapping d_a on X by

$$d_a(x, y) = a^{d(x,y)} = \begin{cases} 1 & \text{if } x = y, \\ a & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$, where $a > 1$. Then d_a is a multiplicative metric and (X, d_a) is known as the discrete multiplicative metric space.

Example 1.6. ([1]) Let $X = C^*[a, b]$ be the collection of all real-valued multiplicative continuous functions on $[a, b] \subset \mathbb{R}_+$. Then (X, d) is a multiplicative metric space with d defined by $d(f, g) = \sup_{x \in [a,b]} \left|\frac{f(x)}{g(x)}\right|$ for arbitrary $f, g \in X$.

Remark 1.7. ([7]) We note that multiplicative metric and metric spaces are independent.

Indeed, the mapping d^* defined in Example 1.2 is multiplicative metric but not metric as it does not satisfy triangular inequality. Consider

$$d^*\left(\frac{1}{3}, \frac{1}{2}\right) + d^*\left(\frac{1}{2}, 3\right) = \frac{3}{2} + 6 = 7.5 < 9 = d^*\left(\frac{1}{3}, 3\right).$$

On the other hand the usual metric on $\mathbb{R}$ is not multiplicative metric as it doesnt satisfy multiplicative triangular inequality, since

$$d(2,3) \cdot d(3,6) = 3 < 4 = d(2,6).$$

One can refer to [3, 6] for detailed multiplicative metric topology.

Definition 1.8. Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

Remark 1.9. The set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. Let $X = \mathbb{R}_+$ and the sequence $\{x_n\} = \{\frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to usual metric and X is not a complete metric space, since $0 \notin \mathbb{R}_+$. In case of a multiplicative metric space, we take a sequence $\{x_n\} = \{a^{\frac{1}{n}}\}$, where $a > 1$. Then $\{x_n\}$ is a multiplicative Cauchy sequence since for $n \geq m$,

$$\begin{aligned} d(x_n, x_m) &= \left|\frac{x_n}{x_m}\right| = \left|\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}}\right| = \left|a^{\frac{1}{n} - \frac{1}{m}}\right| \\ &\leq a^{\frac{1}{m} - \frac{1}{n}} < a^{\frac{1}{m}} < \epsilon \quad \text{if } m > \frac{\log a}{\log \epsilon}, \end{aligned}$$

where $|a| = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$ Also, $\{x_n\} \to 1$ as $n \to \infty$ and $1 \in \mathbb{R}_+$. Hence (X, d) is a complete multiplicative metric space.

In 2012, Özavsar and Çevikel [6] gave the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings in a multiplicative metric space.

Definition 1.10. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative contraction* if there exists a real constant $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^{\lambda}(x, y) \quad \text{for all } x, y \in X.$$

Also they proved the Banach Contraction Principle in the setting of multiplicative metric spaces as follows:

Theorem 1.11. *Let f be a multiplicative contraction mapping of a complete multiplicative metric space (X, d) into itself. Then f has a unique fixed point.*

In 1996, Jungck [4] introduce the notion of weakly compatible mappings in a metric space.

Now, we introduce the notions in multiplicative metric spaces

Definition 1.12. Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are said to be *weakly compatible* if they commute at coincidence points, that is, if $ft = gt$ for some $t \in X$ implies that $fgt = gft$.

Khan et al. [5] initiated the use of the control function as follows:

Definition 1.13. A function $\phi : [1, \infty) \to [1, \infty)$ is called an *alternating distance function* if the following properties are satisfied:

(1) ϕ is increasing and continuous,

(2) $\phi(t) = 1$ if and only if $t = 1$.

In our results we will use the following class of functions.

$\Phi = \{\phi : [1, \infty) \to [1, \infty) : \phi \text{ is an alternating distance function}\}$,

$\Psi = \{\psi : [1, \infty) \to [1, \infty) :$ for any sequence $\{x_n\}$ in $[1, \infty)$ with $x_n \to t > 1$, $\lim_{n\to\infty} \psi(x_n) > 1\}$.

We note that Ψ is non-empty since $\psi(t) = e^t$ for $t \in [1, \infty)$. Thus $\psi \in \Psi$.

Remark 1.14. Clearly for $\psi \in \Psi$, $\psi(t) > 1$ for $t > 1$ and $\psi(1)$ need not be equal to 1.

2. Main Results

Now we prove some fixed point results as follows:

Theorem 2.1. *Let f be a continuous mapping of a complete multiplicative metric space (X, d) into itself such that for all $x, y \in X$*

$$\phi(d(fx, fy)) \leq \frac{\phi(M(x, y))}{\psi(N(x, y))}, \tag{2.1}$$

where $\phi \in \Phi$ and $\psi \in \Psi$,

$$M(x, y) = \max\left\{\frac{d(x, fx)d(y, fy)}{d(x, y)}, \frac{d(y, fx)d(x, fy)}{d(x, y)}, d(x, y)\right\}$$

and

$$N(x, y) = \max\left\{\frac{d(x, fx)d(y, fy)}{d(x, y)}, d(x, y)\right\}.$$

Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Then there exists $x_1 \in X$ such that $x_1 = fx_0$. So we can define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$ for $n \geq 0$.

If there exists some $n \in \mathbb{N}$ such that $x_(n + 1) = x_n$. Then we have $x_{n+1} = fx_n = x_n$, which implies that x_n is a fixed point of f.

Suppose that $x_{n+1} \neq x_n$, that is, $d(x_{n+1}, x_n) \neq 1$ for all n. Let $R_n = d(x_{n+1}, x_n)$ for all $n \geq 0$. From (2.1), we have

$$\begin{aligned}\phi(d(x_n, x_{n+1}) &= \phi(d(fx_{n-1}, fx_n)) \\ &\leq \frac{\phi(M(x_{n-1}, x_n))}{\psi(N(x_{n-1}, x_n))},\end{aligned}$$

where

$$\begin{aligned}&M(x_{n-1}, x_n) \\ &= \max\left\{\frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{d(x_{n-1}, x_n)}, \frac{d(x_n, fx_{n-1})d(x_{n-1}, fx_n)}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\} \\ &= \max\left\{\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, \frac{d(x_n, x_n)d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{R_n, R_{n-1}\}\end{aligned}$$

and

$$\begin{aligned} N(x_{n-1}, x_n) &= \max\left\{\frac{d(x_{n-1}, fx_{n-1})d(x_n, fx_n)}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\} \\ &= \max\left\{\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{R_n, R_{n-1}\}. \end{aligned}$$

Therefore, we have

$$\phi(R_n) \leq \frac{\phi(\max\{R_n, R_{n-1}\})}{\psi(\max\{R_n, R_{n-1}\})}. \tag{2.2}$$

If $R_n > R_{n-1}$, then from (2.2), we have

$$\phi(R_n) \leq \frac{\phi(R_n)}{\psi(R_n)},$$

that is, $\psi(R_n) \leq 1$, which is a contradiction. So $R_n \leq R_{n-1}$, that is, $\{R_n\}$ is a decreasing sequence. Then the inequality (2.2) yields that

$$\phi(R_n) \leq \frac{\phi(R_{n-1})}{\psi(R_{n-1})}. \tag{2.3}$$

Since $\{R_n\}$ is a decreasing sequence of positive real numbers and it is bounded below, there exists $r \geq 1$ such that

$$R_n = d(x_{n+1}, x_n) \to r \tag{2.4}$$

as $n \to \infty$.

Now we shall show that $r = 1$. Assume that $r > 1$. Taking limit on both sides (2.3) and using (2.4), the property of ψ and the continuity of ϕ, we get

$$\phi(r) \leq \frac{\phi(r)}{\lim\limits_{n\to\infty} \psi(R_{n-1})},$$

which implies that $\lim\limits_{n\to\infty} \psi(R_{n-1}) \leq 1$, which, by the property of ψ, is a contradiction. Therefore,

$$R_n = d(x_{n+1}, x_n) \to 1 \tag{2.5}$$

as $n \to \infty$.

Next to show that $\{x_n\}$ is a multiplicative Cauchy sequence. Suppose that $\{x_n\}$ is not a multiplicative Cauchy sequence. Then there exists an $\epsilon > 1$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) \geq k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon.$$

Assume that $n(k)$ is the smallest such positive integer, we get $n(k) > m(k) \geq k$,

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \quad \text{and} \quad d(x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) \cdot d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ and using (2.5), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.6}$$

Again,

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) \cdot d(x_{m(k)}, x_{n(k)}) \cdot d(x_{n(k)}, x_{n(k)-1})$$

and

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) \cdot d(x_{m(k)-1}, x_{n(k)-1}) \cdot d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ in above inequalities and using (2.5) and (2.6), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.7}$$

Again

$$d(x_{n(k)-1}, x_{m(k)}) \leq d(x_{m(k)-1}, x_{n(k)-1}) \cdot d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) \cdot d(x_{n(k)-1}, x_{m(k)}).$$

Letting $k \to \infty$ in the above inequalities and using (2.5) and (2.7), we have

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \epsilon. \tag{2.8}$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_n(k)) = \epsilon. \tag{2.9}$$

Let

$$\begin{aligned} &M(x_{n(k)-1}, x_{m(k)-1}) \\ &= \max\left\{\frac{d(x_{n(k)-1}, fx_{n(k)-1})d(x_{m(k)-1}, fx_{m(k)-1})}{d(x_{m(k)-1}, x_{n(k)-1})},\right. \\ &\quad \left.\frac{d(x_{n(k)-1}, fx_{m(k)-1})d(x_{m(k)-1}, fx_{n(k)-1})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\} \\ &= \max\left\{\frac{d(x_{n(k)-1}, x_{n(k)})d(x_{m(k)-1}, x_{m(k)})}{d(x_{m(k)-1}, x_{n(k)-1})},\right. \\ &\quad \left.\frac{d(x_{n(k)-1}, x_{m(k)})d(x_{m(k)-1}, x_{n(k)})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &N(x_{n(k)-1}, x_{m(k)-1}) \\ &= \max\left\{\frac{d(x_{n(k)-1}, fx_{n(k)-1})d(x_{m(k)-1}, fx_{m(k)-1})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\} \\ &= \max\left\{\frac{d(x_{n(k)-1}, x_{n(k)})d(x_{m(k)-1}, x_{m(k)})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}. \end{aligned} \tag{2.11}$$

Letting $k \to \infty$ in (2.10) and (2.11), using (2.5), (2.7), (2.8) and (2.9), we have

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \max\{1, \epsilon, \epsilon\} = \epsilon \tag{2.12}$$

and

$$\lim_{k\to\infty} N(x_{n(k)-1}, x_{m(k)-1}) = \max\{1, \epsilon\} = \epsilon. \tag{2.13}$$

From (2.1), using (2.10) and (2.11), we have

$$\begin{aligned} \phi(d(x_{m(k)}, x_{n(k)})) &= \phi(d(fx_{m(k)-1}, fx_{n(k)-1})) \\ &\le \frac{\phi(M(x_{n(k)-1}, x_{m(k)-1}))}{\psi(N(x_{n(k)-1}, x_{m(k)-1}))}. \end{aligned}$$

Taking limit on both the sides and using (2.6), (2.12) and (2.13), the property of ψ and the continuity of ϕ, we have

$$\phi(\epsilon) \le \frac{\phi(\epsilon)}{\lim\limits_{k\to\infty} \psi(N(x_{n(k)-1}, x_{m(k)-1}))},$$

that is, $\lim_{k\to\infty} \psi(N(x_{n(k)-1}, x_{m(k)-1})) \le 1$, which is a contradiction by property of ψ. Thus $\{x_n\}$ is a multiplicative Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Then using continuity of f, we get

$$fu = f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} x_{n+1} = u.$$

Hence u is a fixed point of f.

Finally, we shall prove the uniqueness of the fixed point of f. Suppose that u and v $(u \ne v)$ be two fixed points of f. Consider

$$\phi(d(u,v)) = \phi(d(fu,fv)) \le \frac{\phi(M(u,v))}{\psi(N(u,v))},$$

where $\phi \in \Phi$, $\psi \in \Psi$ and

$$\begin{aligned} M(u,v) &= \max\left\{\frac{d(u,fu)d(v,fv)}{d(u,v)}, \frac{d(v,fu)d(u,fv)}{d(u,v)}, d(u,v)\right\} \\ &= \max\left\{\frac{d(u,u)d(v,v)}{d(u,v)}, \frac{d(v,u)d(u,v)}{d(u,v)}, d(u,v)\right\} \\ &= d(u,v) \end{aligned}$$

and

$$N(u,v) = \max\left\{\frac{d(u,fu)d(v,fv)}{d(u,v)}, d(u,v)\right\} = d(u,v).$$

Therefore, we have

$$\phi(d(u,v)) = \phi(d(fu,fv)) \le \frac{\phi(d(u,v))}{\psi(d(u,v))},$$

which implies that $\psi(d(u,v)) \le 1$, which is a contraction by definition of ψ. Hence $u = v$. Therefore f has a unique fixed point. This completes the proof. □

Next we prove the following result without the condition of continuity of f.

Theorem 2.2. *Let f be a mapping of a complete multiplicative metric space (X,d) into itself such that for all $x, y \in X$*

$$\phi(d(fx,fy)) \le \frac{\phi(M(x,y))}{\psi(N(x,y))},$$

where $\phi \in \Phi$ *and* $\psi \in \Psi$,

$$M(x,y) = \max\left\{\frac{d(x,fx)d(y,fy)}{d(x,y)}, \frac{d(y,fx)d(x,fy)}{d(x,y)}, d(x,y)\right\}$$

and

$$N(x,y) = \max\left\{\frac{d(x,fx)d(y,fy)}{d(x,y)}, d(x,y)\right\}.$$

Then f has a unique fixed point.

Proof. From the proof of Theorem 2.1 $\{x_n\}$ is a multiplicative Cauchy sequence and hence there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Suppose that $fu \neq u$, that is, $d(u,fu) > 1$. Consider

$$\phi(d(fx_n,fu)) \leq \frac{\phi(M(x_n,u))}{\psi(N(x_n,u))}, \tag{2.14}$$

where

$$\begin{aligned} &M(x_n,u)\\ &= \max\left\{\frac{d(x_n,fx_n)d(u,fu)}{d(x_n,u)}, \frac{d(u,fx_n)d(x_n,fu)}{d(x_n,u)}, d(x_n,u)\right\}\\ &= \max\left\{\frac{d(x_n,x_{n+1})d(u,fu)}{d(x_n,u)}, \frac{d(u,x_{n+1})d(x_n,fu)}{d(x_n,u)}, d(x_n,u)\right\} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} N(x_n,u) &= \max\left\{\frac{d(x_n,fx_n)d(u,fu)}{d(x_n,u)}, d(x_n,u)\right\}\\ &= \max\left\{\frac{d(x_n,x_{n+1})d(u,fu)}{d(x_n,u)}, d(x_n,u)\right\}. \end{aligned} \tag{2.16}$$

Letting $n \to \infty$ in (2.15) and (2.16),we have

$$\lim_{n\to\infty} M(x_n,u) = d(u,fu) > 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n,u) = d(u,fu) > 1.$$

Again letting $n \to \infty$ in (2.14), using (2.15), (2.16) and property of ϕ and ψ, we have

$$\phi(d(u,fu)) \leq \frac{\phi(d(u,fu))}{\lim_{n\to\infty} \psi(N(x_n,u))},$$

which implies that $\lim_{n\to\infty} \psi(N(x_n,u)) \leq 1$, which is a contradiction by property of ψ. Therefore, $fu = u$ and hence u is a fixed point of f.

Uniqueness easily follows from Theorem 2.1. This completes the proof. □

Corollary 2.3. *Let f be a mapping of a complete multiplicative metric space (X,d) into itself such that for all $x,y \in X$*

$$\phi(d(fx,fy)) \leq \frac{\phi(N(x,y))}{\psi(N(x,y))},$$

where $\phi \in \Phi$, $\psi \in \Psi$, and

$$N(x,y) = \max\left\{\frac{d(x,fx)d(y,fy)}{d(x,y)}, d(x,y)\right\}.$$

Then f has a unique fixed point.

Corollary 2.4. *Let f be a mapping of a complete multiplicative metric space (X,d) into itself such that for all $x,y \in X$ and for some $k \in (0,1)$*

$$\phi(d(fx,fy)) \leq k \max\left\{\frac{d(x,fx)d(y,fy)}{d(x,y)}, d(x,y)\right\}.$$

Then f has a unique fixed point.

Example 2.5. Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}_+$ be a multiplicative metric defined as $d(x,y) = a^{|x-y|,}$ where $a > 1$. Then (X,d) is a complete multiplicative metric space. Define a mapping $f : X \to X$ as follows $fx = 1$ for all $x \in X$.

Define $\phi, \psi : [1,\infty) \to [1,\infty)$ as $\phi(t) = \psi(t) = t$. Clearly ϕ be increasing and $\phi(1) = 1$. Also for any sequence $\{x_n\}$ in $[1,\infty)$ with $x_n \to t > 1$, $\lim_{n\to\infty} \psi(x_n) > 1$.

Also $\phi(d(fx,fy)) \leq \frac{\phi(M(x,y))}{\psi(N(x,y))}$ holds. Hence all the conditions of Theorem 2.1 and Theorem 2.2 are satisfied. Also f has a fixed point 1.

Theorem 2.6. *Let f and g be mappings of a multiplicative metric space (X,d) into itself such that $f(X) \subset g(X)$ and for all $x,y \in X$*

$$\phi(d(fx,fy)) \leq \frac{\phi(M(x,y))}{\psi(N(x,y))},$$

where $\phi \in \Phi$ and $\psi \in \Psi$,

$$M(x,y) = \max\left\{\frac{d(fx,gx)d(fy,gy)}{d(gx,gy)}, \frac{d(gy,fx)d(gx,fy)}{d(gx,gy)}, d(gx,gy)\right\}$$

and

$$N(x,y) = \max\left\{\frac{d(fx,gx)d(fy,gy)}{d(gx,gy)}, d(gx,gy)\right\}.$$

Assume that $g(X)$ is complete. Then f and g have a coincidence fixed point. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Again for $x_1 \in X$, there exists $x_2 \in X$ such that $fx_1 = gx_2$. Continuing like this, we can define a sequence $\{y_n\}$ in X such that

$$y_n = fx_n = gx_{n+1}.$$

If $y_n = y_{n+1}$, then clearly f and g have a coincidence point. Since

$$y_n = fx_n = gx_{n+1} = y_{n+1} = fx_{n+1} = gx_{n+2},$$

which implies that x_{n+1} is a coincidence point of f and g.

Now we assume that $y_n \neq y_{n+1}$, that is, $d(y_n, y_{n+1}) > 1$.

Consider

$$\phi(d(y_n, y_{n+1})) = \phi(d(fx_n, fx_{n+1})) \leq \frac{\phi(M(x_n, x_{n+1}))}{\psi(N(x_n, x_{n+1}))},$$

where $\phi \in \Phi$, $\psi \in \Psi$, and

$$\begin{aligned} &M(x_n, x_{n+1}) \\ &= \max\left\{\frac{d(fx_n, gx_n)d(fx_{n+1}, gx_{n+1})}{d(gx_n, gx_{n+1})},\right. \\ &\qquad \left.\frac{d(gx_{n+1}, fx_n)d(gx_n, fx_{n+1})}{d(gx_n, gx_{n+1})}, d(gx_n, gx_{n+1})\right\} \\ &= \max\left\{\frac{d(y_n, y_{n-1})d(y_{n+1}, y_n)}{d(y_{n-1}, y_n)}, \frac{d(y_n, y_n)d(y_{n-1}, y_{n+1})}{d(y_{n-1}, y_n)}, d(y_{n-1}, y_n)\right\} \\ &= \max\{d(y_{n+1}, y_n), d(y_{n+1}, y_n), d(y_{n-1}, y_n)\} \end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\left\{\frac{d(fx_n, gx_n)d(fx_{n+1}, gx_{n+1})}{d(gx_n, gx_{n+1})}, d(gx_n, gx_{n+1})\right\} \\ &= \max\left\{\frac{d(y_n, y_{n-1})d(y_{n+1}, y_n)}{d(y_{n-1}, y_n)}, d(y_{n-1}, y_n)\right\} \\ &= \max\{d(y_{n+1}, y_n), d(y_{n-1}, y_n)\}. \end{aligned}$$

If $d(y_{n+1}, y_n) > d(y_{n-1}, y_n)$, then $M(x_n, x_{n+1}) = N(x_n, x_{n+1}) = d(y_{n+1}, y_n)$. Then we have

$$\phi(d(y_n, y_{n+1})) \leq \frac{\phi(d(y_{n+1}, y_n))}{\psi(d(y_{n+1}, y_n))},$$

which implies that $\psi(d(y_{n+1}, y_n)) \le 1$, which is a contradiction. Therefore, $M(x_n, x_{n+1}) = N(x_n, x_{n+1}) = d(y_{n-1}, y_n)$ and we have

$$\phi(d(y_n, y_{n+1})) \le \frac{\phi(d(y_{n-1}, y_n))}{\psi(d(y_{n-1}, y_n))}, \tag{2.17}$$

which implies that

$$\phi(d(y_n, y_{n+1})) \le \frac{\phi(d(y_{n-1}, y_n))}{\psi(d(y_{n-1}, y_n))} \le \phi(d(y_{n-1}, y_n)).$$

Since ϕ is increasing, therefore $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$, which implies that $\{d(y_n, y_{n+1})\}$ is decreasing sequence and bounded below. Hence it converges to some positive number $r \ge 1$.

Now we prove that $r = 1$. Letting $n \to \infty$ in (2.17), and using the continuity of ϕ, we have

$$\phi(r) \le \frac{\phi(r)}{\lim_{n\to\infty} \psi(d(y_{n-1}, y_n))},$$

which implies that $\lim_{n\to\infty} \psi(d(y_{n-1}, y_n)) \le 1$, which is is a contradiction by property of ψ. Hence $r = 1$. Therefore, we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 1. \tag{2.18}$$

Next to show that $\{y_n\}$ is a multiplicative Cauchy sequence. Suppose that $\{y_n\}$ is not a multiplicative Cauchy sequence. Then there exists an $\epsilon > 1$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) \ge k$ and $d(y_{m(k)}, y_{n(k)}) \ge \epsilon$. Assume that $n(k)$ is the smallest such positive integer, we get $n(k) > m(k) \ge k$,

$$d(y_{m(k)}, y_{n(k)}) \ge \epsilon \quad \text{and} \quad d(y_{m(k)}, y_{n(k)-1}) < \epsilon.$$

Now,

$$\epsilon \le d(y_{m(k)}, y_{n(k)}) \le d(y_{m(k)}, y_{n(k)-1}) \cdot d(y_{n(k)-1}, y_{n(k)}).$$

Letting $k \to \infty$ and using (2.18), we have

$$\lim_{n\to\infty} d(y_{m(k)}, y_{n(k)}) = \epsilon. \tag{2.19}$$

Again,

$$d(y_{m(k)-1}, y_{n(k)-1}) \le d(y_{m(k)-1}, y_{m(k)}) \cdot d(y_{m(k)}, y_{n(k)}) \cdot d(y_{n(k)}, y_{n(k)-1})$$

and

$$d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{m(k)-1}) \cdot d(y_{m(k)-1}, y_{n(k)-1}) \cdot d(y_{n(k)-1}, y_{n(k)}).$$

Letting $k \to \infty$ in above inequality and using (2.18) and (2.19), we have

$$\lim_{n\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = \epsilon. \tag{2.20}$$

Again

$$d(y_{n(k)-1}, y_{m(k)}) \leq d(y_{m(k)-1}, y_{n(k)-1}) \cdot d(y_{m(k)-1}, y_{m(k)})$$

and

$$d(y_{m(k)-1}, y_{n(k)-1}) \leq d(y_{m(k)-1}, y_{m(k)}) \cdot d(y_{n(k)-1}, y_{m(k)}).$$

Letting $k \to \infty$ in the above inequalities and using (2.20), we have

$$\lim_{k\to\infty} d(y_{n(k)-1}, y_{m(k)}) = \epsilon.$$

Similarly, we have

$$\lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)}) = \epsilon.$$

Consider

$$\phi(d(y_{n(k)}, y_{m(k)})) = \phi(d(fx_{n(k)}, fx_{m(k)})) \leq \frac{\phi(M(x_{n(k)}, x_{m(k)}))}{\psi(N(x_{n(k)}, x_{m(k)}))}, \tag{2.21}$$

where

$$\begin{aligned} &M(x_{n(k)}, x_{m(k)}) \\ &= \max\left\{\frac{d(fx_{n(k)}, gx_{n(k)})d(fx_{m(k)}, gx_{m(k)})}{d(gx_{n(k)}, gx_{m(k)})},\right. \\ &\qquad \left.\frac{d(gx_{m(k)}, fx_{n(k)})d(gx_{n(k)}, fx_{m(k)})}{d(gx_{n(k)}, gx_{m(k)})}, d(gx_{n(k)}, gx_{m(k)})\right\} \\ &= \max\left\{\frac{d(y_{n(k)}, y_{n(k)-1})d(y_{m(k)}, y_{m(k)-1})}{d(y_{n(k)-1}, y_{m(k)-1})},\right. \\ &\qquad \left.\frac{d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)})}{d(y_{n(k)-1}, y_{m(k)-1})}, d(y_{n(k)-1}, y_{m(k)-1})\right\} \end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
&N(x_{n(k)}, x_{m(k)}) \\
&= \max\left\{\frac{d(fx_{n(k)}, gx_{n(k)})d(fx_{m(k)}, gx_{m(k)})}{d(gx_{n(k)}, gx_{m(k)})}, d(gx_{n(k)}, gx_{m(k)})\right\} \\
&= \max\left\{\frac{d(y_{n(k)}, y_{n(k)-1})d(y_{m(k}, y_{m(k)-1})}{d(y_{n(k)-1}, y_{m(k)-1})}, d(y_{n(k)-1}, y_{m(k)-1})\right\}.
\end{aligned} \tag{2.23}$$

Letting $n \to \infty$ in (2.22) and (2.23), we have

$$\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}) = \epsilon \quad \text{and} \quad \lim_{k\to\infty} N(x_{n(k)}, x_{m(k)}) = \epsilon.$$

Letting $k \to \infty$ in (2.21), we have

$$\phi(\epsilon) \leq \frac{\phi(\epsilon)}{\lim\limits_{n\to\infty} \psi(N(x_{n(k)}, x_{m(k)}))},$$

which implies that $\lim\limits_{k\to\infty} \psi(N(x_{n(k)}, x_{m(k)})) \leq 1$, which is a contradiction. Hence $\{y_n\}$ is a multiplicative Cauchy sequence.

Since $g(X)$ is complete, there exists $z \in g(X)$ such that $y_n \to z$ as $n \to \infty$. Let $u \in X$ such that $gu = z$.

Now consider

$$\phi(d(fx_n, fu)) \leq \frac{\phi(M(x_n, u))}{\psi(N(x_n, u))}, \tag{2.24}$$

where

$$\begin{aligned}
&M(x_n, u) \\
&= \max\left\{\frac{d(fx_n, gx_n)d(fu, gu)}{d(gx_n, gu)}, \frac{d(gu, fx_n)d(gx_n, fu)}{d(gx_n, gu)}, d(gx_n, gu)\right\} \\
&= \max\left\{\frac{d(y_n, y_{n-1})d(fu, gu)}{d(y_{n-1}, gu)}, \frac{d(gu, y_n)d(y_{n-1}, fu)}{d(y_{n-1}, gu)}, d(y_{n-1}, gu)\right\}
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
N(x_n, u) &= \max\left\{\frac{d(fx_n, gu)d(fu, gu)}{d(gx_n, gu)}, d(gx_n, gu)\right\} \\
&= \max\left\{\frac{d(y_n, y_{n-1})d(fu, gu)}{d(y_{n-1}, gu)}, d(y_{n-1}, gu)\right\}.
\end{aligned} \tag{2.26}$$

Letting $n \to \infty$ in (2.25) and (2.26), we have

$$\lim_{n\to\infty} M(x_n, u) = d(gu, fu) \quad \text{and} \quad \lim_{n\to\infty} N(x_n, u) = d(gu, fu).$$

Letting $n \to \infty$ in (2.24), we have

$$\phi(d(gu, fu)) \leq \frac{\phi(d(gu, fu))}{\lim_{n\to\infty} \psi(N(x_n, u))},$$

which implies that $\lim_{n\to\infty} \psi(N(x_n, u)) \leq 1$, which is a contradiction. Thus $d(gu, fu) = 1$, that is, $gu = fu$. Hence u is a coincidence point of f and g.

Now we prove that the coincidence point of f and g is unique. Suppose u and v $(u \neq v)$ is two coincidence points of f and g.

Consider

$$\phi(d(fv, fu)) \leq \frac{\phi(M(v, u))}{\psi(N(v, u))}, \tag{2.27}$$

where

$$\begin{aligned} &M(x_n, u) \\ &= \max\left\{\frac{d(fv, gv)d(fu, gu)}{d(gv, gu)}, \frac{d(gu, fv)d(gx_n, fu)}{d(gv, gu)}, d(gv, gu)\right\} \\ &= d(v, u) \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} N(v, u) &= \max\left\{\frac{d(fv, gu)d(fu, gu)}{d(gv, gu)}, d(gv, gu)\right\} \\ &= d(v, u). \end{aligned} \tag{2.29}$$

Using (2.28) and (2.29) in (2.27), we have

$$\phi(d(v, u)) \leq \frac{\phi(d(v, u))}{\psi(d(v, u))},$$

which implies that $\psi(d(v, u)) \leq 1$, which is a contradiction. Hence $u = v$. Therefore the coincidence point of f and g is unique.

Finally suppose that f and g are weakly compatible. Since $fu = gu = p$, hence $fgu = gfu$, that is, $fp = gp$. Using uniqueness of the coincidence point of f and g, we have $p = u$. Thus $u = fu = gu$. Hence u is a unique common fixed point of f and g. This completes the proof. □

Remark 2.7. In Theorem 2.6, if $g = I$ (: the identity mapping), then we obtain Theorem 2.2.

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