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BRUSH NUMBERS OF CERTAIN MYCIELSKI GRAPHS

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Abstract: The concept of the brush number was introduced for a simple connected undirected graph G. The concept will be applied to the Mycielskian graph of a simple connected graph G to find the brush number in terms of an *optimal orientation* of G. We also apply the concept to a special family of directed graphs called, finite LinearJaco Graphs and describe a recursive formula for the brush number. Finally the concept is applied to the Mycielski Jaco graph in respect of an *optimal orientation*. Further to that, the concept of a *brush centre* of a simple connected graph is introduced. Because brushes themselves may be technology of kind, the technology in real world applications will normally be the subject of maintenance or calibration or virus vetting or alike. Therefore, finding a *brush centre* of a graph will allow for well located maintenance centres of the brushes prior to a next cycle of cleaning.

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1. Introduction

This paper consolidates the preliminary work found in [3, 4, 5]. For a general

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reference to notation and concepts of graph theory see [1]. Unless mentioned otherwise, a graph $G = G(V, E)$ on $\nu(G)$ vertices with $\epsilon(G)$ edges will be a finite undirected and connected simple graph. If a graph G has been assigned an orientation, the corresponding directed graph will be denoted, $G^{\rightarrow}$. For ease of self-containess we briefly re-introduce the concepts of *brush numbers* and *Mycielskian graphs.*

1.1. The Brush Number of a Graph G

The concept of the brush number $b_r(G)$ of a simple connected graph G was introduced by McKeil [10] and Messinger et al.[9]. The problem is: *Initially set all edges of a simple connected undirected graph G dirty.* Then allocate a finite number of brushes, $\beta_G(v) \geq 0$ to each vertex $v \in V(G)$. Sequentially any vertex which has $\beta_G(v) \geq d_G(v)$ brushes allocated may clean the vertex v and send exactly one brush along a dirty edge and in doing so allocate an additional brush to the adjacent vertex (neighbour). The reduced graph $G' = G - vu$, $\forall vu \in E(G), \beta_G(v) \geq d_G(v)$ is considered for the next iterative cleaning step. Note that a neighbour of vertex v in G say vertex u, now have $\beta_{G'}(u) = \beta_G(u) + 1$.

Clearly for any simple connected undirected graph G the first step of cleaning can begin if and only if at least one vertex v is allocated, $\beta_G(v) = d_G(v)$ brushes. The minimum number of brushes that is required to allow the first step of cleaning is, $\beta_G(u) = d_G(u) = \delta(G)$. Note that this condition does not guarantee that the graph will be cleaned. The condition merely assures at least the first step of cleaning initiates.

If a simple connected graph G is orientated to become a directed graph, brushes may only clean along an out-arc from a vertex. Cleaning may initiate from a vertex v if and only if $\beta_G(v) \geq d^+_G(v)$ and $d^-_G(v) = 0$. The sequence in which vertices sequentially initiate cleaning is called the *cleaning sequence* in respect of the orientation $\alpha_i(G)$. The minimum number of brushes to be allocated to clean a graph for a given orientation $\alpha_i(G)$ is denoted $b_r^{\alpha_i}(G)$. If an orientation $\alpha_i(G)$ renders cleaning of the graph undoable, define $b_r^{\alpha_i}(G) = \infty$. An orientation $\alpha_i(G)$ for which $b_r^{\alpha_i}(G)$ is a minimum over all possible orientations is called *optimal.* This minimum number of brushes is called the brush number of G denoted, $b_r(G)$. To ensure well-defineness, $b_r(\mathfrak{N}_{o,n}) = n$, $\mathfrak{N}_{0,n}$ the null graph (edgeless) of order n.

Now, since the graph G having $\epsilon(G)$ edges can have $2^{\epsilon(G)}$ orientations, the *optimal orientation* is not necessary unique. Define the set $\mathbb{A} = \{\alpha_i(G) : \alpha_i \text{ an orientation of } G\}$.

Lemma 1.1. *For a simple connected directed graph G, we have:*

$$b_r(G) = \min\{ \sum_{v \in V(G)} \max\{0, d^+(v) - d^-(v)\} : \text{over all } \alpha_i(G) \in \mathbb{A}\}$$
$$= \min\{b_r^{\alpha_i}(G) : \forall \alpha_i(G)\}\}.$$

Proof. See [11]. ☐

Although we mainly deal with simple connected graphs it is easy to see that for set of simple connected graphs $\{G_1, G_2, G_3, ..., G_n\}$ we have, $b_r(\cup_{\forall i} G_i) = \sum\limits_{i=1}^{n} b_r(G_i)$.

1.2. Mycielskian Graph $\mu(G)$ of a Graph, G

Mycielski [10] introduced an interesting graph transformation in 1955. The transformation can be described as follows:

(i) Consider any simple connected graph G on $n \geq 2$ vertices labeled $v_1, v_2, v_3, ..., v_n$ and edge set $E(G)$.

(ii) Consider the extended vertex set $V(G) \cup \{x_1, x_2, x_3, ..., x_n\}$ and add the edges $\{v_i x_j, v_j x_i |$ if and only if $v_i v_j \in E(G)\}$.

(iii) Add one more vertex w together with the edges $\{w x_i : \forall i\}$.

The transformed graph *(Mycielskian graph of G or Mycielski G)* denoted $\mu(G)$, is the simple connected graph with $V(\mu(G)) = V(G) \cup \{x_1, x_2, x_3, ..., x_n\} \cup \{w\}$ and $E(\mu(G)) = E(G) \cup \{v_i x_j, v_j x_i |$ iff $v_i v_j \in E(G)\} \cup \{w x_i : \forall i\}$.

2. Brush Numbers of Mycielskian Graphs

Note that if $\beta_{G'}(v) > d_{G'}(v)$ at a particular cleaning step then, exactly $\beta_{G'}(v) - d_{G'}(v)$ brushes are left redundant or can clean along new edges linked to vertex v. It is known that for $b_r(G)$ an optimal orientation exists and brushes may only clean along out-arcs of a vertex. Construct the following directed Mycielskian graph of G, denoted $\mu^{\rightarrow}(G)$.

(i) Consider any simple connected graph G on $n \geq 2$ vertices labeled $v_1, v_2, v_3, ..., v_n$ and edge set $E(G)$.

(ii) Orientate G corresponding to an optimal orientation associated with $b_r(G)$, denoted $G^{\rightarrow}_{b_r(G)}$.

(iii) Consider the extended vertex set $V(G) \cup \{x_1, x_2, x_3, ..., x_n\}$ and add the arcs $\{(v_i, x_j), (v_j, x_i) :$ if and only if $v_iv_j \in E(G)\}$.

(iv) Add one more vertex w together with the arcs $\{(x_i, w) : \forall i\}$.

Knowing that after adding an edge e (*or arc*) to a graph G, $b_r(G+e) \geq b_r(G)$ enables us to determine the brush number of the directed Mycielskian graph, $\mu^{\rightarrow}(G)$.

Theorem 2.1. *(Tshegofatso's theorem)*[1] *For a simple connected graph G of order $n \geq 2$, the brush number of the Mycielskian graph of G is given by:*
$b_r(\mu(G)) = b_r(\mu^{\rightarrow}(G)) = 2\sum\limits_{i=1}^{n} d^+_{G^{\rightarrow}_{b_r(G)}}(v_i)$.

Proof. Allocating the $b_r(G)$ brushes to the corresponding vertices of G implies that the same allocations to $G^{\rightarrow}_{b_r(G)}$ will ensure cleaning $G^{\rightarrow}_{b_r(G)}$ with minimum brushes. Consider the directed Mycielski G, $\mu^{\rightarrow}(G)$ and any vertex $v \in V(G)$. Note that $d_{G^{\rightarrow}_{b_r(G)}}(v) = d^+_{G^{\rightarrow}_{b_r(G)}}(v) + d^-_{G^{\rightarrow}_{b_r(G)}}(v)$.

Case 1: Assume $d^-_{G^{\rightarrow}_{b_r(G)}}(v) = d^+_{G^{\rightarrow}_{b_r(G)}}(v)$. Clearly *zero* brushes are initially allocated to v and at some iterative cleaning step exactly $d^-_{G^{\rightarrow}_{b_r(G)}}(v)$ brushes reaches v. These brushes will exit from v along the $d^+_{G^{\rightarrow}_{b_r(G)}}(v)$ arcs if and only if a minimum of $d_{G^{\rightarrow}_{b_r(G)}}(v) = d^+_{G^{\rightarrow}_{b_r(G)}}(v) + d^-_{G^{\rightarrow}_{b_r(G)}}(v) = 2d^+_{G^{\rightarrow}_{b_r(G)}}(v)$ brushes are added to v to clean the $2d^+_{G^{\rightarrow}_{b_r(G)}}(v)$ arcs linking v with $2d^+_{G^{\rightarrow}_{b_r(G)}}(v)$ vertices $x_i \in \{x_1, x_2, x_3, ..., x_n\}$. It follows that for all vertices satisfying this case we have the partial minimum sum of brushes:

$$2\sum_{v \in V(G), d^-_{G^{\rightarrow}_{b_r(G)}}(v) = d^+_{G^{\rightarrow}_{b_r(G)}}(v)} d^+_{G^{\rightarrow}_{b_r(G)}}(v).$$

Case 2: Assume $d^-_{G^{\rightarrow}_{b_r(G)}}(v) < d^+_{G^{\rightarrow}_{b_r(G)}}(v)$. Clearly a minimum of $d^+_{G^{\rightarrow}_{b_r(G)}}(v) - d^-_{G^{\rightarrow}_{b_r(G)}}(v)$ brushes must be added to v to clean all out-arcs from v in $G^{\rightarrow}$. In addition a minimum of $d^-_{G^{\rightarrow}_{b_r(G)}}(v) + 2(d^+_{G^{\rightarrow}_{b_r(G)}}(v) - d^-_{G^{\rightarrow}_{b_r(G)}}(v))$ brushes must be allocated to v to clean the $d^+_{G^{\rightarrow}_{b_r(G)}}(v) - d^-_{G^{\rightarrow}_{b_r(G)}}(v)$ arcs linking v with vertices $x_i \in \{x_1, x_2, x_3, ..., x_n\}$. It follows that the minimum number of additional brushes

[1]The first author dedicates this theorem to Miss Tshegofatso Katlego Manell and he hopes this young lady will grow up with a deep fondness for mathematics.

is given by:

$$2(d^+_{G^{\rightarrow}_{b_r(G)}}(v) + d^-_{G^{\rightarrow}_{b_r(G)}}(v)) + 2(d^+_{G^{\rightarrow}_{b_r(G)}}(v) - d^-_{G^{\rightarrow}_{b_r(G)}}(v)) = 2d^+_{G^{\rightarrow}_{b_r(G)}}(v).$$

It follows that for all vertices satisfying this case we have the partial minimum sum of brushes:

$$2 \sum_{v \in V(G), d^-_{G^{\rightarrow}_{b_r(G)}}(v) < d^+_{G^{\rightarrow}_{b_r(G)}}(v)} d^+_{G^{\rightarrow}_{b_r(G)}}(v).$$

Case 3: Assume $d^-_{G^{\rightarrow}_{b_r(G)}}(v) > d^+_{G^{\rightarrow}_{b_r(G)}}(v)$. The proof follows similar to Case 2.[2]

Since all cases have been settled and all vertices are accounted for, the result:

$$b_r(\mu(G)) = b_r(\mu^{\rightarrow}(G)) = 2 \sum_{v \in V(G), d^-_{G^{\rightarrow}_{b_r(G)}}(v) = d^+_{G^{\rightarrow}_{b_r(G)}}(v)} d^+_{G^{\rightarrow}_{b_r(G)}}(v)$$

$$+2 \sum_{v \in V(G), d^-_{G^{\rightarrow}_{b_r(G)}}(v) < d^+_{G^{\rightarrow}_{b_r(G)}}(v)} d^+_{G^{\rightarrow}_{b_r(G)}}(v) + 2 \sum_{v \in V(G), d^-_{G^{\rightarrow}_{b_r(G)}}(v) > d^+_{G^{\rightarrow}_{b_r(G)}}(v)} d^+_{G^{\rightarrow}_{b_r(G)}}(v)$$

$$= 2\sum_{i=1}^{n} d^+_{G^{\rightarrow}_{b_r(G)}}(v_i),$$

follows conclusively. □

3. Brush Numbers of Linear Jaco Graphs

Despite earlier definitions in respect of the family of Jaco graphs [2], the definitions found in [6] serve as the unifying definitions. For ease of reference the most important definitions are repeated here.

Definition 3.1. [6] Let $f(x) = mx + c; x \in \mathbb{N}$, $m, c \in \mathbb{N}_0$. The family of infinite linear Jaco graphs denoted by $\{J_\infty(f(x)) : f(x) = mx + c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is defined by $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$, $A(J_\infty(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

[2]The reader can formalise the proof of Case 3.

Definition 3.2. [6] The family of finite linear Jaco graphs denoted by $\{J_n(f(x)) : f(x) = mx + c; x \in \mathbb{N}$ and $m, c \in \mathbb{N}_0\}$ is defined by $V(J_n(f(x))) = \{v_i : i \in \mathbb{N}, i \leq n\}$, $A(J_n(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

The reader is referred to [6] for the definition of the *prime Jaconian vertex* and the *Hope graph*. Recall a result from [6].

Lemma 3.1. [6] *For $m = 0$ and $c \geq 0$ two special classes of disconnected linear Jaco graphs exist. For $c = 0$ the Jaco graph $J_n(0)$ is a null graph* (edgeless graph) *on n vertices hence, $b_r(\mathfrak{N}_{0,n}) = n$. For $c > 0$, the Jaco graph $J_n(c) = \bigcup\limits_{\lfloor \frac{n}{c+1} \rfloor - copies} K^{\rightarrow}_{c+1} \bigcup K^{\rightarrow}_{n-(c+1)\cdot\lfloor \frac{n}{c+1} \rfloor}$, hence:*

$$b_r(J_n(c)) = \lfloor \frac{n}{c+1} \rfloor \cdot \lfloor \frac{(c+1)^2}{4} \rfloor + \lfloor \frac{t^2}{4} \rfloor,$$

$t = n - (c+1) \cdot \lfloor \frac{n}{c+1} \rfloor$, $n \geq 2$.

Proof. Part 1 follows from the definition and the fact that $b_r(\cup_{\forall i} G_i) = \sum\limits_{i=1}^{n} b_r(G_i)$. From [9] we recall that $b_r(K_n) = \lfloor \frac{n^2}{4} \rfloor$, $n \geq 2$ hence, Part 2. $\square$

It is important to note that Definition 3.2 read together with Definition 3.1, prescribes a well-defined orientation of a Jaco graph. So we have one defined orientation of the $2^{\epsilon(J_n(x))}$ possible orientations. For this study the underlying Jaco Graph denoted $J^*_n(x)$, is considered the initial graph.

We further the study for the case $m = 1, c = 0$.

Theorem 3.2. *For the finite Jaco Graph $J_n(x), n \in \mathbb{N}$, with prime Jaconian vertex v_i we have that:*

$$b_r(J_n(x)) = \sum_{j=1}^{i}(d^+(v_j) - d^-(v_j)) + \sum_{j=i+1}^{n} \max\{0, (n-j) - d^-(v_j)\}.$$

Proof. Consider a Jaco Graph $J_n(x), n \in \mathbb{N}$ having the prime Jaconian vertex v_i. From the definition of a Jaco Graph it follows that $d^+(v_j) - d^-(v_j) \geq 0, 1 \leq j \leq i$. Hence, $\max\{0, d^+(v_j) - d^-(v_j)\}_{1 \leq j \leq i} = d^+(v_j) - d^-(v_j)$. So from Lemma 3.1 it follows that the first term must be $\sum\limits_{j=i}^{i}(d^+(v_j) - d^-(v_j))$ for the defined orientation. Similarly, it follows from the definition of a Jaco Graph that in the *finite case*, $\ell = (n-j) - d^-(j), i+1 \leq j \leq n$ represents the shortage of brushes to initiate cleaning from vertex v_j or, the surplus of brushes at v_j. Hence, $\ell > 0$ or $\ell \leq 0$. So from Lemma 3.1 it follows that the second term must be $\sum\limits_{j=i+1}^{n} \max\{0, (n-j) - d^-(v_j)\}$ for the defined orientation. So to settled the result we must show that no other orientation improves on the minimality of $\sum\limits_{j=1}^{i}(d^+(v_j) - d^-(v_j)) + \sum\limits_{j=i+1}^{n} \max\{0, (n-j) - d^-(v_j)\}$.

Case 1: Consider the Jaco Graph, $J_1(x)$. Clearly be default, $b_r(J_1(x)) = 1$.

Case 2: Consider the Jaco Graphs, $J_n(x), 2 \leq n \leq 4$. Label the edges of the underlying graph of $J_4^*(1)$, as $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4$. Now clearly, because we are considering paths, P_2, P_3 or P_4 only, the orientations $\{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$ or $\{(v_4, v_3), (v_3, v_2), (v_2, v_1)\}$ or respectively *lesser* thereof, provide *optimal orientations*. Thus the defined orientations of Jaco graphs, $J_n(x), 2 \leq n \leq 4$ are optimal.

Case 3: Consider the Jaco Graph, $J_5(x)$. Label the edges of the underlying graph $J_5^*(x)$ as $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_3v_5, e_5 = v_4v_5$. We know that 2^5 cases need to be considered to exhaust all cases. Consider the orientations tabled below.

e_1	e_2	e_3	e_4	e_5	e_1	e_2	e_3	e_4	e_5
(v_1, v_2)	(v_2, v_3)	(v_3, v_4)	(v_3, v_5)	(v_4, v_5)	(v_2, v_1)	(v_2, v_3)	(v_3, v_4)	(v_3, v_5)	(v_4, v_5)
(v_1, v_2)	(v_2, v_3)	(v_3, v_4)	(v_3, v_5)	(v_5, v_4)	(v_2, v_1)	(v_2, v_3)	(v_3, v_4)	(v_3, v_5)	(v_5, v_4)
(v_1, v_2)	(v_2, v_3)	(v_3, v_4)	(v_5, v_3)	(v_4, v_5)	(v_2, v_1)	(v_2, v_3)	(v_3, v_4)	(v_5, v_3)	(v_4, v_5)
(v_1, v_2)	(v_2, v_3)	(v_3, v_4)	(v_5, v_3)	(v_5, v_4)	(v_2, v_1)	(v_2, v_3)	(v_3, v_4)	(v_5, v_3)	(v_5, v_4)
(v_1, v_2)	(v_2, v_3)	(v_4, v_3)	(v_3, v_5)	(v_4, v_5)	(v_2, v_1)	(v_2, v_3)	(v_4, v_3)	(v_3, v_5)	(v_4, v_5)
(v_1, v_2)	(v_2, v_3)	(v_4, v_3)	(v_3, v_5)	(v_5, v_4)	(v_2, v_1)	(v_2, v_3)	(v_4, v_3)	(v_3, v_5)	(v_5, v_4)
(v_1, v_2)	(v_2, v_3)	(v_4, v_3)	(v_5, v_3)	(v_4, v_5)	(v_2, v_1)	(v_2, v_3)	(v_4, v_3)	(v_5, v_3)	(v_4, v_5)
(v_1, v_2)	(v_2, v_3)	(v_4, v_3)	(v_5, v_3)	(v_5, v_4)	(v_2, v_1)	(v_2, v_3)	(v_4, v_3)	(v_5, v_3)	(v_5, v_4)
(v_1, v_2)	(v_3, v_2)	(v_3, v_4)	(v_3, v_5)	(v_4, v_5)	(v_2, v_1)	(v_3, v_2)	(v_3, v_4)	(v_3, v_5)	(v_4, v_5)
(v_1, v_2)	(v_3, v_2)	(v_3, v_4)	(v_3, v_5)	(v_5, v_4)	(v_2, v_1)	(v_3, v_2)	(v_3, v_4)	(v_3, v_5)	(v_5, v_4)
(v_1, v_2)	(v_3, v_2)	(v_3, v_4)	(v_5, v_3)	(v_4, v_5)	(v_2, v_1)	(v_3, v_2)	(v_3, v_4)	(v_5, v_3)	(v_4, v_5)
(v_1, v_2)	(v_3, v_2)	(v_3, v_4)	(v_5, v_3)	(v_5, v_4)	(v_2, v_1)	(v_3, v_2)	(v_3, v_4)	(v_5, v_3)	(v_5, v_4)
(v_1, v_2)	(v_3, v_2)	(v_4, v_3)	(v_3, v_5)	(v_4, v_5)	(v_2, v_1)	(v_3, v_2)	(v_4, v_3)	(v_3, v_5)	(v_4, v_5)
(v_1, v_2)	(v_3, v_2)	(v_4, v_3)	(v_3, v_5)	(v_5, v_4)	(v_2, v_1)	(v_3, v_2)	(v_4, v_3)	(v_3, v_5)	(v_5, v_4)
(v_1, v_2)	(v_3, v_2)	(v_4, v_3)	(v_5, v_3)	(v_4, v_5)	(v_2, v_1)	(v_3, v_2)	(v_4, v_3)	(v_5, v_3)	(v_4, v_5)
(v_1, v_2)	(v_3, v_2)	(v_4, v_3)	(v_5, v_3)	(v_5, v_4)	(v_2, v_1)	(v_3, v_2)	(v_4, v_3)	(v_5, v_3)	(v_5, v_4)

For all possible orientations of $J_5^*(x)$ as tabled, we have: $b_r^{\alpha_1} = 2, b_r^{\alpha_2} = 2, b_r^{\alpha_3} = \infty, b_r^{\alpha_4} = 3, b_r^{\alpha_5} = 3, b_r^{\alpha_6} = \infty, b_r^{\alpha_7} = 3, b_r^{\alpha_8} = 3, b_r^{\alpha_9} = 4, b_r^{\alpha_{10}} = 4, b_r^{\alpha_{11}} = \infty, b_r^{\alpha_{12}} = 4, b_r^{\alpha_{13}} = 4, b_r^{\alpha_{14}} = \infty, b_r^{\alpha_{15}} = 3, b_r^{\alpha_{16}} = 3, b_r^{\alpha_{17}} = 3, b_r^{\alpha_{18}} = 3, b_r^{\alpha_{19}} = \infty, b_r^{\alpha_{20}} = 4, b_r^{\alpha_{21}} = 4, b_r^{\alpha_{22}} = \infty, b_r^{\alpha_{23}} = 3, b_r^{\alpha_{24}} = 4, b_r^{\alpha_{25}} = 3, b_r^{\alpha_{26}} = 3, b_r^{\alpha_{27}} = \infty, b_r^{\alpha_{28}} = 3, b_r^{\alpha_{29}} = 3, b_r^{\alpha_{30}} = \infty, b_r^{\alpha_{31}} = 2, b_r^{\alpha_{32}} = 2$

It follows that the defined orientation of the Jaco graph $J_5(x)$, tabled as α_1 has $b_r^{\alpha_1} = 2 = \min\{b_r^{\alpha_i} : \forall \alpha_i\}$. Since the prime Jaconian vertex of $J_5(x)$ is v_3, the result:

$$\sum_{j=1}^{3}(d^+(v_j) - d^-(v_j)) + \sum_{j=4}^{5} \max\{0, (5-j) - d^-(v_j)\},$$

holds.

We settle the theorem through induction. Assume the results holds for $J_k(x)$ having prime Jaconian vertex v_i. Consider the Jaco graph $J_{k+1}(x)$. Clearly $J_{k+1}(x) = J_k(x) + (v_j, v_{k+1}),\ i+1 \leq j \leq k$. So the minimum number of additional brushes to be added to the $b_r(J_k(x))$ brushes to clean $J_{k+1}(x)$ is given by $\sum\limits_{j=i+1}^{k} \max\{0, d^+(v_j) - d^-(v_j)\}$. So the minimum number of brushes to be allocated to clean $J_{k+1}(x)$ is given by:

$$b_r(J_{k+1}(x)) = \sum_{j=1}^{i}(d^+(v_j) - d^-(v_j))_{in\ J_k(x)} + \sum_{j=i+1}^{k} \max\{0, (k-j) - d^-(v_j)\}_{in\ J_k(x)}$$
$$+ \sum_{j=i+1}^{k} \max\{0, d^+(v_j) - d^-(v_j)\}_{in\ J_{k+1}(x)}$$
$$= \sum_{j=1}^{i+1}(d^+(v_j) - d^-(v_j))_{in\ J_{k+1}(x)} + \sum_{j=i+2}^{k+1} \max\{0, ((k+1)-j) - d^-(v_j)\}_{in\ J_{k+1}(x)}.$$

Since a re-orientation of any one, or more of the arcs $(v_j, v_{k+1}),\ i+1 \leq j \leq k$ in $J_{k+1}(x)$ does not require more brushes, but could in some instances render the cleaning process undoable, the result holds in general. □

For illustration the adapted table below follows from the Fisher Algorithm [2] for $J_n(x), n \in \mathbb{N}, n \leq 15$. Note that the Fisher Algorithm determines $d^+(v_i)$ on the assumption that the Jaco Graph is always sufficiently large, so at least $J_n(x), n \geq i + d^+(v_i)$. For a smaller graph the degree of vertex v_i is given by $d(v_i)_{J_n(x)} = d^-(v_i) + (n-i)$. In [2, 6] Bettina's theorem describes an arguably, closed formula to determine $d^+(v_i)$. Since $d^-(v_i) = n - d^+(v_i)$ it is then easy to determine $d(v_i)_{J_n(x)}$ in a smaller graph $J_n(x), n < i + d^+(v_i)$.

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	v_j^*	$b_r(J_i(x))$,	$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	v_j^*	$b_r(J_i(x))$,
1	0	1	v_1	1	9	3	6	v_5	6
2	1	1	v_1	1	10	4	6	v_6	7
3	1	2	v_2	1	11	4	7	v_7	8
4	1	3	v_2	1	12	4	8	v_7	9
5	2	3	v_3	2	13	5	8	v_8	11
6	2	4	v_3	3	14	5	9	v_8	12
7	3	4	v_4	4	15	6	9	v_9	14
8	3	5	v_5	5	16	6	10	v_{10}	16

Table 1: Vertex v_j^* the prime Jaconian vertex

From Theorem 2.1 and Lemma 1.1 the brush allocations can easily be determined.

Example 1. $J_9(x)$ requires the minimum brush allocations, $\beta_{J_9(x)}(v_1) = 1, \beta_{J_9(x)}(v_2) = 0, \beta_{J_9(x)}(v_3) = 1, \beta_{J_9(x)}(v_4) = 2, \beta_{J_9(x)}(v_5) = 1, \beta_{J_9(x)}(v_6) = 1, \beta_{J_9(x)}(v_7) = 0, \beta_{J_9(x)}(v_8) = 0, \beta_{J_9(x)}(v_9) = 0$.

4. Brush Numbers of Mycielski Jaco Graphs, $\mu^{\rightarrow}(J_n(x)), n \in \mathbb{N}$

In general we have that if $\beta_{G'}(v)$ at a particular cleaning step has $\beta_{G'}(v) > d_{G'}(v)$, exactly $\beta_{G'}(v) - d_{G'}(v)$ brushes are left redundant and can clean along new edges linked to vertex v if such are added through transformation of the graph G. The latter observation allows for an adaption of Theorem 2.1 to obtain the *brush number* of $\mu^{\rightarrow}(J_n(x))$, $n \geq 3$. Note that $\mu(J_1^*(x)) \simeq K_1 \cup P_2$, hence a disconnected graph. Easy to see that $\mu(J_2^*(x)) \simeq C_5$ hence $b_r(\mu^{\rightarrow}(J_2(x)) = 2$.

Theorem 4.1. *For the Jaco graph $J_n(x), n \in \mathbb{N}, n \geq 2$ the brush number of the Mycielski Jaco graph is given by:*

$$b_r(\mu^{\rightarrow}(J_n(x)) = 2 \sum_{i=1}^{n} d^+_{J_n(x)}(v_i).$$

Proof. Consider the Jaco graph, $J_3(x)$. From Theorem 3.2, $b_r(J_3(x)) = 1$ with brush allocations $\beta_{J_3(x)}(v_1) = 1, \beta_{J_3(x)}(v_2) = 0, \beta_{J_3(x)}(v_3) = 0$. In the graph $\mu^{\rightarrow}(J_3(x))$, add the set of vertices $\{x_1, x_2, x_3\} \cup \{w\}$ and add the arcs $\{(v_1, x_2), (v_2, x_1), (v_2, x_3), (v_3, x_2)\} \cup \{(x_1, w), (x_2, w), (x_3, w)\}$.

Clearly v_1 has degree, $d^+_{\mu^{\rightarrow}(J_3(x))}(v_1) = 2$. So besides the normal cleaning sequence within $J_3(x)$, vertex v_1 must either dispatch a second brush along arc (v_1, x_2) or await a brush dispatched from vertex x_2. In the latter case a brush will be left redundant at vertex v_1. So without loss of generality and to ensure optimality, allocate a second brush to v_1. At this stage we have the *first brush number term* of $\mu^{\rightarrow}(J_3(x))$, $2d^+_{J_3(x)}(v_1) = 2$.

On initiating the cleaning process one brush will be dispatched to v_2 along the arc (v_1, v_2). However, in $\mu^{\rightarrow}(J_3(1))$ we have the additional arcs $(v_2, x_1), (v_2, x_3)$, so two additional brushes must be allocated to vertex v_2. So, the *second brush number term* of $\mu^{\rightarrow}(J_3(x))$, $2d^+_{J_3(x)}(v_2) = 2$.

In the third step of cleaning the brush, initially dispatched from v_1 to v_2 can be dispatched to vertex v_3. The third step of cleaning may proceed with no further allocation to v_3. Hence v_3 requires *zero* additional brushes. Now, the *third and final brush number term* of $\mu^{\rightarrow}(J_3(x))$, $2d^+_{J_3(x)}(v_3) = 0$. Hence we have the result:

$b_r(\mu^{\rightarrow}(J_3(x))) = 2\sum\limits_{i=2}^{3} d^+_{J_3(x)}(v_i)$.

Now the result is settled through induction. Assume the the result holds for $J_k(x)$, with prime Joconian vertex v_j. Assume that:

$b_r(\mu^{\rightarrow}(J_k(x))) = 2\sum\limits_{i=2}^{k} d^+_{J_k(x)}(v_i)$, holds.

Now consider the Jaco graph $J_{k+1}(x)$ to begin with. This extension adds the vertex v_{k+1} and the set of arcs,

$$\{(v_{i+1}, v_{k+1}), (v_{i+2}, v_{k+1}), (v_{i+3}, v_{k+1}), ..., (v_k, v_{k+1})\}$$

to $J_k(x)$ to obtain $J_{k+1}(x)$.

Each vertex $v_j, i+1 \leq j \leq k$ receives two additional arcs namely (v_j, v_{k+1}) and (v_j, x_{k+1}) in the transformed Mycielski Jaco graph. The minimum additional brushes per such vertex is thus two. Now the respective brush number terms of $\mu^{\rightarrow}(J_{k+1}(x))$ are $(d^+_{J_k(x)}(v_j) + 2) = 2d^+_{J_{k+1}(x)}(v_j)$. This implies we have the partial brush number:

$2\sum\limits_{i=1}^{k} d^+_{J_{k+1}(x)}(v_i)$.

With regards to the vertex v_{k+1} we have, $d_{J_{k+1}(x)}(v_{k+1}) = d^-(v_{k+1})$ stemming from the $k-j$ vertices, $v_{j+1}, v_{j+2}, v_{j+3}, ..., v_k$. Hence, $k-i$ brushes will be allocated to vertex v_{k+1} through the iterative cleaning process. In the Mycielski Jaco graph the additional arcs

$$(v_{k+1}, x_k), (v_{k+1}, x_{k-1}), (v_{k+1}, x_{k-2}), ..., (v_{k+1}, x_{j+1}),$$

(*exactly $k-j$ arcs*) have been added to vertex v_{k+1} so no additional brushes are needed. It means that the *final brush number term* is, $2d^+_{J_{k+1}(1)}(v_{k+1}) = 2.0 = 0$. Since all the brush number terms of $\mu^{\rightarrow}(J_{k+1}(x))$ have now been determined at the absolute minimum, the result, $b_r(\mu^{\rightarrow}(J_{k+1}(x)) = 2\sum\limits_{i=1}^{k+1} d^+(v_i)$ holds. Through induction the result is settled for $b_r(\mu^{\rightarrow}(J_n(x))$, $n \in \mathbb{N}$. □

5. Brush Centre of a Graph

From Theorem 2.1 and Lemma 1.1 the brush allocations can easily be determined for Jaco graphs. See Example 1. So far cleaning was restricted to a brush transversing a dirty edge only once. If the latter restriction is relaxed to, after the first complete cleaning sequence a brush may transverse an edge for a second time for a complete reversed cleaning sequence, the initial allocation of brushes or a deviation thereof can be obtained.This observation leads to the concept of a *brush centre*. The question is: *What is the minimium set of vertices,* $\mathbb{B}_r(G) \subseteq V(G)$ (*primary condition*) *to allocate the* $b_r(G)$ *brushes to, to ensure cleaning of graph* G *and on return* (*second cleaning*) *the brushes are clustered as centrally as possible for maintenance* (*secondary condition is the min(max(distance between vertices of the brush centre)))*? Finding a *brush centre* of a graph will allow for well located maintenance centres of the brushes prior to a next cycle of cleaning. Because brushes themselves may be technology of kind, the technology in real worl application will normally be the subject of maitenance or calibration or virus vetting or alike.

It is easy to see that for the path P_n the allocation of one brush to either $\{v_1\}$ or $\{v_n\}$ is a minimum set and clustered absolutely centrally so both represent a *brush centre*. So P_n has two possible *brush centres*. Similarly, the allocation of two brushes to any set $\mathbb{B}_r(C_n) = \{v_i\}, v_i \in V(C_n)$ of the cycle cycle C_n represents a *brush centre*. So C_n has n possible *brush centres*. Hence the *brush centre* of a graph G is not necessary unique. However, for the star $K_{1,n}$ the *brush centre* is indeed unique, namely, $\mathbb{B}_r(K_{1,n}) = \{v_1\}_{(v_1\ central)}$, with $\beta_{K_{1,n}}(v_1)_{(v_1\ central)} = n$.

5.1. Brush Centre of the Mycielski Jaco Graph

Let us immediately jump paths and consider $J_5(x)$. In the defined Jaco graph $J_5(x)$ the brush number is $b_r(J_5(x)) = 2$, with the brush allocation $\beta_{J_5(x)}(v_1) = 1, \beta_{J_5(x)}(v_2) = 0, \beta_{J_5(x)}(v_3) = 1, \beta_{J_5(x)}(v_4) = 0, \beta_{J_5(x)}(v_5) = 0$. Note that after the first cleaning sequence both brushes are allocated to the vertex v_5. The latter allocation of brushes with an appropriate re-orientation of $J_5^*(x)$ also clean the Jaco graph. On a second cleaning sequence the brushes can *park* at v_5 for *maintenance*. Clearly the set $\{v_5\}$ with $\beta_{J_5^*(x)}(v_5) = 2$ is a (*the*) brush centre.

Theorem 5.1. *Consider the initial minimal brush allocation of* $b_r(J_n(x))$ *brushes to the finite Jaco graph,* $J_n(x), n \in \mathbb{N}$. *The location of the brushes at the end of the cleaning sequence represents a brush centre of* $J_n^*(x), n \in \mathbb{N}$.

Proof. For the paths $J_1(x), J_2(x), J_3(x), J_4(x)$ the result is obvious. As observed in the introduction, the set $\mathbb{B}_r(J^*_5(x)) = \{v_5\}$ with $\beta_{J^*_5(x)}(v_5) = 2$ is a (*the*) brush centre of $J^*_5(x)$.

We prove the result through induction. Assume the result holds for $J_k(x)$ which has the prime Jaconian vertex v_i. The assumption implies that after the first cleaning sequence the brushes are clustered amongst vertices in the vertex set $\mathbb{B}_r \subseteq \{v_{i+1},, v_{i+2}, v_{i+3}, .., v_k\}$ in compliance with both the *primary condition* and the *secondary condition* namely, the min(max(distance between vertices of the brush centre)). Note that $min(max(d_{v,u\in\mathbb{B}_r}(v,u))) = 1$ because $\mathbb{B}_r(J^*_k(x)) \subseteq \mathbb{H}(J^*_k(x))$.

Now consider the Jaco graph $J_{k+1}(x)$ to begin with. This extension adds the vertex v_{k+1} and the set of arcs,

$$\{(v_{j+1}, v_{k+1}), (v_{j+2}, v_{k+1}), (v_{j+3}, v_{k+1}), ..., (v_k, v_{k+1})\}$$

to $J_k(x)$ to obtain $J_{k+1}(x)$.

Each vertex $v_j, i+1 \leq j \leq k$ receives one additional arc namely (v_j, v_{k+1}). In addition the prime Jaconian vertex may, or may not change to v_{i+1}. Exactly $k-i$ new arcs were added in the extension from $J_k(x))$ to $J_{k+1}(x)$. Since $b_r(J_{k+1}(x)) \geq b_r(J_k(x))$ additional brushes might be needed at some of the vertices $v_{i+1}, v_{i+2}, ..., v_{k_1}$ if v_i remains the prime Jaconian vertex of $J_{k+1}(x)$. Else, additional brushes might be needed at some of the vertices $v_{i+2}, ..., v_{k_1}$. Note that the vertex v_k will not require additional brushes. Since the minimum additional brushes to be allocated is always possible (*primary condition*) and $\mathbb{B}_r(J^*_{k+1}(x)) \subseteq \mathbb{H}(J^*_{k+1}(x))$ the min$(max(d_{v,u\in\mathbb{B}_r(J^*_{k+1}(x))}(v,u))) = 1$ (*secondary condition*), the result follows for $J_{k+1}(x)$. Hence the result is settled for all Jaco graphs, $J_n(x), n \in \mathbb{N}$. □

6. Conclusion

Besides the study of brush numbers for a Mycielski graph and its application to Jaco graphs, the concept of a brush centre is introduced. If brushes are categorised into types, a study on brush centers with constraints such as, cleaning may initiate at vertex v if and only if at least one brush of types $t_i, t_j \ldots, t_k$ are located at v or, types $t_i, t_j \ldots, t_k$ may not be located simultaneously at v is worthy of research. Technology inclusion or exclusion has application in many fields of science in general. In chemical systems the principal of inclusion or exclusion of chemical elements is a fundamental principle for the enhancement or neutralisation of particular chemical reactions. If the transmission corridor and direction of transmission from one chemical chamber to another is modeled an arc, perhaps brush centers could find application within the field of chemical engineering.

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