International Journal of Pure and Applied Mathematics
Volume 106 No. 2 2016, 389-400
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v106i2.4

ijpam.eu

THE UPPER VERTEX MONOPHONIC NUMBER OF A GRAPH

P. Titus[1 §], K. Iyappan[2]

[1]Department of Mathematics
University College of Engineering Nagercoil
Anna University
Tirunelveli Region, Nagercoil, 629 004, INDIA

[2]Department of Mathematics
V.V. College of Engineering
Tirunelveli, 627 657, INDIA

Abstract: For any vertex x in a connected graph G of order $p \geq 2$, a set $S \subseteq V(G)$ is an *x-monophonic set* of G if each vertex $v \in V(G)$ lies on an $x-y$ monophonic path for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$. An x-monophonic set S is called a *minimal x-monophonic set* if no proper subset of S is an x-monophonic set. The *upper x-monophonic number*, denoted by $m_x^+(G)$, is defined as the maximum cardinality of a minimal x-monophonic set of G. We determine bounds for it and find the same for some special classes of graphs. For any two positive integers a and b with $1 \leq a \leq b$, there exists a connected graph G with $m_x(G) = a$ and $m_x^+(G) = b$ for some vertex x in G. Also, it is shown that for any three positive integers a, b and n with $a \geq 2$ and $a \leq n \leq b$, there exists a connected graph G with $m_x(G) = a$, $m_x^+(G) = b$ and a minimal x-monophonic set of cardinality n.

AMS Subject Classification: 05C12
Key Words: monophonic path, vertex monophonic set, vertex monophonic number, minimal vertex monophonic set, upper vertex monophonic number

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q

Received: October 9, 2015
Published: February 9, 2016

§Correspondence author

respectively. For basic graph theoretic terminology we refer to Harary [3]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *neibourhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neibourhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete. The *closed interval* $I[x,y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup\limits_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1, 4] and further studied in [2].

The concept of vertex geodomination number was introduced in [5] and further studied in [6]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x - y$ geodesic for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a $g_x - set$.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. A vertex v of G such that $d_m(u, v) = e_m(u)$ is called a *monophonic eccentric vertex* of u. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $diam_m(G)$ of G is $diam_m(G) = \max\{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [7].

The concept of vertex monophonic number was introduced by Santhakumaran and Titus in [8]. For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S \subseteq V(G)$ is an *x-monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$. An x-monophonic set of cardinality $m_x(G)$ is called a $m_x - set$ of G.

These concepts have interesting applications in Channel Assignment Problem in radio technologies. Also, there are useful applications of these concepts to security based communication network design. In the case of designing the channel for a communication network, although all the vertices are covered by the network when considering detour monophonic sets, some of the edges may be left out. This drawback is rectified in the case of detour monophonic sets so that considering detour monophonic sets is more advantageous to real life application of communication networks.

The following theorems will be used in the sequel.

Theorem 1. [8] *Let x be any vertex of a connected graph G. Every simplicial vertex of G other than the vertex x (whether x is simplicial or not) belongs to every m_x- set.*

Theorem 2. [8] *A graph G is complete if and only if $m_x(G) = p - 1$ for every vertex x in G.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Minimal Vertex Monophonic Sets

Definition 3. *Let x be any vertex of a connected graph G. An x-monophonic set S_x is called a minimal x-monophonic set if no proper subset of S_x is an x-monophonic set. The upper x-monophonic number, denoted by $m_x^+(G)$, is defined as the maximum cardinality of a minimal x-monophonic set of G.*

It is clear from the definition that for any vertex x in G, x does not belong to any minimal x-monophonic set of G.

Example 4. For the graph G given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimal vertex monophonic sets and the upper vertex monophonic numbers are given in Table 2.1.

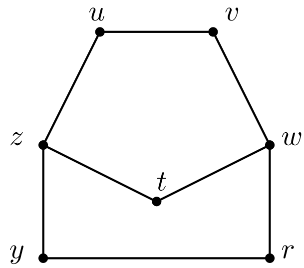


Figure 2.1: G

Remark 5. *For any vertex x in a connected graph G, every minimum x-monophonic set is a minimal x-monophonic set, but the converse is not true. For the graph G given in Figure 2.1, $\{t, y\}$ is a minimal u-monophonic set but it is not a minimum u-monophonic set of G.*

Vertex x	Minimum x-monophonic sets	$m_x(G)$	Minimal x-monophonic sets	$m_x^+(G)$
u	$\{w\}, \{r\}$	1	$\{w\}, \{r\}, \{t, y\}$	2
v	$\{z\}, \{y\}$	1	$\{z\}, \{y\}, \{t, r\}$	2
w	$\{z\}, \{y\}, \{u\}$	1	$\{z\}, \{y\}, \{u\}$	1
r	$\{z\}, \{u\}$	1	$\{z\}, \{u\}, \{v, t\}$	2
y	$\{v\}, \{w\}$	1	$\{v\}, \{w\}, \{u, t\}$	2
z	$\{v\}, \{w\}, \{r\}$	1	$\{v\}, \{w\}, \{r\}$	1
t	$\{r, u\}, \{r, v\}, \{y, u\}, \{y, v\}$	2	$\{r, u\}, \{r, v\}, \{y, u\}, \{y, v\}$	2

Table 2.1

Theorem 6. *Let x be any vertex of a connected graph G.*
(i) Every simplicial vertex of G other than x (whether x is simplicial vertex or not) belongs to every minimal x-monophonic set.
(ii) No cut-vertex of G belongs to any minimal x-monophonic set.

Proof. (i) Let x be any vertex of G. Since x does not belong to any minimal x-monophonic set, let $y \neq x$ be a simplicial vertex of G. Then y is not an internal vertex of any monophonic path and so y belongs to every minimal x-monophonic set of G.

(ii) Let $y \neq x$ be a cut-vertex of G. Let U and W be two components of $G - \{y\}$. For any vertex x in G, let S_x be a minimal x-monophonic set of G. Suppose that $x \in U$. Now, suppose that $S_x \bigcap W = \emptyset$. Let $w_1 \in W$. Then $w_1 \notin S_x$. Since S_x is an x-monophonic set, there exists an element z in S_x such that w_1 lies in some $x - z$ monophonic path $P : x = z_0, z_1, ..., w_1, ..., z_k = z$ in G. Since $S_x \bigcap W = \emptyset$ and y is a cut-vertex of G, it follows that the $x - w_1$ subpath of P and the $w_1 - z$ subpath of P both contain y so that P is not a path in G. Hence $S_x \bigcap W \neq \emptyset$. Let $w_2 \in S_x \bigcap W$. Then $w_2 \neq y$ so that y is an internal vertex of an $x - w_2$ monophonic path. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an $x-y$ monophonic path also lies on an $x-w_2$ monophonic path. Hence it follows that S is an x-monophonic set of G, which is a contradiction to S_x a minimal x-monophonic set of G. Thus y does not belong to any minimal x-monophonic set of G. Similarly, if $x \in W$, then y does not belong to any minimal x-monophonic set of G. □

Since every end-vertex is a simplicial vertex, the following theorem is an easy consequence of the definition of the upper vertex monophonic number of a graph and Theorem 6.

Theorem 7. *(i) For any non-trivial tree T with k end-vertices, $m_x^+(T) = k-1$ or k according as x is an end-vertex or not.*
(ii) For any vertex x in the complete graph K_p of order $p \geq 2$, $m_x^+(K_p) = p-1$.

Theorem 8. *(i) For any vertex x in the cycle C_p of order $p \geq 4$, $m_x^+(C_p) = 1$.*
(ii) For the wheel $W_p = K_1 + C_{p-1}(p \geq 5)$, $m_x^+(W_p) = p-1$ or 1 according as x is K_1 or x is in C_{p-1}.

Proof. (i) Let C_p be a cycle. For any vertex x in C_p, let y be a non-adjacent vertex of x. Clearly every vertex of C_p lies on an $x-y$ monophonic path and so $\{y\}$ is a minimal x-monophonic set of C_p. Since no adjacent vertex of x lies on a minimal x-monophonic set of C_p, we have $m_x^+(C_p) = 1$.

(ii) Let x be the vertex of K_1. Clearly $S = V(C_{p-1})$ is the unique minimal x-monophonic set of W_p and so $m_x^+(W_p) = p-1$.

Let $C_{p-1} : u_1, u_2, ..., u_{p-1}, u_1$ be the cycle in W_p. Let x be any vertex in C_{p-1}. Let y be a non-adjacent vertex of x in W_p. Then any vertex v in W_p lies on an $x-y$ monophonic path and so $\{y\}$ is a minimal x-monophonic set of W_p. Since no adjacent vertex of x lies on a minimal x-monophonic set of W_p, we have $m_x^+(W_p) = 1$. □

Theorem 9. *Let $K_{m,n}(m, n \geq 2)$ be a complete bipartite graph with bipartition (V_1, V_2). Then* $m_x^+(K_{m,n}) = \begin{cases} m-1 & \text{if } x \in V_1 \\ n-1 & \text{if } x \in V_2 \end{cases}$

Proof. Let $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{w_1, w_2, ..., w_n\}$ be a partition of G. Let $x \in V_1$, say $x = u_1$. Since the vertex $u_i(2 \leq i \leq m)$ does not lie on any monophonic path starting from x and every vertex of V_2 lies on an $x-u_2$ monophonic path, we have $S_x = V_1 - \{x\}$ is the unique minimal x-monophonic set of G. Hence $m_x^+(G) = |S_x| = m-1$. Let $x \in V_2$. Then by a similar argument, we get $m_x^+(G) = n-1$. □

Theorem 10. *For $n \geq 2$, $m_x^+(Q_n) = 1$ for any vertex x in Q_n.*

Proof. Let x be any vertex in Q_n. Let y be a non-adjacent vertex of x in Q_n. It is easily seen that every vertex of Q_n lies on an $x - y$ monophonic path in Q_n. Hence $\{y\}$ is a minimal x-monophonic set of G and so $m_x^+(G) = 1$. □

Theorem 11. *Let G be a connected block graph with number of cut-vertices k. Then $m_x^+(G) = p - k$ or $p - k - 1$ for any vertex x in G.*

Proof. Let G be a connected block graph. Then every vertex of G is either a cut-vertex or a simplicial vertex and hence by Theorem 6, $m_x^+(G) = p - k$ or $p - k - 1$ for any vertex x in G. □

Theorem 12. *For every non-trivial tree T, $m_x^+(T) = p - d_m$ or $p - d_m + 1$ for any vertex x in T if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, ..., v_{d_m} = v$ be a monophonic diametral path. Let k be the number of end-vertices of T and let l be the number of internal vertices of T other than $v_1, v_2, ..., v_{d_m - 1}$. Then $d_m - 1 + l + k = p$. By Theorem 7(i), $m_x^+(T) = k$ or $k - 1$ for any vertex x in T and so $m_x^+(T) = p - d_m - l + 1$ or $p - d_m - l$ for any vertex x in T. Hence $m_x^+(T) = p - d_m + 1$ or $p - d_m$ for any vertex x in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P, if and only if T is a caterpillar. □

3. Bounds and Realization Results for $m_x^+(G)$

Theorem 13. *For any vertex x in G, $1 \leq m_x(G) \leq m_x^+(G) \leq p - 1$.*

Proof. It is clear from the definition of x-monophonic set that $m_x(G) \geq 1$. Since every minimum x-monophonic set is a minimal x-monophonic set, $m_x(G) \leq m_x^+(G)$. Also, since the vertex x does not belong to any minimal x-monophonic set, it follows that $m_x^+(G) \leq p - 1$. □

Remark 14. *The bounds for $m_x(G)$ and $m_x^+(G)$ in Theorem 13 are sharp. For the cycle $C_p (p \geq 4), m_x^+(C_p) = 1$ for any vertex x in C_p. For the graph G given in Figure 2.1, $m_t(G) = m_t^+(G) = 2$. Also, for the complete graph K_p, $m_x^+(K_p) = p - 1$ for every vertex x in K_p. All the inequalities in Theorem 13 can be strict. For the graph G given in Figure 2.1, $m_v(G) = 1, m_v^+(G) = 2$ and $p = 7$. Thus $m_v(G) < m_v^+(G) < p - 1$.*

Theorem 15. *Let x be any vertex in a connected graph G of order $p \geq 3$. If $m_x(G) = 1$, then $m_x^+(G) \leq p - 2$.*

Proof. Let $S_x = \{y\}$ be a minimum x-monophonic set of G and let T_x be a minimal x-monophonic set of G with maximum cardinality. Then $y \neq x$. If $y \in T_x$, then $T_x = S_x$ and so $m_x^+(G) = 1 \leq p - 2$. If $y \notin T_x$, then $m_x^+(G) = |T_x| \leq p - 2$. □

Theorem 16. *Let x be any vertex in a connected graph G. Then $m_x(G) = p - 1$ if and only if $m_x^+(G) = p - 1$.*

Proof. Let $m_x(G) = p - 1$. Since $m_x(G) \leq m_x^+(G) \leq p - 1$, we have $m_x^+(G) = p - 1$. Conversely, let $m_x^+(G) = p - 1$. Then $T = V(G) - \{x\}$ is the minimal x-monophonic set of G. Now, claim that $m_x(G) = p - 1$. Otherwise, G has a minimum x-monophonic set T_1 with $|T_1| \leq p - 2$. Since x does not belong to any minimum x-monophonic set, T_1 is a proper subset of T and so T is not a minimal x-monophonic set of G, which is a contradiction. □

Theorem 17. *For any two integers a and b with $1 \leq a \leq b$, there is a connected graph G with $m_x(G) = a$ and $m_x^+(G) = b$ for some vertex x in G.*

Proof. We prove this theorem by considering two cases.

Case 1. $1 \leq a = b$. Let $G = K_{a+1}$. Then by Theorems 2 and 7(ii), $m_x(G) = m_x^+(G) = a$.

Case 2. $1 \leq a < b$. Let $P_4 : w, x, y, z$ be a path of order 4. Now, let G be a graph obtained from P_4 by adding b new vertices $\{u_1, u_2, ..., u_{a-1}, v_1, v_2, ..., v_{b-a+1}\}$ and joining each $u_i (1 \leq i \leq a-1)$ with x; and joining each $v_i (1 \leq i \leq b-a+1)$ with w and z. The graph G is shown in Figure 3.1. Let $S = \{u_1, u_2, ..., u_{a-1}\}$ be the set of all simplicial vertices of G.

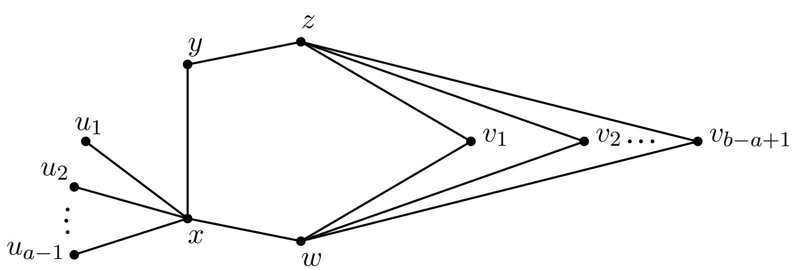


Figure 3.1 : G

First, we show that $m_x(G) = a$ for the vertex x in G. By Theorem 1, every minimum x-monophonic set of G contains S. Since S is not an x-monophonic set of G, $S_1 = S \bigcup \{z\}$ is a minimum x-monophonic set of G so that $m_x(G) = |S_1| = a$.

Next, we show that $m_x^+(G) = b$. Let $M = \{u_1, u_2, ..., u_{a-1}, v_1, v_2, ..., v_{b-a+1}\}$. It is clear that M is an x-monophonic set of G. We claim that M is a minimal x-monophonic set of G. Assume that M is not a minimal x-monophonic set of G. Then there exists a proper subset T of M such that T is an x-monophonic set of G. Let $s \in M$ and $s \notin T$. By Theorem 6(i), $s \neq u_i$ for all $i = 1, 2, ..., a-1$. Then $s = v_i (1 \leq i \leq b-a+1)$. Clearly v_i does not lie on any $x - v_j$ monophonic path, where $j \neq i$, it follows that T is not an x-monophonic set of G, which is a contradiction. Thus M is a minimal x-monophonic set of G and so $m_x^+(G) \geq |M| = b$. Also, it is clear that every minimal x-monophonic set of G contains at most b elements and hence $m_x^+(G) \leq b$. Thus $m_x^+(G) = b$. □

Remark 18. *The graph G given in Figure 3.1 contains exactly two minimal x-monophonic sets, namely $S \bigcup \{z\}$ and M. This example shows that there is no "'Intermediate Value Theorem" for minimal x-monophonic sets, that is, if n is an integer such that $m_x(G) < n < m_x^+(G)$, then there need not exist a minimal x-monophonic set of cardinality n in G.*

Theorem 19. *For any three positive integers a, b and n with $a \geq 2$ and $a \leq n \leq b$, there exists a connected graph G with $m_x(G) = a$, $m_x^+(G) = b$ and a minimal x-monophonic set of cardinality n.*

Proof. Let $P : z_1, z_2, z_3, z_4$ and $Q : v_1, v_2, v_3, v_4$ be two paths. Let H be the graph obtained from P and Q by identifying the vertices z_2 in P and v_2 in Q. Let G be the graph obtained from H by adding b new vertices $u_1, u_2, ..., u_{a-2}, y_1, y_2, ..., y_{b-n+1}, x_1, x_2, ..., x_{n-a+1}$ and joining each $u_i (1 \leq i \leq a-2)$ with z_2; joining each $y_i (1 \leq i \leq b-n+1)$ with z_1 and z_4; and joining each $x_i (1 \leq i \leq n-a+1)$ with v_1 and v_4 in H. The graph G is shown in Figure 3.2.

Let $S = \{u_1, u_2, ..., u_{a-2}\}$ be the set of all simplicial vertices of G and let $x = z_2$. Then by Theorem 1, every x-monophonic set of G contains S and also for any vertex $y \in V(G) - S$, $S \bigcup \{y\}$ is not an x-monophonic set of G. It is clear that $S_1 = S \bigcup \{z_4, v_4\}$ is a minimum x-monophonic set of G and so $m_x(G) = |S_1| = a$.

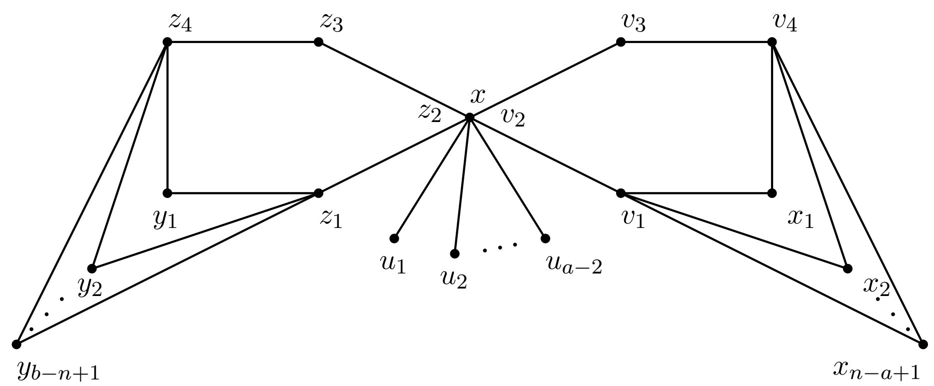


Figure 3.2 : G

Next we show that $m_x^+(G) = b$. Let $M = S\bigcup\{y_1, y_2, ..., y_{b-n+1}, x_1, x_2, ..., x_{n-a+1}\}$. It is clear that M is an x-monophonic set of G. We claim that M is a minimal x-monophonic set of G. Assume that M is not a minimal x-monophonic set of G. Then there exists a proper subset M_1 of M such that M_1 is an x-monophonic set of G. Let $w \in M$ and $w \notin M_1$. By Theorem 6(i), either $w = y_i (1 \leq i \leq b-n+1)$ or $w = x_j (1 \leq j \leq n-a+1)$. If $w = y_i (1 \leq i \leq b-n+1)$, then w does not lie on any $x - z$ monophonic path for any $z \in M_1$, which is a contradiction. Similarly, if $w = x_j (1 \leq j \leq n-a+1)$, then w does not lie on any $x - z$ monophonic path for any $z \in M_1$, which is a contradiction. Thus M is a minimal x-monophonic set of G and so $m_x^+(G) \geq |M| = b$. Also, it is clear that every minimal x-monophonic set of G contains at most b elements and hence $m_x^+(G) \leq b$. Hence $m_x^+(G) = b$.

Finally we show that there is a minimal x-monophonic set of cardinality n. Let $T = S\bigcup\{z_4, x_1, x_2, ..., x_{n-a+1}\}$. It is clear that T is an x-monophonic set of G. We claim that T is a minimal x-monophonic set of G. Assume that T is not a minimal x-monophonic set of G. Then there is a proper subset T_1 of T such that T_1 is an x-monophonic set of G. Let $t \in T$ and $t \notin T_1$. By Theorem 6(i), clearly $t = z_4$ or $t = x_j (1 \leq j \leq n-a+1)$. If $t = z_4$, then $y_i (1 \leq i \leq b-n+1)$ does not lie on any $x - y$ monophonic path for some $y \in T_1$, which is a contradiction. If $t = x_j (1 \leq j \leq n-a+1)$, then x_j does not lie on any $x - y$ monophonic path for some $y \in T_1$, which is a contradiction. Thus T is a minimal x-monophonic set of G with cardinality n. □

For every connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [7] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. It can be extended so that the upper vertex monophonic number can be prescribed.

Theorem 20. *For positive integers r, d and n with $r \leq d$, there exists a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $m_x^+(G) = n$ for some vertex x in G.*

Proof. We prove this theorem by considering three cases.

Case 1. $r = d = 1$. Let $G = K_{n+1}$. It is easily seen that $e_m(x) = 1$ for every vertex x in G and so $rad_m(G) = diam_m(G) = 1$. Also, by Theorem 7(ii), $m_x^+(G) = n$ for any vertex x in G.

Case 2. $1 = r < d$. Let $C_{d+2} : v_1, v_2, ..., v_{d+2}, v_1$ be a cycle of order $d + 2$. Let G be the graph obtained by adding $n - 1$ new vertices $u_1, u_2, ..., u_{n-1}$ to C_{d+2} and joining each of the vertices $u_1, u_2, ..., u_{n-1}$ to the vertex v_1 and also joining each vertex $v_i (3 \leq i \leq d + 1)$ to the vertex v_1. The graph G is shown in Figure 3.3. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1$, $e_m(v_2) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$. Let $S = \{v_2, v_{d+2}, u_1, u_2, ..., u_{n-1}\}$ be the set of all simplicial vertices of G and let $x = v_2$. Then by Theorem 6(i), every minimal x-monophonic set of G contains $S - \{v_2\}$. Clearly S is the unique minimal x-monophonic set of G and so $m_x^+(G) = |S| = n$.

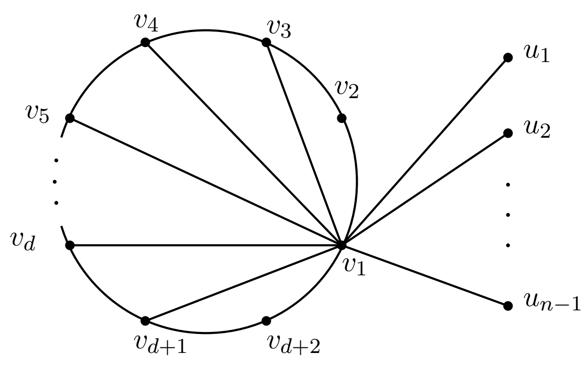
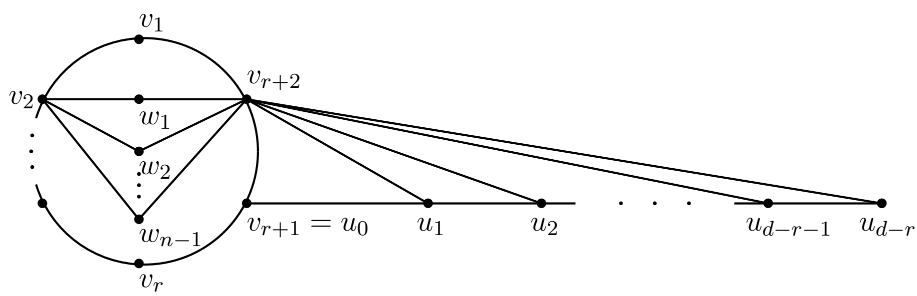


Figure 3.3 : G

Case 3. $2 \leq r \leq d$. Let H be a graph obtained from a cycle C_{r+2} : $v_1, v_2, ..., v_{r+2}, v_1$ of order $r + 2$ and a path P_{d-r+1} : $u_0, u_1, ..., u_{d-r}$ of order $d - r + 1$ by identifying the vertex v_{r+1} in C_{r+2} and u_0 in P_{d-r+1}; also join each vertex $u_i (1 \leq i \leq d - r)$ in P_{d-r+1} with v_{r+2} in C_{r+2}. Now, let G be the graph obtained from H by adding $n-1$ new vertices $w_1, w_2, ..., w_{n-1}$ and join each $w_i(1 \leq i \leq n-1)$ with v_2 and v_{r+2} in H. The graph G is shown in Figure 3.4.

It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G. Also, $e_m(v_{r+2}) = r$ and $e_m(v_1) = d$. It follows that $rad_m(G) = r$ and $diam_m(G) = d$. Now, let $x = u_{d-r}$ and let $S = \{v_1, w_1, w_2, ..., w_{n-1}\}$. Since every vertex of G lies on an $x-y$, where $y \in S$, monophonic path, S is an x-monophonic set of G. Suppose that S_1 is a proper subset of S such that S_1 is an x-monophonic set of G. Then there exists a vertex z in S such that $z \notin S_1$. It is clear that z is either v_1 or $w_i(1 \leq i \leq n-1)$. In all cases z does not lie on any $x-u$, where $u \in S_1$, monophonic path, it follows that S_1 is not an x-monophonic set of G. This shows that S is a minimal x-monophonic set of G and so $m_x^+(G) \geq n$. Also, it is clear that any minimal x-monophonic set of G contains at most n elements and hence $m_x^+(G) \leq n$. Thus $m_x^+(G) = n$.



□

Figure 3.4 : G

References

[1] F. Buckley, F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, (1990).

[2] G. Chartrand, F. Harary, P. Zhang, On the Geodetic Number of a Graph, *Networks*, **39** (2002), 1-6.

[3] F. Harary, *Graph Theory*, Addison-Wesley, (1969).

[4] F. Harary, E. Loukakis, C. Tsouros, The Geodetic Number of a Graph, *Math. Comput. Modeling*, **17**, No. 11 (1993), 87-95.

[5] A.P. Santhakumaran, P. Titus, Vertex Geodomination in Graphs, *Bulletin of Kerala Mathematics Association*, **2**, No. 2 (2005), 45-57.

[6] A.P. Santhakumaran, P. Titus, On the Vertex Geodomination Number of a Graph, *Ars Combinatoria*, **101** (2011), 137-151.

[7] A.P. Santhakumaran, P. Titus, Monophonic Distance in Graphs, *Discrete Mathematics, Algorithms and Applications*, **3**, No. 2 (2011), 159-169.

[8] A.P. Santhakumaran, P. Titus, The Vertex Monophonic number of a Graph, *Discussiones Mathematicae Graph Theory*, **32** (2012), 191-204.