ON T -REPRESENTABILITY OF THE CYCLIC SUBGROUPS OF THE SYMMETRIC GROUP-I

P. Sini¹, P.T. Ramachandran²

^{1,2}Department of Mathematics

University of Calicut

Malappuram Dt., Pin: 673635, Kerala, INDIA

Abstract: In this paper we investigate the group of homeomorphisms of topological spaces. A subgroup K of the group $S(X)$ of all permutations of a set X is called t -representable on X if there exists a topology T on X such that the group of homeomorphisms of $(X, T) = K$. We determine the t -representability of groups generated by a permutation which is a product of disjoint cycles having equal lengths.

AMS Subject Classification: 54H15, 20B35

Key Words: cyclic groups, group of homeomorphisms, permutation groups, t -representable

1. Introduction

Let $H(X, T)$ denote the group of all homeomorphisms of a topological space (X, T) and $H(X, T)$ is a subgroup of the symmetric group $S(X)$. A subgroup K of the group $S(X)$ of all permutations of a set X is called t -representable on X if there exists a topology T on X such that the group of homeomorphisms of $(X, T) = K$. The determination of t -representable subgroups of $S(X)$ is a very difficult and interesting problem. In [3], [4], P. T. Ramachandran showed that

no nontrivial proper normal subgroups of the group of all permutations of a set X can be the group of homeomorphisms of (X, T) for any topology T on X . If $X = \{a_1, a_2, \dots, a_n\}$, $n \geq 3$, the group of permutations of X generated by the cycle $(a_1, a_2, \dots, a_n)$ cannot be represented as the group of homeomorphisms of (X, T) for any topology T on X whereas if X is an infinite set, then the cyclic group generated by an infinite cycle can be represented as the group of homeomorphisms of (X, T) for a topology T on X see [3] and [5]).

In [6] it is proved that the direct sum of finite t -representable permutation groups is t -representable, every permutation group of order two is t -representable and also determined the t -representability of finite transitive permutation groups.

Basic terminology about topological spaces and groups will be consistent with [2], [7]. In this paper we continue the study carried out in [6].

2. Main Results

We investigate the t -representability of cyclic subgroups of $S(X)$ generated by a product of disjoint cycles having equal length.

We need the following theorem.

Theorem 1. [4] *Let $X = \{a_1, a_2, \dots, a_n\}$, $n \geq 3$ and H be the group of permutations on X generated by the cycle $(a_1, a_2, \dots, a_n)$. Then H is not t -representable on X .*

Now we determine the t -representability of permutation groups generated by a permutation which is a product of two disjoint cycles that have equal length n where $n \geq 3$.

Susan J. Andima and W. J Thron [1] associated with each topology T on a set X with a preorder relation ' $\leq$ ' on X defined by $a \leq b$ if and only if every open set containing b contains a . Then the group of homeomorphisms of (X, T) onto itself is equal to the group of order isomorphisms of $(X, \leq)$, which is denoted by $G(X, \leq)$.

The next definition appears in [7].

Definition 2. T_0 Space: A topological space (X, T) is said to be a T_0 space if given any two distinct points in X , there exist an open set which contains one of them but not the other.

So (X, T) is a T_0 space if and only if $(X, \leq)$ is a partially ordered set. If X is a finite nonempty set, then the partially ordered set $(X, \leq)$ has both maximal

and minimal elements. Also an order isomorphism of $(X, \leq)$ maps maximal elements to maximal elements and minimal elements to minimal elements.

Theorem 3. *If σ is a permutation on X which is a product of two disjoint cycles having equal length n where $n \geq 3$, then the group generated by σ is not t -representable on X .*

Proof. Let $\sigma = (a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n)$ and K be the group generated by σ . Decompose X as $X_1 \cup X_2$ where $X_1 = \{a_1, a_2, \dots, a_n\}$ and $X_2 = \{b_1, b_2, \dots, b_n\}$. Assume that K is t -representable on X . Then there exist a topology T on X such that the group of homeomorphisms on X , $H(X, T) = K$. Since there exist no transposition on X as homeomorphism of (X, T) , the space must be T_0 . Let $(X, \leq)$ be the corresponding partially ordered set. Since X is a finite set, $(X, \leq)$ has both maximal and minimal elements. If an element is both minimal and maximal, then all the elements are minimal and maximal and hence $H(X, T) = S(X)$, which is not possible. So a minimal element can not be a maximal element. Then either X_1 or X_2 is the set of all minimal elements.

Assume that X_1 is the set of all minimal elements. Then the elements of X_2 is the set of all maximal elements. Thus the elements of X_i are incomparable to each other for $i = 1, 2$. So there exist atleast one $a_j \in X_1$ and $b_i \in X_2$ such that b_i succeeds a_j . Otherwise $G(X, \leq) = S(X)$, which is a contradiction. Since $G(X, \leq) = K$, if b_i succeeds a_j , then $\sigma^h(b_i)$ succeeds $\sigma^h(a_j)$ for $h = 1, 2, \dots, n-1$ and this gives $a_{j \oplus (h-1)} < b_{i \oplus (h-1)}$ where $\oplus$ denotes addition modulo n . So without loss of generality we can assume that b_1 atleast succeeds a_1 . If b_1 succeeds only a_1 , then $(a_1, a_i)(b_1, b_i)$, where $i = 2, 3, \dots, n$ are order isomorphisms, which is not possible. Now assume that b_1 succeed k elements in X_1 namely $a_1, a_2, \dots, a_k$ where $1 < k \leq n$. Then each $b_i \in X_2$ exactly succeed k elements in X_1 namely $a_i, a_{i \oplus 1}, \dots, a_{i \oplus (k-1)}$. Then

$$\mathcal{B} = \{\{a_i\}, \{a_i, a_{i \oplus 1}, \dots, a_{i \oplus k-1}, b_i\}, i = 1, 2, \dots, n\}$$

form a base for T .

If $k = n$, then $H(X, T) = S(X_1) \oplus S(X_2)$, which is not possible. So assume that $1 < k < n$

For $j = 1, 2, \dots, n$, define $h : X \rightarrow X$ as

$$h(x) = \begin{cases} a_{j \oplus n - (i-1)} & \text{if } x = a_i, 1 \leq i \leq n \\ b_{j \oplus n - (k+i-2)} & \text{if } x = b_i, 1 \leq i \leq n. \end{cases}$$

Now we claim that h is a homeomorphism on (X, T) onto itself. For let $U \in \mathcal{B}$. Then either U is a singleton or $U = \{a_l, a_{l \oplus 1}, \dots, a_{l \oplus (k-1)}, b_l\}$, where

$1 \leq l \leq n$. If U is a singleton, then $h(U) \in \mathcal{B}$. Now let

$$U = \{a_l, a_{l \oplus 1}, \dots, a_{l \oplus (k-1)}, b_l\}.$$

Then

$$\begin{aligned} h(U) &= \{a_{j \oplus n - (l-1)}, a_{j \oplus n - l}, a_{j \oplus n - (l+1)}, \dots, \\ &\quad a_{j \oplus n - (l+k-2)}, b_{j \oplus n - (l+k-2)}\} \\ &= \{a_{j \oplus (n-l)+1}, a_{j \oplus n - l}, a_{j \oplus (n-l)-1}, \dots, \\ &\quad a_{j \oplus (n-(l+k)+2)}, b_{j \oplus (n-(l+k)+2)}\} \\ &= \{a_{j \oplus n - (l+k)+2}, a_{j \oplus n - (l+k)+3}, \dots, \\ &\quad a_{j \oplus n - (l+k)+k}, a_{j \oplus n - (l+k)+(k+1)}, b_{j \oplus n - (l+k)+2}\} \\ &= \{a_p, a_{p \oplus 1}, a_{p \oplus 2}, \dots, a_{p \oplus (k-1)}, b_p\}, \\ &\quad \text{where } p = j \oplus n - (l+k) + 2. \end{aligned}$$

Thus $h(U) \in \mathcal{B}$. Observe that h is the product of transpositions and hence $h^{-1} = h$. Consequently, we have $h^{-1}(U) \in \mathcal{B}$. This is true for all $U \in \mathcal{B}$. So h is a homeomorphism on (X, T) , which is a contradiction to the fact that $H(X, T) = K$. This completes the proof. $\square$

Example 4. Let $X = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and

$$\sigma = (a_1, a_2, a_3)(b_1, b_2, b_3).$$

Then the cyclic group generated by σ is,

$$\langle \sigma \rangle = \{I, (a_1, a_2, a_3)(b_1, b_2, b_3), (a_1, a_3, a_2)(b_1, b_3, b_2)\}.$$

Suppose $\langle \sigma \rangle$ is t -representable on X , then there exist a topology T on X such that $H(X, T) = \langle \sigma \rangle$. Consider $(X, \leq)$, then arguing as in the Theorem 3 either $X_1 = \{a_1, a_2, a_3\}$ or $X_2 = \{b_1, b_2, b_3\}$ is the set of all minimal elements. Without loss of generality we can assume that X_1 is the set of all minimal elements. So the possible partial orders on X are essentially as in the following figure

$$(X, \leq_1)$$

$$(X, \leq_2)$$

$$(X, \leq_3)$$

Then

$$G(X, \leq_1) = \{I, (a_1, a_2)(b_1, b_2), (a_1, a_3)(b_1, b_3), (a_2, a_3)(b_2, b_3), \\ (a_1, a_2, a_3)(b_1, b_2, b_3), (a_1, a_3, a_2)(b_1, b_3, b_2)\},$$

$$G(X, \leq_2) = \{I, (a_1, a_2)(b_1, b_3), (a_1, a_3)(b_2, b_3), (a_2, a_3)(b_1, b_2), \\ (a_1, a_2, a_3)(b_1, b_2, b_3), (a_1, a_3, a_2)(b_1, b_3, b_2)\},$$

$$\text{and } G(X, \leq_3) = S(X_1) \oplus S(X_2).$$

So there exist no partial orders on X such that the group of order isomorphisms is the group generated σ . Hence $\langle \sigma \rangle$ is not t -representable on X .

Now we consider the t -representability of the cyclic group generated by a permutation on X which is an arbitrary product of more than two disjoint cycles having equal lengths n where $n > 2$. Here we prove that such cyclic subgroups of symmetric group $S(X)$ are t -representable on X .

Theorem 5. *If σ is a permutation on X which is an arbitrary product of more than two disjoint cycles having equal length n where $n > 2$, then the group generated by σ , $\langle \sigma \rangle$ is t -representable on X .*

Proof. Let

$$\sigma = \prod_{i \in I} C_i$$

where $\{C_i, i \in I\}$ be an indexed family of disjoint cycles having equal length n where $n > 2$ and $|I| > 2$. Let $C_i = (a_{i1}, a_{i2}, \dots, a_{in})$ for every $i \in I$ and $X = \bigcup_{i \in I} X_i$.

By assuming axiom of choice, well-order the set I by the order relation $<$. Let i_0 be the first element of I . Now consider the set $\{i \in I : i_0 < i\}$ and i_1 denote the first element of the set $I \setminus \{i_0\}$. In general i_j denote the first element of the set $I \setminus \{i_0, i_1, i_2, \dots, i_{j-1}\}$.

Define a base $\mathcal{B}$ by

$$\mathcal{B} = \{A_{ij} : i \in I, j = 1, 2, \dots, n\}$$

where

$$A_{ij} = \begin{cases} \{a_{kj} : k \leq i\} & \text{if } i < i_2 \\ \{a_{kj} : k \leq i\} \cup \{a_{i_0 j \oplus (n-1)}\} & \text{if } i \geq i_2. \end{cases}$$

and T be the topology on X having base $\mathcal{B}$. It can be verified that σ is a homeomorphism of (X, T) . Then all powers of σ are homeomorphisms of (X, T) . Hence $<\sigma> \subseteq H(X, T)$

Conversely let $h \in H(X, T)$. We have to prove that h is of the form σ^m for some m such that $1 \leq m \leq n$. Since h is a homeomorphism, $h(A_{i_0 1}) = A_{i_0 m}$ for some m , $1 \leq m \leq n$. Thus $h(a_{i_0 1}) = a_{i_0 m}$. If $h(a_{i_0 1}) = a_{i_0 m}$, then $h(a_{i_1 1}) = a_{i_1 m}$ for all $i \in I$ and $h(a_{i_0 n}) = a_{i_0 m \oplus (n-1)}$. Now let $h(a_{i_0 2}) = a_{i_0 p}$. Then $h(a_{i_2 2}) = a_{i_2 p}$ for all $i \in I$ and $h(a_{i_0 2 \oplus (n-1)}) = a_{i_0 p \oplus (n-1)}$. That is $h(a_{i_0 1}) = a_{i_0 p \oplus (n-1)}$. By the fact that $h(a_{i_0 1}) = a_{i_0 m}$ we have $m = p \oplus (n-1)$. This implies that $p = m \oplus 1$.

Now assume that $h(a_{i_0 \alpha}) = a_{i_0 \beta}$ for some α and β , thus $h(a_{i \alpha}) = a_{i \beta}$ for all $i \in I$ and $h(a_{i_0 \alpha \oplus (n-1)}) = a_{i_0 \beta \oplus (n-1)}$. Now let $h(a_{i_0 \alpha \oplus 1}) = a_{i_0 \gamma}$. Then $h(a_{i_0 \alpha \oplus 1 \oplus (n-1)}) = a_{i_0 \gamma \oplus (n-1)}$, which implies $h(a_{i_0 \alpha}) = a_{i_0 \gamma \oplus (n-1)}$ and we obtain that $\beta = \gamma \oplus (n-1)$ and so $\gamma = \beta \oplus 1$. Consequently we get $h(a_{i \alpha \oplus 1}) = a_{i \beta \oplus 1}$ for all $i \in I$ and so $h(a_{i k}) = a_{i k \oplus (m-1)}$ for all $i \in I$ and $1 \leq k \leq n$. Hence $h(a_{i k}) = \sigma^m(a_{i k})$ for all $i \in I$ and $1 \leq k \leq n$, which implies that $h = \sigma^m$. Thus $h \in <\sigma>$. So $H(X, T) \subseteq <\sigma>$. This completes the proof. $\square$

Theorem 6. Let X be any set and H be the group on X generated by $\sigma = \prod_{i \in I} C_i$ where $\{C_i, i \in I\}$ be an indexed family of disjoint cycles having equal length n . Then

1. H is t -representable if $|I| > 2$ or $n < 3$.
2. H is not t -representable if $|I| \leq 2$ and $n \geq 3$.

Proof. In [6], it is proved that every permutation group of order two is t -representable. So if the length of each cycle of the permutation σ is less than three, then $\langle \sigma \rangle$ is t -representable. Thus H is t -representable if $n < 3$. Now H is t -representable if $I > 2$ by Theorem 5 and H is not t -representable if $|I| \leq 2$ and $n \geq 3$ by Theorems 1 and 3. $\square$

3. Conclusion

It is convenient to point out that we determined the t -representability of groups generated by a permutation which is a product of disjoint cycles having equal lengths.

Acknowledgements

The first author acknowledges the financial support from University Grants Commission, Govt. of India for awarding the teacher fellowship for doing the research and expresses sincere thanks to Dr. Ramachandran P. T, Associate Professor, Department of Mathematics, University of Calicut for his guidance and critical suggestions through out the preparation of this paper.

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