International Journal of Pure and Applied Mathematics
Volume 106 No. 3 2016, 923-936
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v106i3.17

ijpam.eu

CONNECTIVITY IN THE CASE OF AN IDEMPOTENT PRETOPOLOGY

Monique Dalud-Vincent[1], Michel Lamure[2]
[1]MEPS - Max Weber Center
FASSP
University Lyon 2
5 Avenue Pierre Mendès-France
69676 Bron cedex, FRANCE
[2]Professor emeritus
University Lyon 1, FRANCE

Abstract: This work focuses on connectivity in a pretopological space where the pseudo-closure mapping is an idempotent one.

AMS Subject Classification: 54A05, 54B05, 54B15
Key Words: graph, pretopology, connectivity, idempotent pseudoclosure

1. Introduction

We already illustrated how pretopology generalizes both graph theory and topology [4]. We also established links between matroids and pretopology as well as between hypergraphs and pretopology [6].

In this paper, we present results concerning strong connectivity [4] and connectivity [7] in the case of a pretopology defined by an idempotent pseudoclosure (we now say an idempotent pretopology).

Received: December 10, 2015
Published: March 2, 2016

2. Different Types of Pretopological Spaces, see [1] [2] [3] [4]

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such as:

$$a\ (\varnothing) = \varnothing,$$

$$\forall\ \mathrm{A} \subset\ \mathrm{X}, \quad \mathrm{A} \subset\ a\ (\mathrm{A}),$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1. (X, a) is a V-type pretopological space if and only if

$$\forall\ \mathrm{A} \subset\ \mathrm{X}, \forall\ \mathrm{B} \subset\ \mathrm{X}, \mathrm{A} \subset\ \mathrm{B} \Rightarrow\ a\ (\mathrm{A}) \subset\ a\ (\mathrm{B}).$$

2. (X, a) is a V_D-type pretopological space if and only if

$$\forall\ \mathrm{A} \subset\ \mathrm{X}, \forall\ \mathrm{B} \subset\ \mathrm{X},\ a\ (\mathrm{A} \cup\ \mathrm{B}) =\ a\ (\mathrm{A}) \cup a\ (\mathrm{B}).$$

3. (X, a) is a V_S-type pretopological space if and only if

$$\forall\ \mathrm{A} \subset\ \mathrm{X}, \quad a(A) = \bigcup_{x \in A} a(\{x\}).$$

4. (X, a) a V_D-type pretopological space, is a topological space if and only if $\forall\ \mathrm{A} \subset\ \mathrm{X},\ a\ (a\ (\mathrm{A})) = a\ (\mathrm{A})$.

Property 1. *If (X, a) is a V_S-type space then (X, a) is a V_D-type space. If (X, a) is a V_D-type space then (X, a) is a V-type space.*

Example. Let X be a non empty set and R be a binary relationship defined on X. Let x $\in$ X. We note

$$R(x) = \{y \in X/xRy\}$$

and

$$R^{-1}(x) = \{y \in X/yRx\}.$$

The collateral pretopology, denoted a_c, is defined by:

$$\forall A \subset X, a_c(A) = \{x \in X/\forall y \in R^{-1}(x), R(y) \cap A \neq \emptyset\} \cup A.$$

This pretopology is only a V-type one, see [9].

Definition 2. Let (X, a) be a V-type pretopological space. Let A $\subset$ X. A is a closed subset if and only if a (A) = A.

We note $\forall$ A $\subset$ X, a^0 (A) = A and $\forall$ n, n$\geq$ 1, a^n(A) = $a(a)^{n-1}$(A).

We name closure of A the subset of X, denoted F_a(A), which is the smallest closed subset which contains A.

F'_a, the inverse of the closure generated by a, is defined by:

$$\forall A \subset X, F'_a(A) = \{x \in X / F_a(\{x\}) \cap A \neq \emptyset\}.$$

We note a" = $F'_a F_a$ (a" is the composed of the mapping F'_a and F_a) and F''_a the closure according to a".

a' inverse of pseudoclosure a is defined as follows:

$$\forall A \subset X, a'(A) = \{x \in X / \ a(\{x\}) \cap A \neq \emptyset\}.$$

a' defines a pretopology on X. We denote $F_{a'}$ the closure according to a' and $F'_{a'}$ the inverse of the closure according to a' (i.e. $F_{a'}$). We also denote $F''_{a'}$ the closure according to (a')" = $F'_{a'}F_{a'}$.

Remark 1. If a is of V-type then a^n, F_a, a", F''_a also are of V-type and a' and F'_a are of V_S-type. If a is of V_S-type then a^n, F_a, a", F''_a, a', F'_a are also V_S-type.

Definition 3. Let (X,a) a V-type pretopological space. a is idempotent if and only if a^2 = a.

3. Results Related to (X, a) and (X, a')

Proposition 1. *Let (X,a) a V-type pretopological space.*

a is idempotent $\Leftrightarrow$ $F_a = a$.

Proof. Let us suppose that a is idempotent.

Let A $\subset$ X. F_a(A) is the smallest closed subset including A (by definition of F_a).

But, by idempotence of a, a(A) is a closed subset.

Let us prove that a(A) is the smallest closed subset including A:

If a(A) is not the smallest closed subset including A, there exists B closed subset different of a(A) such as A $\subset$ B and a(B) = B. As A $\subset$ B and a is of V-type, we get a(A) $\subset$ a(B) = B, which leads to a contradiction.

Then, we can say that a(A) is the smallest closed subset including A and $F_a = a$.

The converse is obvious.

Proposition 2. *Let (X,a) a V-type pretopological space. If a is idempotent then a' is idempotent.*

Proof. $\forall A \subset X$,

$$\begin{aligned} a'(A) &\subset a'(a'(A)) \\ &\subset (a^2)'(A) \quad \text{(see [8])} \\ &\subset \{x \in X / a^2(\{x\}) \cap A \neq \emptyset\} \quad \text{(by definition)} \\ &\subset \{x \in X / a(\{x\}) \cap A \neq \emptyset\} \quad (a \text{ is idempotent}) \\ &\subset a'(A) \quad \text{(by definition).} \end{aligned}$$

So, we can say: a'(A) = a'(a'(A)).

Proposition 3. *Let (X,a) a V-type pretopological space. If a is idempotent then* F'_a = a' = $F_{a'}$.

Proof. If a is idempotent then a' is idempotent (see Proposition 2).
Therefore, $F_{a'}$ = a' (see Proposition 1).
On the other hand, ∀ A ⊂ X,

$$\begin{aligned} F'_a(A) =&\{y \in X / F_a(\{y\}) \cap A \neq \emptyset\} \\ =&\{y \in X / a(\{y\}) \cap A \neq \emptyset\} \quad (a \text{ is idempotent}) \\ =&a'(A). \quad \text{(by definition)} \end{aligned}$$

Definition 4. Let (X,a) a V-type pretopological space. Let A ⊂ X. We define the induced pretopology by a on A, denoted a_A, as follows:

$$\forall C \subset A, \quad a_A(C) = a(C) \cap A.$$

(A, a_A) (or more simply A) is called a pretopological subspace of (X, a).

Proposition 4. *Let (X,a) a V-type pretopological space with a idempotent. Let A ⊂ X, A being a non empty set. We denote* F_{aA} *the closure according to* a_A*,* F'_{aA} *the inverse of the closure according to* a_A*,* $F_{(aA)'}$ *the closure according to* $(a_A)'$*, and* $F_{(a')A}$ *the closure according to* $(a')_A$.

i. $F_{aA} = a_A$.

ii. $F'_{aA} = F_{(aA)'} = F_{(a')A} = (a')_A = (a_A)'$.

Proof. i. Let C ⊂ A.

$$a_A(C) = a(C) \cap A = F_a(C) \cap A,$$

(see Proposition 1).

Moreover

$$a_A(C) \subset F_{aA}(C) \subset F_a(C) \cap A$$

(see [1]).

Therefore $F_{aA} = a_A$.

ii. We have $(a_A)' = (a')_A$ (see [8]).

Then $F_{(aA)'} = F_{(a')A}$.

On the other hand, whatever $C \subset A$,

$$\begin{aligned} F'_{aA}(C) =& \{y \in A / F_{aA}(\{y\}) \cap C \neq \emptyset\} \quad \text{(by definition)} \\ =& \{y \in A / a_A(\{y\}) \cap C \neq \emptyset\} \quad \text{(Proposition 4-i)} \\ =& (a_A)'(C). \end{aligned}$$

At last, according to Proposition 4-i, a_A is idempotent.

Then (a_A)' is idempotent (see Proposition 2), and moreover $F_{(aA)'} = (a_A)$' according to Proposition 1.

Ultimately, we get

$$F'_{aA} = F_{(aA)'} = F_{(a')A} = (a')_A = (a_A)'.$$

4. Subsets and Subspaces *x*-connected in (X, a)

We shall speak about x-connectivity to indicate one of the following types of connectivity:

Definition 5. (see [1] [6]) Let (X,a) a V-type pretopological space.

1- (X, a) is strongly connected if and only if $\forall$ C $\subset$ X, C $\neq \varnothing$, F_a(C) = X.

2- (X, a) is connected if and only if $\forall$C $\subset$ X, C$\neq \varnothing$, F_a(C) = X or F_a(X-F_a(C)) $\cap$ F_a(C) $\neq \varnothing$.

Definition 6. (see [1] [6]) Let (X,a) a V-type pretopological space. Let A a non empty subset of X. Let B a non empty subset of X. There exists a path in (X, a) from B to A if and only if B $\subset$ F_a(A). There exists a chain in (X, a) from B to A if and only if B $\subset$ F''_a(A).

Proposition 5. (see [6]) *Let (X,a) a V-type pretopological space.*

i- If $\forall$x $\in$ X and $\forall$y $\in$ X, there exists a chain in (X, a) from $\{y\}$ to $\{x\}$ then (X, a) is connected.

ii-(X, a) is strongly connected $\Leftrightarrow$ $\forall x \in X$ *and* $\forall y \in X$, *there exists a path in (X, a) from* { y } *to* { x }.

Proposition 6. (see [6]) *Let (X,a) a* V_S*-type pretopological space.*

(X, a) is connected $\Leftrightarrow$ $\forall x \in X$ *and* $\forall y \in X$, *there exists a chain from* $\{y\}$ *to* $\{x\}$ *in (X, a).*

Property 2. (see [6]) *Let (X,a) a V-type pretopological space. Let* $x \in X$ *and let* $y \in X$.

i- If there exists a sequence $x_0 \ldots x_n$ *of elements of X such as* $x_0 = x$, $x_n = y$ *with* $\forall j = 0, \ldots, n-1$, $x_{j+1} \in a(\{x_j\})$ *or* $x_j \in a(\{x_{j+1}\})$ *then there exists a chain in (X, a) from* $\{y\}$ *to* $\{x\}$.

ii- If there exists a sequence $x_0 \ldots x_n$ *of elements of X such as* $x_0 = x$, $x_n = y$ *with* $\forall j = 0, \ldots, n-1$, $x_{j+1} \in a(\{x_j\})$ *then there exists a path in (X, a) from* $\{y\}$ *to* $\{x\}$.

Property 3. (see [6]) *Let (X,a) a* V_S*-type pretopological space. Let* $x \in X$ *and let* $y \in X$.

i-There exists a chain in (X, a) from $\{y\}$ *to* $\{x\}$ $\Leftrightarrow$ *there exists a sequence* $x_0 \ldots x_n$ *of elements of X such as* $x_0 = x$, $x_n = y$ *with* $\forall j = 0, \ldots, n-1$, $x_{j+1} \in a(\{x_j\})$ *or* $x_j \in a(\{x_{j+1}\})$.

ii-There exists a path in (X, a) from $\{y\}$ *to* $\{x\}$ $\Leftrightarrow$ *there exists a sequence* $x_0 \ldots x_n$ *of elements of X such as* $x_0 = x$, $x_n = y$ *with* $\forall j = 0, \ldots, n-1$, $x_{j+1} \in a(\{x_j\})$.

Property 4. (see [5]) *Let (X,a) a* V_S*-type pretopological space.*
$\forall x \in X, \forall y \in X, x \in F''_a(\{y\}) \Leftrightarrow y \in F''_a(\{x\})$.

Définition 7. (see [1] [3]) Let (X,a) a V-type pretopological space. Let A $\subset$ X, A being a non empty set.

A is a pretopological subspace x-connected of (X,a) if and only if (A, a_A), as a pretopological space, is x-connected.

(A, a_A) is a greatest x-connected subspace of (X, a) if and only if (A, a_A) is a x-connected subspace of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X, A$\neq$ B, (B, a_B) is not a x-connected subspace of (X, a).

Definition 8. (see [1]) Let (X,a) a V-type pretopological space. Let A $\subset$ X, A being a non empty set. We note $(F_a)_A$ the closure obtained by restriction of the closure F_a on A. $(F_a)_A$ is such as $\forall$ C $\subset$ A, $(F_a)_A$ (C) = F_a(C) $\cap$A.

A is a x-connected subset of (X, a) if and only if A, endowed with $(F_a)_A$, is x-connected. We shall say that A is x-connected to mean that A is a x-connected subset of (X, a).

A is a x-connected component of (X, a) if and only if A is a x-connected subset of (X, a) and ∀B, A ⊂ B ⊂ X with A≠ B, B is not a x-connected subset of (X, a).

Proposition 7. (see [1] [4]) *Let (X, a) be a V-type pretopological space. Let A ⊂ X with A non empty.*

i- If A is a x-connected subspace of (X,a) then A is a x-connected subset of (X,a).

ii- If a is an idempotent function, then:

A is a x-connected subspace of (X,a) ⇔ A is a x-connected subset of (X, a).

Proposition 8. (see [7]) *Let (X, a) be a V_s-type pretopological space. Let { U_i, $i \in I$ } a family of connected subsets of (X, a) which is a partition of X.*

∀ $i \in I$, $F"_a(U_i)$ is a connected component of (X, a).

Proposition 9. *Let (X, a) be a V-type pretopological space.*

Let { S_i, $i \in I$ } a family of non empty subsets of X such as:

1- $\bigcup_{i\in I} S_i = X$

2- $\forall\ i \in I, \forall\ x \in S_i, \forall\ y \in S_i$, *there exists a sequence* $x_0 \ldots x_n$ *of elements of X such as:*

$x_0 = x$, $x_n = y$ *with* $\forall j = 0, \ldots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ *or* $x_j \in a(\{x_{j+1}\})$

3- $\forall\ i \in I, \forall\ k \in I$, $i \neq k, \forall\ x \in S_i, \forall\ y \in S_k$, *it does not exist a sequence* $x_0 \ldots x_n$ *of elements of X such as :*

$x_0 = x$, $x_n = y$ *with* $\forall j = 0, \ldots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ *or* $x_j \in a(\{x_{j+1}\})$

i- { S_i, $i \in I$ } is a partition of X.

ii-∀ $i \in I$, S_i is a connected subspace of (X,a).

Proof. i- By definition.

ii-∀ i ∈ I, S_i is a connected subspace of (X,a).

We can say : ∀ i ∈ I,∀ x ∈ S_i,∀ y ∈ S_i, there exists a sequence $x_0 \ldots x_n$ of elements of X such as x_0= x, x_n= y with ∀j = 0,. . . , n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$ and ∀ i ∈ I,∀ k ∈ I, i≠ k,∀ x ∈ S_i,∀ y ∈ S_k, it does not exist a sequence $x_0 \ldots x_n$ of elements of X such as x_0= x, x_n= y with ∀j = 0,. . . , n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$

then, ∀ x ∈ S_i,∀ y ∈ S_i, there exists a sequence $x_0 \ldots x_n$ of elements of S_i such as x_0= x, x_n= y with ∀j = 0,. . . , n-1, $x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$

because, otherwise there exists $x_{j'}$ element of $x_0 \ldots x_n$ such as $x_{j'}$ does not belong to S_i

then, there exists k ∈ I such as $x_{j'} \in S_k$ (from hypothesis 1) .

In this case, there exists a sequence $x_0 \ldots x_n$ of elements of X such as

x_0= x, x_n= $x_{j'}$ with ∀j = 0,. . . , n-1, x_{j+1} ∈ a ({ x_j}) or x_j ∈ a ({ x_{j+1}})
which contradicts hypothesis 3 because x ∈ S_i and $x_{j'}$ ∈ S_k.

At last, ∀ x ∈ S_i, ∀ y ∈ S_i, there exists a chain from { y } to { x } in (S_i,a_{Si}) (Property 2-i)

Which implies S_i is a connected subspace of (X,a) (Proposition 5-i) .

Proposition 10. *Let (X,a) be a V_s-type pretopological space. We assume the same conditions as in Proposition 9.*

i- { S_i, $i \in I$ } is the set of connected components of (X,a).

ii- { S_i, $i \in I$ } is the set of the greatest connected subspaces of (X,a).

Proof. i- ∀ i ∈ I, S_i is a connected subspace of (X,a) (Proposition 9). Thus, S_i is a connected subset of (X,a) (Proposition 7-i) .

Let us prove that ∀ i ∈ I, S_i = F"$_a$(S_i):

From Proposition 9 and Property 3-i,

∀ i ∈ I,∀ k ∈ I, i≠ k,∀ x ∈ S_i,∀ y ∈ S_k, it does not exist a chain from {y} to {x} in (X, a)

then ∀ i ∈ I,∀ k ∈ I, i≠ k,∀ x ∈ S_i,∀ y ∈ S_k, {y} is not included in F"$_a$({x})

and ∀ x ∈ S_i, S_i ⊂ F"$_a$({ x}) = F"$_a$(Si) (Property 3-i)

then ∀ i ∈ I,∀ k ∈ I, i≠ k,∀ y ∈ S_k, {y} is not included in F"$_a$(S_i)

which implies F"$_a$(S_i) = S_i

and leads to the result from Proposition 8.

ii- see [4].

Proposition 11. *Let (X,a) be a V_s-type pretopological space. We assume the same conditions as in Proposition 9.*

Let $x \in X$ and $y \in X$. We get:

i- There exists $i \in I$, $x \in S_i$ and $y \in S_i$ ⇔ $x \in$ F"$_a$({y})

ii- There exists $i \in I$, there exists $j \in I$, $i \neq j$, $x \in S_i$ and $y \in S_j$ ⇔ $x \notin$ F"$_a$({y}).

Proof. i- and ii- From Property 3-i and Property 4.

5. Decomposing (X, a) with a Idempotent for Strong Connectivity

Proposition 12. *Let (X, a) be a V-type pretopological space with a idempotent.*

Let $x \in X$ and $y \in X$.

There exists a sequence $x_0 \dots x_n$ of elements of X such as x_0= x, x_n= y with ∀j = 0,. . . , n-1, $x_{j+1} \in a$ ({x_j}) ⇔ $y \in a$ ({ x}).

Proof. There exists a sequence $x_0 \dots x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0,\dots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ implies $y \in a^n(\{x\})$ hence $y \in a(\{x\})$ by idempotence of a.
Conversely: obvious.

Proposition 13. *Let (X, a) be a V-type pretopological space with a idempotent.*

If $\{ C_i,\ i \in I \}$ is a family of non empty subsets of X such as:

1- $\bigcup_{i\in I} C_i = X$

2- $\forall i \in I, \forall x \in C_i, \forall y \in C_i$, there exists a sequence $x_0 \dots x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0,\dots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$

3- $\forall i \in I, \forall k \in I, i \neq k, \forall x \in C_i, \forall y \in C_k$, it does not exist a sequence $x_0 \dots x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0,\dots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ or it does not exist a sequence $x_0 \dots x_n$ of elements of X such as $x_0 = y$, $x_n = x$ with $\forall j = 0,\dots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$

then for any $x \in X$ and $y \in X$.

i- There exists $i \in I$, $x \in C_i$ and $y \in C_i \Leftrightarrow y \in a(\{x\})$ and $x \in a(\{y\})$.

ii- There exists $i \in I$, there exists $k \in I$, $i \neq k$, $x \in C_i$ and $y \in C_k \Leftrightarrow y \notin a(\{x\})$ or $x \notin a(\{y\})$.

Proof. i- There exists $i \in I$, $x \in C_i$ and $y \in C_i \Leftrightarrow$ there exists a sequence $x_0 \dots x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0,\dots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$

and there exists a sequence $x_0 \dots x_n$ of elements of X such as $x_0 = y$, $x_n = x$ with $\forall j = 0,\dots, n\text{-}1$, $x_{j+1} \in a(\{x_j\})$ (by definition of C_i)

which means that $y \in a(\{x\})$ and $x \in a(\{y\})$ (Proposition 12)

ii- By converse of i.

Definition 9. Let (X, a) be a V-type pretopological space. Let $A \subset X$ with A non empty.

i- A is a clique of (X, a) if and only if $\forall x \in A, \forall y \in A, y \in a(\{x\})$.

ii- A is a maximal clique of (X, a) if and only if A is a clique of (X, a) and $\forall$ A', $A \subset A' \subset X$ and $A \neq A'$, A' is not a clique of (X, a).

Proposition 14. *Let (X, a) be a V-type pretopological space with a idempotent.*

Conditions are the same as in Proposition 13.

$\{C_i,\ i \in I\}$ is the set of maximal cliques of (X, a).

Proof. Obvious from Proposition 13.

Proposition 15. *Let (X, a) be a V-type pretopological space with a idempotent.*

Conditions are the same as in Proposition 13.
i- ∀ $i \in I$, C_i is greatest strongly connected subspace of (X, a).
ii- ∀ $i \in I$, C_i is a strongly connected component of (X, a).

Proof. i- C_i is strongly connected subspace of (X, a) (Proposition 13 and Proposition 5-ii) . Let us prove that C_i is greatest strongly connected subspace of (X, a):

If not, there exists A such as $C_i \subset A \subset X$ and $C_i \neq A$ and A is greatest strongly connected subspace of (X, a) with $A = \bigcup_{j \in J_A} C_j$ ($J_A \subset I$) and $F_{aA}(C_j) = A$, $\forall j \in J_A$ (see [4]).

But $F_{aA} = a_A$ (Proposition 4-i)
then $\forall j \in J_A$, $a_A(C_j) = a(C_j) \cap A = A$.
In particular, $a(C_i) \cap A = A$
then $A \subset a(C_i)$
then if $C_j \subset A$ with $i \neq j$, we get $C_j \subset a(C_i)$
hence $\forall x \in C_j$, $\forall y \in C_i$, $x \in a(\{y\})$ (because $a(\{y\}) = a(C_i)$) .
From Proposition 13-ii, we can say $\forall x \in C_j, \forall y \in C_i$, $y \notin a(\{x\}) = a(C_j)$
then $a(C_j) \cap A \neq A$
which contredicts A greatest strongly connected subspace of (X, a).
That means C_i is greatest strongly connected subspace of (X, a).
ii- From Proposition 7-ii.

Consequence: In a V-type pretopological space with a idempotent, in particular in the usual topological, maximal cliques provide the greatest strongly connected subspaces which also are the strongly connected components.

6. Decomposing (X, a) with a Idempotent for Connectivity

Proposition 16. *Let (X,a) be a V_s-type pretopological space with a idempotent. Let $A \subset X$. $F"_a(A) = \bigcup_{n \geq 0} (a'a)^n(A)$.*

Proof. Let $A \subset X$, $F"_a(A) = \bigcup_{n \geq 0} (F'_a F_a)^n(A)$ (see [5]).
$= \bigcup_{n \geq 0} (a'a)^n(A)$ (from Propositions 1 and 3).

Proposition 17. *Let (X,a) be a V_s-type pretopological space with a idempotent.*
Conditions are the same as in Proposition 9.
For any $x \in X$ and $y \in X$.

i- There exists $i \in I$, $x \in S_i$ *and* $y \in S_i \Leftrightarrow x \in \bigcup_{n \geq 0} (a'a)^n(\{y\})$.

ii- There exists $i \in I$, *there exists* $k \in I$, $i \neq k$, $x \in S_i$ *and* $y \in S_k \Leftrightarrow x \notin \bigcup_{n \geq 0} (a'a)^n(\{y\})$.

Proof. i- and ii- From Proposition 11 and Proposition 16.

7. Application for Pretopology of Collateral

Remark 2. $\forall x \in X, \forall z \in X$ *with* $x \neq z$,
- *If* $R^{-1}(x) \neq \emptyset$ *then* $x \in a_c(\{z\}) \Leftrightarrow R^{-1}(x) \subset R^{-1}(z)$
- *If* $R^{-1}(x) = \emptyset$ *then* $x \notin a_c(\{z\})$.

Proof. $a_c(\{z\}) = \{x \in X / \forall y \in R^{-1}(x), R(y) \cap \{z\} \neq \emptyset\} \cup \{z\} = \{x \in X / \forall y \in R^{-1}(x), z \in R(y)\} \cup \{z\}$
- If $R^{-1}(x) = \emptyset$ then $x \notin a_c(\{z\})$ by definition.
- If $R^{-1}(x) \neq \emptyset$
then $x \in a_c(\{z\}) \Leftrightarrow \forall y \in R^{-1}(x), z \in R(y)$
$\Leftrightarrow \forall y \in R^{-1}(x), y \in R^{-1}(z)$
$\Leftrightarrow R^{-1}(x) \subset R^{-1}(z)$.

Remark 3. a_c *is idempotent.*

Proof. Let us prove that $\forall A \subset X$, $a_c(a_c(A)) = a_c(A)$:
$a_c(A) = \{x \in X / \forall y \in R^{-1}(x), R(y) \cap A \neq \emptyset\} \cup A$
then $a_c(a_c(A)) = a_c(A) \cup \{x \in X / \forall y \in R^{-1}(x), R(y) \cap a_c(A) \neq \emptyset\}$.
If $x \in a_c(a_c(A))$ and $x \notin a_c(A)$
then $\forall y \in R^{-1}(x)$, $R(y) \cap a_c(A) \neq \emptyset$ and there exists $y \in R^{-1}(x)$, $R(y) \cap A = \emptyset$ and $x \notin A$
then $\forall y \in R^{-1}(x)$ such as $R(y) \cap A = \emptyset$, we get:
there exists y' such as $y' \notin A$, $y' \in R(y)$ and $y' \in a_c(A)$
then there exists y' such as $y' \notin A$, $y' \in R(y)$ and $\forall y'' \in R^{-1}(y')$, $R(y'') \cap A \neq \emptyset$
then there exists y' such as $y' \notin A$, $y \in R^{-1}(y')$ and $\forall y'' \in R^{-1}(y')$, $R(y'') \cap A \neq \emptyset$
Which implies $R(y) \cap A \neq \emptyset$; Not possible.

Remark 4. *For any* $x \in X$ *and* $y \in X$.
Following assertions are equivalent:
i- $y \in a_c(\{x\})$

ii- There exists a sequence $x_0 \ldots x_n$ of elements of X such as $x_0 = x$, $x_n = y$ with $\forall j = 0, \ldots, n-1$, $x_{j+1} \in a_c(\{x_j\})$

iii- $R^{-1}(y) \subset R^{-1}(x)$ with $R^{-1}(y) \neq \emptyset$ or $x = y$.

Proof. From Remarks 2 and 3 and from Proposition 12.

Proposition 18. *Let (X,a_c) a pretopological space with a_c collateral pretopology.*

Conditions are the same as in Proposition 13.

For any $x \in X$ and $y \in X$.

i- 1- There exists $i \in I$, $x \in C_i$ and $y \in C_i \Leftrightarrow x = y$ or ($R^{-1}(y) \neq \emptyset$ and $R^{-1}(x) \neq \emptyset$ with $R^{-1}(x) = R^{-1}(y)$).

2- There exists $i \in I$, there exists $k \in I$, $i \neq k$, $x \in C_i$ and $y \in C_k \Leftrightarrow x \neq y$ and ($R^{-1}(y) = \emptyset$ or $R^{-1}(x) = \emptyset$ or ($R^{-1}(x) \neq \emptyset$ and $R^{-1}(y) \neq \emptyset$ with $R^{-1}(x) \neq R^{-1}(y)$)).

ii- $\{C_i, i \in I\}$ is the set of the greatest strongly connected subspaces of (X,a) and also the set of maximal cliques of (X,a).

Proof. i- 1- There exists $i \in I$, $x \in C_i$ and $y \in C_i$

$\Leftrightarrow$ $y \in a_c(\{x\})$ and $x \in a_c(\{y\})$ (Proposition 13)

$\Leftrightarrow$ ($R^{-1}(y) \subset R^{-1}(x)$ with $R^{-1}(y) \neq \emptyset$ or $x = y$)

and ($R^{-1}(x) \subset R^{-1}(y)$ with $R^{-1}(x) \neq \emptyset$ or $x = y$) (Remark 4)

$\Leftrightarrow$ ($R^{-1}(y) \subset R^{-1}(x)$ and $R^{-1}(x) \subset R^{-1}(y)$ with $R^{-1}(x) \neq \emptyset$ and $R^{-1}(y) \neq \emptyset$) or ($x = y$)

$\Leftrightarrow$ ($R^{-1}(y) = R^{-1}(x)$ with $R^{-1}(x) \neq \emptyset$ and $R^{-1}(y) \neq \emptyset$) or ($x = y$).

2- By converse of i-1.

ii- From Propositions 14 and 15.

Proposition 19. *Let (X,a_c) a pretopological space with ac collateral pretopology.*

Let $x \in X$.

Let C(x) greatest strongly connected subspace of (X, a) including x.

If $R^{-1}(x) \neq \emptyset$, $C(x) = \{y \in X / R^{-1}(x) = R^{-1}(y)\}$.

If $R^{-1}(x) = \emptyset$, $C(x) = \{x\}$.

Proof. From Proposition 18.

Example. *Let (X,a_c) a pretopological space with $X = \{a,b,c,d,e,f,g,h,i,j\}$ and a_c collateral pretopology defined according to data in the following table:*

x	R^{-1}(x)	C(x)
a	{e,f,g}	{a,b}
b	{e,f,g}	{a,b}
c	{g,h}	{c,d}
d	{g,h}	{c,d}
e	ø	{e}
f	ø	{f}
g	{i}	{g,h}
h	{i}	{g,h}
i	ø	{ i }
j	{ e }	{ j }

Then, we get 7 greatest strongly connected subspaces (or strongly connected components) or maximal cliques : C(a) = {a,b}, C(d) = {c,d}, C(e) = {e}, C(f) = {f}, C(g) = {g,h}, C(i) = {i}, C(j) = {j}.

8. Application Starting from a Transitive Binary Relationship

Let X a non empty set and R a binary relationship defined on X. The pretopology of successors, noted a_d, is defined by the following pseudoclosure:

$$\forall\ A \subset\ X,\ a_d(A) = \{x \in\ X/\ R(x) \cap A \neq \varnothing\} \cup A$$
$$\text{with } R(x) = \{y \in\ X/\ x\ R\ y\}.$$

The pretopology of antecedents, noted a_a, is defined by:
$\forall$ A $\subset$ X, a_a(A) = {x $\in$ X/ R^{-1}(x) $\cap$ A $\neq \varnothing$} $\cup$ A with R^{-1}(x) = {y $\in$ X/ y R x}.

Let R a transitive binary relationship and (X,a_d) the pretopological space of successors defined from R. In this case, a_d is of a V_s-type and also idempotent (because of transitivity of R).

We also can say that (a_d)' = ((a_a)')' = a_a (see [8]).

Example. *Let (X,a_d) the pretopological space of successors, with X = {a,b,c,d,e,f,g}, defined according to data in the following table:*

x	R(x)
a	{a,b,c,d}
b	{a,b,c,d}
c	{a,b,c,d}
d	{a,b,c,d}
e	{a,b,c,d,e}
f	{f,g}
g	{f,g}

$C_1 = \{a,b,c,d\}$, $C_2 = \{e\}$ and $C_3 = \{f,g\}$ are the maximal cliques or also the greatest strongly connected subspaces (or strongly connected components).

$S_1=\{a,b,c,d,e\}$ and $S_2=\{f,g\}$ are the greatest connected subspaces (or connected components).

References

[1] Z. Belmandt, *Manuel de prétopologie et ses applications*, Hermès, Paris (1993).

[2] M. Brissaud, *Chaînes et connexités*, Seminar URA 934, Villeurbanne (1990).

[3] M. Dalud-Vincent, *Modèle prétopologique pour une méthodologie d'analyse de réseaux. Concepts et algorithmes*, Ph.D. Thesis, Lyon 1 University (1994).

[4] M. Dalud-Vincent, M. Brissaud, M. Lamure, Pretopology as an extension of graph theory: the case of strong connectivity, *International Journal of Applied Mathematics*, **5**, N° 4 (2001), 455-472.

[5] M. Dalud-Vincent, M. Brissaud, M. Lamure, Closed sets and closures in pretopology, *International Journal of Pure and Applied Mathematics*, **50**, No. 3 (2009), 391-402.

[6] M. Dalud-Vincent, M. Brissaud, M. Lamure, Pretopology, Matroïdes and Hypergraphs, *International Journal of Pure and Applied Mathematics*, **67**, No. 4 (2011), 363-375.

[7] M. Dalud-Vincent, M. Brissaud, M. Lamure, Connectivities and Partitions in a Pretopological Space, *International Mathematical Forum*, **6**, No. 45 (2011), 2201-2215.

[8] M. Dalud-Vincent, M. Lamure, Connectivities for a Pretopology and its inverse, *International Journal of Pure and Applied Mathematics*, **86**, No. 1 (2013), 43-54. http://dx.doi.org/10.12732/ijpam.v86i1.5.

[9] G. Duru, *Contribution à l'étude des structures des systèmes complexes dans les sciences humaines*, Thèse d'Etat, Lyon I (1980).