A FIXED POINT THEOREM FOR VOLUME PRESERVING LINEAR TRANSFORMATIONS

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Abstract: In this note we derive consequences of the fact that if $g \in \mathrm{SL}(n)$, where $n \geq 2$, and $\sigma_i(g)$ are the coefficients of its characteristic polynomial, then g has 1 as an eigenvalue if and only if $\sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(g) = 0$ or 2 according to the parity of n . These are:

Corollary 2. Let D be a real or complex $n \times n$ matrix of $\mathrm{tr}(D) = 0$. If D is singular, then one of the following equations holds (according to the parity of n).

$$\sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(\mathrm{Exp} D) = 0 \quad \sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(\mathrm{Exp} D) = 2.$$

Conversely, if D has no eigenvalues in $2\pi i\mathbb{Z}$ and the appropriate one of these equations holds, D must be singular.

Corollary 3. Let $A \in M_n(\mathbb{R}) = \mathcal{M}$ which acts on $\mathcal{M}$ by $\mathrm{ad}_A(X) = [A, X]$. If A has only non-negative eigenvalues and one of the following equations holds (according to the parity of n)

$$\sum_{i=1}^{n^2-1} (-1)^{i-1} \sigma_i(\mathrm{Ad}_{\mathrm{Exp}(A)}) = 0 \quad \sum_{i=1}^{n^2-1} (-1)^{i-1} \sigma_i(\mathrm{Ad}_{\mathrm{Exp}(A)}) = 2,$$

then $[A, X] = 0$ for some $X \in \mathcal{M}$, not a linear combination of A and I .

Corollary 4. Let $A \in \mathcal{M}$ and have all non-negative eigenvalues. Then the action of $\mathrm{Exp}(A)$ on $\mathcal{M}$ by conjugation has a fixed point which is not a linear combination of A and I .

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Let K be a field, $g \in \text{GL}(n, K)$, where $n \geq 2$ and

$$\chi_g(x) = x^n - \sigma_1(g)x^{n-1} + \sigma_2(g)x^{n-2} - \dots + (-1)^n \det(g)$$

be its characteristic polynomial. Since g is invertible, it not only acts on K^n , but also on $K^n \setminus (0)$. Here we shall be interested in when g has a fixed point under this action, or equivalently when g has 1 as an eigenvalue. We consider the map $\text{GL}(n, K) \rightarrow K^{n-1}$ given by $g \mapsto (\sigma_1(g), \dots, \sigma_{n-1}(g))$. For this to occur the image of g under this map must lie in a certain hyperplane and conversely. We consider the following equation.

$$\sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(g) = 1 + (-1)^n \det(g). \quad (1)$$

When $g \in \text{SL}(n, K)$ equation 1 becomes

$$\sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(g) = 1 + (-1)^n. \quad (2)$$

Theorem 1. *Let $g \in \text{GL}(n, K)$. Equation 1 holds, if and only if g has a fixed point in this action. (Here the parity of n doesn't matter.) Hence when $g \in \text{SL}(n, K)$, g has a fixed point if and only if equation 2 holds. That is, $\sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(g) = 0$ or 2 according to the parity of n .*

Proof. If $g \in \text{GL}(n, K)$ and the equation holds, then evidently $\chi_g(1) = 0$ so 1 is an eigenvalue and conversely. $\square$

When $g \in \text{SL}(n, K)$ this result seems remarkably similar to the fact that the Euler characteristic, $\chi(S^n)$, of the n sphere is $1 + (-1)^n$ expressing the fact that the Euler characteristic based on chains is the same as that based on homology. We illustrate it with some examples.

- Let $g \in \text{SL}(2, K)$, where $K = \mathbb{R}$ or $\mathbb{C}$ acting as usual on K^2 . Then g has a fixed point if and only if $\text{tr}(g) = 2$. This means g has 1 as an eigenvalue with multiplicity 2 and so is conjugate to a unitriangular matrix and conversely.
- Let $G = \text{SO}(3, \mathbb{R})$. By Euler's theorem every $g \in G$ has a rotation axis and eigenvalue 1. Hence $\text{tr}(g) = \sigma_2(g)$ for all $g \in G$.

- Let $A = (a_{i,j})$ be a stochastic matrix of order n (all $a_{i,j} \geq 0$ and $\sum_j a_{i,j} = 1$ for all i). Then the theorem of Perron-Frobenius (see [2] pg. 17-18) tells us, among other things, that A has 1 as an eigenvalue. Hence equation 1 holds (whether A is invertible, or not).
- One can construct a large number of matrices in $\text{SL}(3, \mathbb{Z})$ all of which have 1 as an eigenvalue. For $g \in \text{SL}(3, \mathbb{Z})$, $\chi_g(x)$ has its coefficients in $\mathbb{Z}$ and, 1 is an eigenvalue if and only if $\text{tr}(g) = \sigma_2(g)$. Since, as we shall see below, $\sigma_2(g) \neq 0$, this means $\frac{\text{tr}(g)}{\sigma_2(g)} = 1$. Let

$$g = \begin{bmatrix} 0 & 0 & 1 \\ 1 & m & p \\ 0 & 1 & n \end{bmatrix}$$

where n , m and p are integers. Then $\chi_g(x) = x^3 - (n+m)x^2 + (mn-p)x - 1$. Thus $\det(g) = 1$, $\text{tr}(g) = n + m$ and $\sigma_2(g) = mn - p$. We consider Pythagorean triples of integers, $r^2 = s^2 + t^2$, and their multiples by $\nu = 1, 2, \dots$, $\nu^2 r^2 = \nu^2 s^2 + \nu^2 t^2$. Let r and ν be chosen so that $\text{tr}(g) = 2\nu r$ and $\sigma_2(g) = \nu^2 s^2 - p$. We choose p so that $\text{tr}(g)/\sigma_2(g) = 1$. Since $\frac{\text{tr}(g)}{\sigma_2(g)} = \frac{2\nu r}{\nu^2 s^2 - p}$, this will be 1 exactly when $p = \nu^2 s^2 - 2\nu r$. This determines the integer p . Hence $2\nu r = \sigma_2(A) = mn - p = mn - (\nu^2 s^2 - 2\nu r)$. Hence $mn = \nu^2 s^2 = \nu^2 r^2 - \nu^2 t^2 = (\nu r + \nu t)(\nu r - \nu t)$. Taking $m = \nu r + \nu t$ and $n = \nu r - \nu t$ (or vice versa), g has entries $m = \nu r + \nu t$ $n = \nu r - \nu t$ $p = \nu^2 s^2 - 2\nu r$ (and $\text{tr}(A) = m + n = 2\nu r$). Thus for each multiple of a Pythagorean triple we have a solution and there are infinitely many Pythagorean triples as well.

Our theorem has the following corollary:

Corollary 2. *Let D be a real or complex $n \times n$ matrix of $\text{tr}(D) = 0$. If D is singular, then one of the following equations holds (depending on whether n is odd or even).*

$$\sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(\text{Exp } D) = 0 \quad \sum_{i=1}^{n-1} (-1)^{i-1} \sigma_i(\text{Exp } D) = 2.$$

Conversely, if D has no eigenvalues in $2\pi i\mathbb{Z}$ and the appropriate one of these equations holds, D must be singular.

In particular, suppose $\mathfrak{g}$ is a real or complex Lie algebra and D is a derivation of trace 0. Let $\alpha = \text{Exp}(D)$ be the corresponding automorphism. If one of the equations above holds for α and α has no non-trivial eigenvalues which are roots of unity, then the derivation D is singular.¹

Proof. Proof. Because $\text{tr}(D) = 0$, it follows that $\det(\text{Exp } D) = e^{\text{tr}(D)} = 1$ so we can apply the theorem for $\text{SL}(n, K)$. If D is singular, then $D(v) = 0$ for some $v \neq 0$. Hence $\text{Exp } D(v) = v$. Since $\text{Exp } D$ has v as a fixed point and determinant 1, one or the other of these two equations holds. Conversely, if equation 2 holds and therefore by Theorem 1 $\text{Exp } D(v) = v$ for some $v \neq 0$, then $\frac{\text{Exp } D - I}{D}(D(v)) = 0$. By our assumption on the eigenvalues of D , it follows that $\frac{\text{Exp } D - I}{D}$ is invertible and hence $D(v) = 0$. $\square$

Now let $A \in M_n(\mathbb{R}) = \mathcal{M}$ and have it act on $\mathcal{M}$ by $\text{ad}_A(X) = [A, X]$. When X is in the linear span of I and A , $\text{ad}_A(X)$ is automatically 0. Here we shall be interested in *non-trivial* such X .²

Taking the matrix units $E_{i,j}$ as a basis for $\mathcal{M}$ we see

$$\text{tr}(\text{ad}_A) = \sum_{i=1}^n \text{ad}_A(E_{i,i}) = \text{ad}_A\left(\sum_{i=1}^n E_{i,i}\right) = \text{ad}_A(I) = 0.$$

Let

$$\mathcal{X} = \{X \in \mathcal{M} : \text{tr}(AX) = 0\}.$$

Then $\mathcal{X}$ is a subspace of $\mathcal{M}$ invariant under ad_A . This is because $\text{tr}(A[A, X]) = \text{tr}(A^2X) - \text{tr}(AXA)$ and since $\text{tr}(YZ) = \text{tr}(ZY)$ for all Y and Z , this is zero (whether $X \in \mathcal{X}$, or not).

Assume $\text{tr}(A^2) \neq 0$. Then

$$\mathcal{A} \oplus \mathcal{X} = \mathcal{M},$$

where $\mathcal{A}$ is the line through A . $\mathcal{A} \cap \mathcal{X} = (0)$ because if $\text{tr}(A\lambda A) = 0$, then $\lambda \text{tr}(A^2) = 0$ and so $\lambda = 0$. The sum of these subspaces is $\mathcal{M}$ since if $X \in \mathcal{M}$, there is a real λ with $\text{tr}((X - \lambda A)A) = 0$. For this we take

$$\lambda = \frac{\text{tr}(XA)}{\text{tr}(A^2)}.$$

¹Since α has no eigenvalues which are roots of unity, a theorem of N. Jacobson tells us $\mathfrak{g}$ is actually nilpotent. See [2], Theorem 5.4.2.

²It is a pleasure for the author to acknowledge a number of useful conversations on this material with Laura Geatti.

Choosing a basis for $\mathcal{X}$ and, by adding A , extending this to a basis of $\mathcal{M}$ we see

$$\mathrm{ad}_A = \left(\begin{array}{ccc|c} \mathrm{ad}_A|_{\mathcal{X}} & & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right). \quad (3)$$

Hence $\mathrm{Spec}(\mathrm{ad}_A) = \mathrm{Spec}(\mathrm{ad}_A|_{\mathcal{X}}) \cup (0)$ so since $\mathrm{tr}(\mathrm{ad}_A) = 0$, $\mathrm{tr}(\mathrm{ad}_A|_{\mathcal{X}})$ is also 0. Moreover, exponentiating equation 3 we get,

$$\mathrm{Exp}(\mathrm{ad}_A) = \left(\begin{array}{ccc|c} \mathrm{Exp}(\mathrm{ad}_A)|_{\mathcal{X}} & & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right). \quad (4)$$

Now since A has all real eigenvalues, it can be put in triangular form over $\mathbb{R}$. Hence $\mathrm{tr}(A^2)$ is the sum of the squares of its eigenvalues. This means $\mathrm{tr}(A^2) > 0$ unless all eigenvalues are zero, i.e. A is nilpotent.

We recall that $\mathrm{Exp} \mathrm{ad}_A = \mathrm{Ad}_{\mathrm{Exp}(A)}$ and first assume A is not nilpotent. In [1] Section 3.3 an equivariant isomorphism ϕ is established,

$$\phi : \mathcal{M} \rightarrow \mathbb{R}^n \otimes \mathbb{R}^n,$$

with $\mathrm{ad}_A = \phi^{-1}(A \otimes I - I \otimes A^t)\phi$. Because $\mathrm{tr}(A) = \mathrm{tr}(A^t)$, it follows that $\mathrm{tr}(\mathrm{ad}_A) = \mathrm{tr}(A)n - n \mathrm{tr}(A^t) = 0$. Since A has only real eigenvalues, the eigenvalues of ad_A acting on $\mathcal{M}$ being differences of the eigenvalues of A , are also real. Because $\mathcal{X}$ is invariant under ad_A , the eigenvalues of its restrictions to $\mathcal{X}$ being a subset of these eigenvalues are also real and so $\mathrm{ad}_A|_{\mathcal{X}}$ has no purely imaginary eigenvalues at all. Since ad_A has trace zero, it follows from equation 3, $\mathrm{tr}(\mathrm{ad}_A|_{\mathcal{X}}) = 0$.

Moreover, from the Jordan form of A one sees easily that in each Jordan block $\mathrm{Exp}(A)$ is a product of a scalar matrix and a unipotent matrix. Hence on each of these subspaces $\mathrm{Ad}_{\mathrm{Exp}(A)}$ acts (by conjugation) as a unipotent matrix. But the eigenvalues of this action by conjugation are quotients of the eigenvalues of this unipotent matrix. Hence all the eigenvalues on each block are all 1 and so $\mathrm{Ad}_{\mathrm{Exp}(A)} = \mathrm{Exp}(\mathrm{ad}_A)$ is itself unipotent. Equation 4 then tells us $\mathrm{Ad}_{\mathrm{Exp}(A)}|_{\mathcal{X}} = \mathrm{Exp}(\mathrm{ad}_A)|_{\mathcal{X}}$ is unipotent as well. By the previous corollary there is a non-zero $X \in \mathcal{X}$ with $[A, X] = 0$. On the other hand, if A were nilpotent, since the eigenvalues of ad_A are differences of those of A , ad_A is a nilpotent operator on $\mathcal{M}$ and so $\mathrm{ad}_A^j = 0$ for some integer $j \geq 1$. If $j \geq 2$,

$\text{ad}_A(X) = 0$ for all $X \in \text{ad}_A^{j-1}$. Thus $[A, X] = 0$ for all these X . If $j = 1$, then $[A, X] = 0$ for all $X \in \mathcal{M}$.

This proves that if $A \in \mathcal{M}$ has only real eigenvalues and one of the following equations holds (according to whether n is odd or even).

$$\sum_{i=1}^{n^2-1} (-1)^{i-1} \sigma_i(\text{Ad}_{\text{Exp}(A)}) = 0 \quad \sum_{i=1}^{n^2-1} (-1)^{i-1} \sigma_i(\text{Ad}_{\text{Exp}(A)}) = 2,$$

then $[A, X] = 0$ for some $X \in \mathcal{M} \setminus \mathcal{A}$.

We now sharpen this result to obtain non trivial $X \in \text{Ker}(\text{ad}_A)$. For this we shall have to assume the eigenvalues of A are not merely real, but actually *non-negative*.

Let $\mathcal{B} = \text{lin.sp.}_{\mathbb{R}}\{A, I\}$. If A and I are linearly dependent, the statement of Corollary 3 below follows from the above so we now assume A and I are linearly independent. Consider $I + \lambda_* A = Y_*$. We choose λ_* so that $Y_* \in \mathcal{X} \cap \mathcal{B}$. Since this is in $\mathcal{B}$ for any λ_* , we need

$$0 = \text{tr}((I + \lambda_* A)A) = \text{tr}(A) + \lambda_* \text{tr}(A^2).$$

That is, $\lambda_* = -\text{tr}(A)/\text{tr}(A^2)$ and, as above, this can be done as long as A is not nilpotent. Then $Y_* \in \mathcal{X} \cap \mathcal{B}$ and is non-zero since A and I are linearly independent. Call $\mathcal{Y}_*$ the 1 dimensional subspace of $\mathcal{X}$ generated by Y_* . Evidently, $\text{ad}_A(Y_*) = 0$ and in a similar manner to the above we will further decompose $\mathcal{X}$ into a direct sum of ad_A -invariant spaces $\mathcal{X} = \mathcal{Y}_* \oplus \mathcal{X}_*$, where $\mathcal{X}_* = \{X \in \mathcal{X} : \text{tr}(X) = 0\}$. This is so since $\text{tr}(X(I + \lambda_* A)) = 0$ if and only if $\text{tr}(X) + \lambda_* \text{tr}(XA) = 0$ and since $X \in \mathcal{X}$, $\text{tr}(XA) = 0$ automatically.

Now these spaces are disjoint because if $I + \lambda_* A$ had trace zero, then $n + \lambda_* \text{tr}(A) = 0$ so $\lambda_* = -n/\text{tr}(A)$ and hence

$$n = \text{tr}(A)^2/\text{tr}(A^2), \tag{5}$$

the denominator being positive since A is not nilpotent. Taking a_i to be the non-negative eigenvalues of A and applying the Cauchy-Schwarz inequality to the vectors $(a_1, \dots, a_n)$ and $(1, \dots, 1)$, we see

$$\sum_i a_i \leq \sqrt{n}(\sum_i a_i^2)^{1/2}.$$

But from equation 5 it follows that $\sum_i a_i \geq \sqrt{n}(\sum_i a_i^2)^{1/2}$. Hence these quantities are equal. By the case of equality of Cauchy-Schwarz it follows that all

the a_i are equal so A is a scalar and hence A and I are linearly dependent, a contradiction. Thus we get a direct sum decomposition of ad_A -invariant subspaces,

$$\mathcal{M} = \mathcal{B} \oplus \mathcal{X}_*.$$

Since $\text{ad}_A|_{\mathcal{X}_*}$ must have trace zero and $\text{Exp}(\text{ad}_A)$ has all its eigenvalues equal to 1 on $\mathcal{X}_*$, the same is true when this operator is restricted to $\mathcal{X}_*$. Applying Corollary 2 (assuming A is not nilpotent) we see

Corollary 3. *Suppose $A \in \mathcal{M}$ has only non-negative eigenvalues and one of the following equations holds (according to whether n is odd or even)*

$$\sum_{i=1}^{n^2-1} (-1)^{i-1} \sigma_i(\text{Ad}_{\text{Exp}(A)}) = 0 \quad \sum_{i=1}^{n^2-1} (-1)^{i-1} \sigma_i(\text{Ad}_{\text{Exp}(A)}) = 2.$$

Then $[A, X] = 0$ for some $X \in \mathcal{M} \setminus \mathcal{B}$.

On the other hand if A is nilpotent then, as above, ad_A is nilpotent and remains so when restricted to any invariant subspace. The argument then proceeds just as before.

Denote by $\mathfrak{t}(n, \mathbb{R})$ the real triangular matrices of order n and $T(n, \mathbb{R})$ those with positive diagonal entries. We first observe

$$\text{Exp} : \mathfrak{t}(n, \mathbb{R}) \rightarrow T(n, \mathbb{R})$$

is a global diffeomorphism because $T(n, \mathbb{R})$ is a connected and simply connected real solvable Lie group with all real Ad roots. Now let $A \in M_n(n, \mathbb{R})$ and assume it has all real eigenvalues. Then, as above, A can be put in triangular form over $\mathbb{R}$ as can $\text{Exp}(A)$, which has all its eigenvalues positive. Since $\text{Ad}_{\text{Exp}(A)} = \text{Exp}(\text{ad}_A)$ and Exp is global diffeomorphism, the action of conjugation by $\text{Exp}(A)$ on $\mathcal{M}$ is equivariantly equivalent to the action of ad_A on $\mathcal{M}$. Now assume further that the eigenvalues of A are actually non-negative. Then by Corollary 3 above there is an $X \in \mathcal{M} \setminus \mathcal{B}$ so that $[A, X] = 0$. Therefore X also commutes with $\text{Exp}(A)$. This yields,

Corollary 4. *Let $A \in \mathcal{M}$ and have all non-negative eigenvalues. Then the action of $\text{Exp}(A)$ on $\mathcal{M}$ by conjugation has a non-trivial fixed point.*

References

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