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SOME FIXED POINT THEOREMS FOR CYCLIC (α, β, ψ)-CONTRACTIVE MAPPING IN PROBABILISTIC MENGER SPACE

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Abstract: In this paper, we define the concepts of (α, β, ψ)-contractive and cyclic (α, β, ψ)-contractive mapping in probabilistic Menger space. We prove some fixed point theorems for such mapping. Some examples are given to support the obtained results.

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Key Words: continuous t-norm, probabilistic Menger space, cyclic representation, contractive mapping

1. Introduction and Preliminaries

The study of fixed points of mappings in a probabilistic metric space satisfying certain contractive conditions has been at the center of vigorous research activity. Probabilistic metric spaces were introduced in 1942 by Menger [7]. Sehgal and Bharuch-Reid in 1972 followed a generalization of Banach contraction principle on a complete Menger space [10]. Altering distance functions have been used by many authors in a number of works. An "altering distance

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function" is a control function which alters the distances between two points in a metric space. This concept was introduced by Khan, Swaleh and Sessa [6]. Some of the other works in this line of research are noted in [13], [14] and [15]. Recently "altering distance functions" have been extended in the context of Menger space by Choudhury and Das in [1]. This extension of altering distance functions, has been further used by many outhors [2] and [8].

The concept of cyclic contractive mapping was initiated by Kirk, Srinivasan and Veeramani [12]. Sahni Mary Roosevelt proved some fixed point theorems for cyclic weak ϕ-contraction in Menger Space [16]. S.M. Roosevelt defined cyclic weak (ϕ,ψ)-contraction in [17].

The main object of this paper is to study (α,β,ψ)-contractive and cyclic (α,β,ψ)-contractive mapping in Menger space. We obtain the unique fixed point theorem for such mappings in Menger space by altering distances between points. Our result generalizes and improves the previous results in fixed point. We also give some examples to illustrate our main theorems.

We first bring notion, definitions and known results, which are related to our work. For more details, we refer the reader to [5]. We denote by $\mathbb{R}$ the set of real numbers.

Definition 1. A distribution function is a function $F:(-\infty,\infty)\to[0,1]$, that is non-decreasing and left continuous on $\mathbb{R}$, moreover, $\inf_{t\in\mathbb{R}} F(t)=0$ and $\sup_{t\in\mathbb{R}} F(t)=1$.

The set of all the distribution functions is denoted by D, and the set of those distribution functions such that $F(0)=0$ is denoted by D^+. We will denote the specific distribution function by

$$H(t)=\begin{cases}1 & \text{if } t>0\\ 0 & \text{if } t\le 0.\end{cases}$$

Definition 2. A probabilistic metric space (briefly, PM-space) is an ordered pair (X,F), where X is a nonempty set and F is a mapping from $X\times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x,y), the following conditions hold:

(PM1) $F_{x,y}(t)=H(t)$ if and only if $x=y$,

(PM2) $F_{x,y}(t)=F_{y,x}(t)$,

(PM3) If $F_{x,y}(t)=1$ and $F_{y,z}(s)=1$, then $F_{x,z}(t+s)=1$, for all $x,y,z\in X$ and $s,t\ge 0$.

Definition 3. A binary operation $T:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if the following conditions hold:

(a) T is commutative and associative,

(b) T is continuous,

(c) $T(a,1) = a$ for all $a \in [0,1]$,

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$, for $a,b,c,d \in [0,1]$.

The following are three basic continuous t-norms.

(i) The minimum t-norm, say T_M, defined by $T_M(a,b) = \min\{a,b\}$.

(ii) The product t-norm, say T_p defined by $T_p(a,b) = a.b$.

(iii) The Lukasiewicz t-norm, say T_L, defined by $T_L(a,b) = \max\{a+b-1,0\}$.

These t- norms are related in the following way: $T_L \leq T_P \leq T_M$.

Definition 4. A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (X,F,T), where (X,F) is a PM-space and T is a continuous t-norm such that for all $x,y,z \in X$ and $s,t \geq 0$

$$F_{x,y}(t+s) \geq T(F_{x,z}(t), F_{z,y}(s)).$$

Definition 5. Let (X,F,T) be a Menger PM-space. Then

(i) A sequence x_n in X is said to be converge to x if, for every $\epsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer N such that $F_{x_n x}(\epsilon) > 1-\lambda$, whenever $n \geq N$.

(ii) A sequence x_n in X is called Cauchy sequence if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N scuh that $F_{x_n x_m}(\epsilon) > 1-\lambda$ whenever $n,m \geq N$.

(iii) A Menger space is said to be complete if and only if every Cauchy sequence in X is converge to a point in X .

(iv) A sequence x_n is called G-Cauchy if $\lim_{n\to\infty} F_{x_n,x_{n+m}}(t) = 1$, for each $m \in \mathbb{N}$ and $t > 0$.

(v) The space (X,F,T) is called G-complete if every G-Cauchy sequence in X is convergent.

It follows immediately that a Cauchy sequence is a G-Cauchy sequence. The convers is not always true. This has been established by an example in [11].

According to [9], the (ϵ,λ)-topology in Menger PM-space (X,F,T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by

$$N_x = \{N_x(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1)\},$$

where

$$N_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

The (ϵ, λ) -topology is a Hausdorff topology. In this topology, a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$, for every sequence $x_n \to x_0$.

Φ-functions in Menger PM-space introduced by Choudhury and Das in [1].

Definition 6. A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,

(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,

(iii) ϕ is left continuous in $(0, \infty)$,

(iv) ϕ is continuous at 0.

In the sequel, the class of all Φ-functions will be denoted by Φ. Also we denote by Ψ the class of all continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$, whenever $a_n \to 0$ as $n \to \infty$.

In this paper, we give a generalization of the concept $\alpha-\psi$ contractive mapping defined in [4] and introduce the notions of (α, β, ψ)- contractive and cyclic (α, β, ψ)- contractive mapping. Also we will prove some fixed point theorems for such contractive mapping.

The following fixed point theorem proved by Dutta et al, in [3].

Theorem 7. *Let (X, F, T) be a G-complete Menger PM-space and $f : X \to X$ be a mapping satisfying the following inequality*

$$\frac{1}{F_{fx,fy}(\varphi(ct))} - 1 \le \psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \tag{1}$$

where $x, y \in X, c \in (0, 1)$, $\varphi \in \Phi, \psi \in \Psi$ and $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$. Then f has a unique fixed point.

The notion of $\alpha - \psi$-type contractive mapping introduced by D. Gopal in [4].

Definition 8. Let (X, F, T) be a Menger PM-space and $f : X \to X$ be a given mapping. We say that f is an $\alpha - \psi$-type contractive mapping if there exist two functions $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ and $\psi \in \Psi$ satisfying the following inequality

$$\alpha(x, y, t)\left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1\right) \le \psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \tag{2}$$

for all $x, y \in X$ and for all $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$, where $c \in (0, 1)$ and $\varphi \in \Phi$.

2. Fixed Point Theorems for Cyclic (α, β, ψ)-Contractive Mappings

In this section, we introduce the new notion of cyclic (α, β, ψ)-contractive mappings in Menger PM-spaces.

Definition 9. Let (X, F, T) be a Menger PM-space and $f : X \to X$ be a given mapping and $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ be two functions, we say that f is $\alpha - \beta$-admissible if

(i) $x, y \in X$, for all $t > 0$, $\alpha(x, y, t) \geq 1 \Rightarrow \alpha(fx, fy, t) \geq 1$,

(ii) $x, y \in X$, for all $t > 0$, $\beta(x, y, t) \leq 1 \Rightarrow \beta(fx, fy, t) \leq 1$.

Definition 10. Let X be a non-empty set, m be a positive integer and $f : X \to X$ is an operator. Then $X = \cup_1^m X_i$ is a cyclic representation of X with respect to f if

(i) $X_i,\ i = 1, 2, ..., m$ are non-empty sets.

(ii) $f(X_1) \subset X_2, f(X_2) \subset X_3, ..., f(X_{m-1}) \subset X_m$ *and* $f(X_m) \subset X_1$.

Definition 11. Let (X, F, T) be a Menger PM-space, $A_1, A_2, ..., A_m$ be closed subsets of X and $Y = \cup_1^m A_i$. An operator $f : X \to X$ is called a cyclic (α, β, ψ)-contractive if the following conditions hold:

(i) $Y = \cup_1^m A_i$ is a cyclic representation of Y with respect to f.

(ii) There exist two functions $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ and $\psi \in \Psi$ satisfying the following inequality

$$\alpha(x, y, t)\left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1\right) \leq \beta(x, y, t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right), \tag{3}$$

for all $x \in A_i, y \in A_{i+1}$ ($i = 1, 2, ..., m$ *and* $A_{m+1} = A_1$) and for all $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$, where $c \in (0, 1)$ and $\varphi \in \Phi$.

Theorem 12. *Let (X, F, T) be a Menger PM-space, $A_1, A_2, ..., A_m$ be closed subsets of $X$and $Y = \cup_1^m A_i$ is G-complete. If the following conditions hold:*

(i) $f : Y \to Y$ is $\alpha - \beta$-admissible.

(ii) There exists $x_0 \in Y$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$, for all $t > 0$.

(iii) f is a cyclic (α, β, ψ)-contractive.

Then f has a fixed point $y^* \in \cap_1^m A_i$.

Proof. Let $x_0 \in Y = \cup_1^m A_i$ be such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Define a sequance $\{x_n\}$ such that $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $y = x_n$ is a fixed point of f. Assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then, by using the fact that f is $\alpha - \beta$-admissible, we write

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Rightarrow \alpha(x_1, x_2, t) = \alpha(fx_0, fx_1, t) \geq 1.$$

Similarly we write

$$\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \leq 1 \Rightarrow \beta(x_1, x_2, t) = \beta(fx_0, fx_1, t) \leq 1.$$

By induction, it follows that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$, for all $t > 0$. For any $n \geq 0$, there exists $i_n \in \{1, 2, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$.

From the properties of function φ, we can find $t > 0$ such that $F_{x_0 x_1}(\varphi(t)) > 0$. Thuse by applying (3), we have

$$\begin{aligned}
\frac{1}{F_{x_1,x_2}(\varphi(ct))} - 1 &= \frac{1}{F_{fx_0,fx_1}(\varphi(ct))} - 1 \\
&\leq \alpha(x_0, x_1, t)\left(\frac{1}{F_{fx_0,fx_1}(\varphi(ct))} - 1\right) \\
&\leq \beta(x_0, x_1, t)\psi\left(\frac{1}{F_{x_0,x_1}(\varphi(t))} - 1\right) \\
&\leq \psi\left(\frac{1}{F_{x_0,x_1}(\varphi(t))} - 1\right). \qquad (4)
\end{aligned}$$

Again $F_{x_0,x_1}(\varphi(t)) > 0$ implies $F_{x_0,x_1}\left(\varphi(\frac{t}{c})\right) > 0$. So again by applying (3), we have

$$\begin{aligned}
\frac{1}{F_{x_1,x_2}(\varphi(t))} - 1 &= \frac{1}{F_{fx_0,fx_1}(\varphi(t))} - 1 \\
&\leq \alpha(x_0, x_1, t)\left(\frac{1}{F_{fx_0,fx_1}(\varphi(t))} - 1\right) \\
&\leq \beta(x_0, x_1, t)\psi\left(\frac{1}{F_{x_0,x_1}(\varphi(\frac{t}{c}))} - 1\right) \\
&\leq \psi\left(\frac{1}{F_{x_0,x_1}(\varphi(\frac{t}{c}))} - 1\right).
\end{aligned}$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{F_{x_n,x_{n+1}}(\varphi(t))} - 1 \leq \psi^n\left(\frac{1}{F_{x_0,x_1}(\varphi(\frac{t}{c^n}))} - 1\right).$$

Also, (4) implies that $F_{x_1,x_2}(\varphi(ct)) > 0$. Using the similar argument, we get

$$\frac{1}{F_{x_n,x_{n+1}}(\varphi(ct))} - 1 \leq \psi^{n-1}\left(\frac{1}{F_{x_1,x_2}(\varphi(\frac{ct}{c^{n-1}}))} - 1\right).$$

Rewrite this sentence for $n > r$,

$$\frac{1}{F_{x_n,x_{n+1}}(\varphi(c^r t))} - 1 \leq \psi^{n-r}\left(\frac{1}{F_{x_n,x_{n+1}}(\varphi(\frac{c^r t}{c^{n-r}}))} - 1\right). \tag{5}$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$, we have from (5), for all $r > 0$

$$F_{x_n,x_{n+1}}(\varphi(c^r t)) \to 1. \tag{6}$$

Now let $\epsilon > 0$ be given, then by virtue of the properties of φ, we can find $r > 0$ such that $\varphi(c^r t) < \epsilon$. Then it follows from (6) that $F_{x_n,x_{n+1}}(\epsilon) \to 1$, as $n \to \infty$ for every $\epsilon > 0$. On the other hand, we know that

$$F_{x_n,x_{n+p}}(\epsilon) \geq T(F_{x_n,x_{n+1}}(\frac{\epsilon}{p}), T(F_{x_{n+1},x_{n+2}}(\frac{\epsilon}{p}), ..., (F_{x_{n+p-1},x_{n+p}}(\frac{\epsilon}{p}))...).$$

Thus, letting $n \to \infty$, we have for any integer p, $F_{x_n,x_{n+p}}(\epsilon) \to 1$, as $n \to \infty$ for every $\epsilon > 0$. Hence $\{x_n\}$ is a G-cauchy sequence. As Y is G-complete, there exists $y^* \in Y$ such that $lim_{n\to\infty} x_n = y^*$. On the other hand by (3), it follows that the iterative sequence, sequence x_n has an infinitive number of terms in A_i, for each $i = 1, 2, ..., m$. From each $A_i, i = 1, 2, ..., m$, one can extract a subsequence of x_n that converges to y^*. By virtue of the fact that each $A_i, i = 1, 2, ..., m$ is closed, we conclude that $y^* \in \cap_1^m A_i$. Now fix $i \in \{1, 2, ..., m\}$ such that $y^* \in A_i$ and $fy^* \in A_{i+1}$. We take a subsequence x_{n_k} of x_n with $x_{n_k} \in A_{i-1}$. Now,

$$\begin{aligned} F_{y^*,fy^*}(t) &\geq T(F_{y^*,x_{n_k+1}}(\frac{t}{2}), F_{x_{n_k+1},fy^*}(\frac{t}{2})) \\ &= T(F_{y^*,x_{n_k+1}}(\frac{t}{2}), F_{fx_{n_k},fy^*}(\frac{t}{2})). \end{aligned}$$

As $k \to \infty$, $F_{y^*,fy^*}(t) \geq 1$ and so $fy^* = y^*$. Thus $y^* \in \cap_1^m A_i$ is a fixed point of f. □

Corollary 13. *Let (X, F, T) be a Menger PM-space, $A_1, A_2, ..., A_m$ be closed subsets of X and $Y = \cup_1^m A_i$ is G-complete. If $f : X \to X$ be a mapping satisfying the following inequality*

$$\frac{1}{F_{fx,fy}(\varphi(ct))} - 1 \leq \psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right),$$

for all $x \in A_i, y \in A_{i+1}$ $(i = 1, 2, ..., m$ *and* $A_{m+1} = A_1)$ *and for all* $t > 0$ *such that* $F_{x,y}(\varphi(t)) > 0$*, where* $c \in (0, 1)$ *and* $\varphi \in \Phi$*. Then* f *has a fixed point* $y \in \cap_1^m A_i$*.*

Theorem 14. *Let* (X, F, T) *be a Menger PM-space,* $A_1, A_2, ..., A_m$ *be closed subsets of* X *and* $Y = \cup_1^m A_i$ *is G-complete. If the following conditions hold:*

(i) $f : Y \to Y$ *is* $\alpha - \beta$*-admissible.*

(ii) There exists $x_0 \in Y$ *such that* $\alpha(x_0, fx_0, t) \geq 1$ *and* $\beta(x_0, fx_0, t) \leq 1$*, for all* $t > 0$*.*

(iii) f *is a cyclic* (α, β, ψ)*-contractive.*

(iv) For any $x \in A_i, y \in A_{i+1}$ $(i = 1, 2, ..., m$ *and* $A_{m+1} = A_1)$*, there exists* $z \in Y$ *such that* $\alpha(x, z, t) \geq 1$ *and* $\beta(x, z, t) \leq 1$*.*

Then f has a unique fixed point $y^* \in \cap_1^m A_i$*.*

Proof. Let $u, v \in Y$ be such that $fu = u$ and $fv = v$. From hypotheses there exists $z \in Y$ such that $\alpha(u, z, t) \geq 1$ and $\alpha(v, z, t) \geq 1$, $\beta(u, z, t) \leq 1$ and $\beta(v, z, t) \leq 1$. Since f is $\alpha - \beta$-admissible, we get $\alpha(u, f^n z, t) \geq 1$ and $\alpha(v, f^n z, t) \geq 1$, $\beta(u, f^n z, t) \leq 1$ and $\beta(v, f^n z, t) \leq 1$, for all $t > 0$. For any $n \geq 0$, there exists $i_n \in \{1, 2, ..., m\}$ such that $f^{n-1}z \in A_{i_n}$. On the other hand $u \in \cap_1^m A_i$, so by using (3) we obtain

$$\begin{aligned}
\frac{1}{F_{u,f^n z}(\varphi(ct))} - 1 &= \frac{1}{F_{fu,f(f^{n-1}z)}(\varphi(ct))} - 1 \\
&\leq \alpha(u, f^{n-1}z, t)(\frac{1}{F_{fu,f(f^{n-1}z)}(\varphi(ct))} - 1) \\
&\leq \beta(u, f^{n-1}z, t)\ \psi(\frac{1}{F_{u,f^{n-1}z}(\varphi(t))} - 1) \\
&\leq \psi(\frac{1}{F_{u,f^{n-1}z}(\varphi(t))} - 1).
\end{aligned}$$

This implies that

$$(\frac{1}{F_{u,f^n z}(\varphi(ct))} - 1) \leq \psi^n(\frac{1}{F_{u,z}(\varphi(\frac{t}{c^n}))} - 1).$$

Finally, making $n \to \infty$, we obtain $f^n z \to u$. A similar argument shows that $f^n z \to v$ as $n \to \infty$. Now, the uniqueness of the limit gives us $u = v$ and hence the proof is complete. □

Example 1. Let $X = R$, $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$, $t > 0$ and the t-norm T is defined as $T(a, b) = ab$. Then (X, F, T) is a Menger space. Assume $A_1 = A_2 = ... = A_m = [0, 1]$ so that $Y = \cup_1^m A_i = [0, 1]$ is G-complete. We define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \psi(t) = t$. Define the mapping $f : Y \to Y$ by

$$fy = \begin{cases} \frac{1}{4} & y = 1 \\ \frac{y}{4} & y \in [0, 1) \end{cases}$$

and the functions $\alpha, \beta : Y \times Y \times (0, \infty) \to [0, \infty)$ by

$$\alpha(x, y, t) = 2, \ \forall x, y \in Y, \qquad \beta(x, y, t) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ 3 & x \in [\frac{1}{2}, 1]. \end{cases}$$

Suppose both x, y are in $[0, \frac{1}{2})$, then $\alpha(x, y, t) = 2$, $\beta(x, y, t) = 1$, so inequaility (3) holds for all $c \in [\frac{1}{2}, 1)$. If $x, y \in [\frac{1}{2}, 1)$, then $\alpha(x, y, t) = 2$ and $\beta(x, y, t) = 3$, so inequaility (3) holds for all $c \in [\frac{1}{6}, 1)$. For $x = y = 1$ holds trivially. In the same way we can show that for other cases inequaility (3) holds. Now, let $x, y \in Y$ be such that $\alpha(x, y, t) \geq 1$ for all $t > 0$, by definition of α this implies that $\alpha(fx, fy, t) \geq 1$. If $\beta(x, y, t) \leq 1$ for all $t > 0$, this implies that $x \in [0, \frac{1}{2})$ and so by definition of f and β, we have $\beta(fx, fy, t) = 1$, that is, f is a $\alpha - \beta$-admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Indeed for $x_0 = \frac{1}{4}$ we have $\alpha(\frac{1}{4}, f(\frac{1}{4}), t) = 2$ and $\beta(\frac{1}{4}, f(\frac{1}{4}), t) = 1$. Hence we conclude that all the conditions of Theorems 12 and 14 hold and so, f has a unique fixed point $x = 0 \in \cap_1^m A_i$.

Definition 15. Let (X, F, T) be a PM-space and $f : X \to X$ be a given mapping. We say that f is an (α, β, ψ)- contractive mapping if there exist two functions $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ and $\psi \in \Psi$ satisfying the following inequality

$$\alpha(x, y, t)\left(\frac{1}{F_{fx,fy}(\varphi(ct))} - 1\right) \leq \beta(x, y, t)\psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \tag{7}$$

for all $x, y \in X$ and for all $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$, where $c \in (0, 1)$ and $\varphi \in \Phi$.

Remark 1 If $\alpha(x, y, t) = 1$ and $\beta(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$, then condition (7) reduce to condition (1). This implies that a mapping satisfying condition (1) is an (α, β, ψ)- contractive mapping but the converse is not necessarily true, (see Example 2). And if $\beta(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$, then condition (7) reduce to condition (2). This implies that a mapping satisfying condition (2) is an (α, β, ψ)- contractive mapping but the converse is not necessarily true. (see Example 3).

Theorem 16. *Let (X, F, T) be a G-complete Menger PM-space and $f : X \to X$ be a (α, β, ψ) contractive mapping satisfying the following conditions:*

(i) f is $\alpha - \beta$-admissible.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$, for all $t > 0$.

(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, t) \geq 1$ and $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.

Then f has a fixed point, i.e, there exists a point $u \in X$ such that $fu = u$.

Proof. Let $x_0 \in X$ be such that

$$\alpha(x_0, fx_0, t) \geq 1 \text{ and } \beta(x_0, fx_0, t) \leq 1$$

for all $t > 0$. Define a sequance $\{x_n\}$ in X so that $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. By the same method used in the proof of Theorem 12 we can show that $\{x_n\}$ is a G-Cauchy sequences.

As (X, F, T) is G-complete, $\{x_n\}$ is convergent and hence $x_n \to u$ as $n \to \infty$ for some $u \in X$. Again

$$F_{fu,u}(\epsilon) \geq T\left(F_{fu,x_{n+1}}(\frac{\epsilon}{2}), F_{x_{n+1},u}(\frac{\epsilon}{2})\right). \tag{8}$$

Using the properties of φ-function, we can find a $t_2 > 0$ such that $\varphi(t_2) < \frac{\epsilon}{2}$. Hence there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $F_{x_n,u}(\varphi(t_2)) > 0$. Then, for $n > n_0$, we write

$$\begin{aligned}
\frac{1}{F_{x_{n+1},fu}(\frac{\epsilon}{2})} - 1 &\leq \frac{1}{F_{fx_n,fu}(\varphi(t_2))} - 1 \\
&\leq \alpha(x_n, u, \varphi(t_2))\left(\frac{1}{F_{fx_n,fu}(\varphi(t_2))} - 1\right) \\
&\leq \beta(x_n, u, \varphi(t_2))\psi\left(\frac{1}{F_{x_n,u}(\varphi(\frac{t_2}{c}))} - 1\right) \\
&\leq \psi\left(\frac{1}{F_{x_n,u}(\varphi(\frac{t_2}{c}))} - 1\right).
\end{aligned}$$

Making $n \to \infty$, utilizing $\psi(0) = 0$ and continuity of ψ, we obtain $F_{x_{n+1},fu}(\frac{\epsilon}{2}) \to 1$ as $n \to \infty$. Making $n \to \infty$ in (8), by continuity of T and the fact that $x_n \to u$ as $n \to \infty$, we have, $F_{fu,u}(\epsilon) = 1$ for every $\epsilon > 0$. Hence $u = fu$. □

The following Examples show the usefulness of Definition 15.

Example 2. Let $X = \mathbb{R}, T(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x,y \in X$ and for all $t > 0$. Clearly (X,F,T) is a G-complete Menger space. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{x^2}{4} & x \in [0,1] \\ 4 & otherwise \end{cases}$$

and the functions $\alpha, \beta : X \times X \times (0,\infty) \to [0,\infty)$ by

$$\alpha(x,y,t) = \begin{cases} 1 & x,y \in [0,1] \\ 0 & otherwise, \end{cases}$$

$$\beta(x,y,t) = \begin{cases} 1 & x,y \in [0,1] \\ 3 & otherwise. \end{cases}$$

If we define $\varphi, \psi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = \psi(t) = t$, then the mapping f satisfies the hypotheses of theorem 16. To view this, suppose both x,y are in $[0,1]$, then $\alpha(x,y,t) = 1$ and $\beta(x,y,t) = 1$ and so inequality (7) holds for all $c \in [\frac{1}{2}, 1)$. If at least one of x and y is in $\mathbb{R} - [0,1]$, then $\alpha(x,y,t) = 0$ and $\beta(x,y,t) = 3$ and so inequality (7) holds trivially. Now, let $x,y \in X$ be such that $\alpha(x,y,t) \geq 1$ and $\beta(x,y,t) \leq 1$ for all $t > 0$, this implies that $x,y \in [0,1]$ and by definition of f and α and β, we have $fx, fy \in [0,1]$ and $\alpha(fx,fy,t) = 1$ and $\beta(fx,fy,t) = 1$, that is, f is a $\alpha - \beta$-admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ and $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Indeed for $x_0 = 1$ we have $\alpha(1, f(1), t) = 1$ and $\beta(1, f(1), t) = 1$. Next, let $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}, t) \geq 1$ and $\beta(x_n, x_{n+1}, t) \leq 1$, for all $n \in \mathbb{N}$ and for all $t > 0$, and $x_n \to x$ as $n \to \infty$, this implies $x_n, x \in [0,1]$ and so $\alpha(x_n, x, t) \geq 1$ and $\beta(x_n, x, t) \leq 1$, for all $n \in \mathbb{N}$ and for all $t > 0$. Hence we conclude that all the conditions of Theorem 16 hold and so, f has two fixed points $x = 0$ and $x = 4$. We show that Theorem 7 is not applicable in this case, since f does not satify the inequality (1). Consider $x \in [0,1]$ and $y = 4$, then by applying inequality (1) we have $x + 4 \leq 4c$, which gives a contradiction to the fact that $c \in (0,1)$.

Example 3. Let $X = \mathbb{R}, T(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x,y \in X$ and for all $t > 0$. Clearly (X,F,T) is a G-complete Menger space. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{1}{4} & x = \frac{1}{4} \\ \frac{x}{4} & otherwise \end{cases}$$

and the functions $\alpha, \beta : X \times X \times (0, \infty) \to [0, \infty)$ by

$$\alpha(x, y, t) = 4, \ \forall x, y \in X,$$

$$\beta(x, y, t) = \begin{cases} 3 & x, y \in [0, 1), x \neq y \\ 1 & x = y \\ 2 & otherwise. \end{cases}$$

If we define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \psi(t) = t$, then the mapping f satisfies the hypotheses of Theorem 16. To view this, suppose both $x, y \in [0, 1)$ and $x \neq y$, then $\alpha(x, y, t) = 4$ and $\beta(x, y, t) = 3$ and so inequaility (7) holds for all $c \in [\frac{1}{3}, 1)$. If $x = y$, then inequaility (7) holds trivially. If at least one of x and y is not in $[0, 1)$ and $x \neq y$, then $\beta(x, y, t) = 2$ and so inequality (7) holds for all $c \in [\frac{1}{3}, 1)$. In the same way we can show that for other cases inequaility (7) holds.

It is easy to show that f is $\alpha - \beta$-admissible. Also for $x_0 = \frac{1}{4}$, we have $\alpha(\frac{1}{4}, f(\frac{1}{4}), t) = 4$ and $\beta(\frac{1}{4}, f(\frac{1}{4}), t) = 1$. Next, let $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x$ as $n \to \infty$, this implies that $\alpha(x_n, x, t) \geq 1$. If let $\{x_n\}$ be such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$, $t > 0$ and $x_n \to x$ as $n \to \infty$, by definition of β this implies that $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$. The other conditions are the same as Example 2. Hence we conclude that all the conditions of Theorem 16 hold and so, f has two fixed points $x = 0$ and $x = \frac{1}{4}$. We show that inequality (2) is not true in this case. Consider $x, y \in [0, 1)$, and $x \neq y$, then by applying inequality (2) we givea contradiction to the fact that $c \in (0, 1)$.

We prove, with next theorem, the uniqueness of the fixed point.

Theorem 17. *With the same hypotheses of Theorem 16, if for all $x \in X$ and for all $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\beta(x, z, t) \leq 1$, then f has a unique fixed point.*

Proof. Let $u, v \in X$ be such that $fu = u$ and $fv = v$. From hypotheses there exists $z \in X$ such that $\alpha(u, z, t) \geq 1$ and $\alpha(v, z, t) \geq 1$, $\beta(u, z, t) \leq 1$ and $\beta(v, z, t) \leq 1$. By the same method as that of used in the proof of Theoream 14, we can see that $u = v$. □

Example 4. Let $X = \mathbb{R}$, $T(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Cleary (X, F, T) is a G-complete Menger space. We define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \psi(t) = t$. Define the mapping $f : X \to X$, by $f(x) = \frac{x}{4}$, and two functions $\alpha, \beta : X \times X \times$

$(0, \infty) \to [0, \infty)$ by $\alpha(x, y, t) = \beta(x, y, t) = 1$, for all $x, y \in X$ and $t > 0$. Then all the conditions of Theorem 17 hold for $c \in [\frac{1}{4}, 1)$, and so f has a unique fixed point $x = 0$.

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