COMMON COUPLED FIXED POINT THEOREMS OF TWO MAPPINGS SATISFYING GENERALIZED CONTRACTIVE CONDITION IN CONE METRIC SPACE

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Abstract: In this paper, we prove the existence and uniqueness of a common coupled fixed point theorem in complete cone metric space for a pair of mappings satisfying a generalised contractive condition. Some results are also given in the form of corollaries. The results are also verified with the help of example.

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1. Introduction

Dajun Guo and V. Lakshmikantham [1] gave the existence theorems of the coupled fixed points for both continuous and discontinuous operators and gave applications to the initial value problems of ordinary differential equations with discontinuous right-hand sides. Bhaskar and Lakshmikantham [2] proved the

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existence of coupled fixed point theorem for a mixed monotone mapping in a metric space with the help of partial order, using a weak contractivity type of assumption. Since then this new concept is extended and used in various directions. This concept is extended to tripled fixed point by Berinde and Borcut [3] to quadrupled fixed point by Karapinar [4]. Coupled fixed point is also extended in various spaces like metric space, G-metric space, b-metric space, partially ordered metric space, fuzzy metric space, cone metric space etc.

The concept of cone metric space introduced by Huang and Zang [5] in 2007 as generalizations of metric space. They generalised metric space by replacing the set of real numbers with an ordering Banach space. Thus, cone naturally induces a partial order in Banach spaces. In recent years many authors established various coupled fixed point theorems in cone metric space (see [6-17] and references there in).

Let E be a real Banach Space. A subset P of E is called a cone if

1. P is closed, non-empty and $P \neq 0$
2. $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$ we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{int}.P$ (interior of P).

There are two kinds of cone. They are normal cone and non-normal cones. A cone $P \subset E$ is normal if there is a number $K > 0$ such that for all $x, y \in P$, $0 \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$. In other words if $x_n \leq y_n \leq z_n$ and $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = x$ imply $\lim_{n \rightarrow \infty} y_n = x$. Also, a cone $P \subset E$ is regular if every increasing sequence which is bounded above is convergent.

The aim of the present work is to prove the existence and uniqueness of a common coupled fixed point theorem satisfying a generalised contractive condition. Some other results are also given in the form of corollaries.

Definition 1.1. ([5]) Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies the following conditions:

1. $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2. ([5]) Let (X, d) be a cone metric space(CMS), x_n a sequence in X and $x \in X$. For every $c \in E$ with $0 \ll c$, we say that x_n is

1. a Cauchy sequence if there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $d(x_n, x_m) \ll c$;
2. a convergent sequence if there is some $k \in \mathbb{N}$ such that, $n \geq k$, $d(x_n, x) \ll c$. Then x is called limit of the sequence x_n .
3. (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent in X .

Let (X, d) be a cone metric space(CMS). Then the following properties are often used :

1. If E is a real Banach space with a cone P and if $a \preceq ha$ where $a \in P$ and $h \in [0, 1)$, then $a = 0$.
2. If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.
3. If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.
4. If $u \preceq v$ and $v \preceq w$, then $u \ll w$.
5. If $c \in \text{int}.P$, $0 \preceq a_n$ and $a_n \rightarrow \theta$, then there exists an k such that for all $n > k$ we have $a_n \ll c$.

It follows from (e) that the sequence x_n converges to $x \in X$ if $d(x_n, x) \rightarrow \theta$ as $n \rightarrow \infty$ and x_n is a Cauchy sequence if $d(x_n, x_m) \rightarrow \theta$ as $s, m \rightarrow \infty$. In the case when the cone is not necessarily normal, then $d(x_n, y_n) \rightarrow d(x, y)$ if $x_n \rightarrow x$ and $y_n \rightarrow y$ is not applicable.

Definition 1.3. ([2]) An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

2. Main Results

We prove the following theorem:

Theorem 2.1. Let (X, d) be a complete cone metric space with a cone P having non-empty interior and let $S, T : X \times X \rightarrow X$ satisfying

$$d(S(x, y), T(u, v)) \leq a_1 d(x, u) + a_2 d(S(x, y), x) + a_3 d((y, v))$$

$$+a_4d(T(u, v), u) + a_5d(S(x, y), u) + a_6d(T(u, v), x),$$

for all $x, y, u, v \in X$, where $a_i, i = 1, 2, \dots, 6$ are non-negative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1$.

Then S and T have a unique common coupled fixed point in X .

Proof. Let x_0 and y_0 be arbitrary points in X . Let

$$x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k})$$

and

$$x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1})$$

for $k = 0, 1, 2, \dots$.

Then

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ &\leq a_1d(x_{2k}, x_{2k+1}) + a_2d(S(x_{2k}, y_{2k}), x_{2k}) + a_3d(y_{2k}, y_{2k+1}) \\ &\quad + a_4d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5d(S(x_{2k}, y_{2k}), x_{2k+1}) \\ &\quad + a_6d(T(x_{2k+1}, y_{2k+1}), x_{2k}) \\ &= a_1d(x_{2k}, x_{2k+1}) + a_2d(x_{2k+1}, x_{2k}) + a_3d(y_{2k}, y_{2k+1}) \\ &\quad + a_4d(x_{2k+2}, x_{2k+1}) + a_5d(x_{2k+1}, x_{2k+1}) + a_6d(x_{2k+2}, x_{2k}) \\ &= a_1d(x_{2k}, x_{2k+1}) + a_2d(x_{2k+1}, x_{2k}) + a_3d(y_{2k}, y_{2k+1}) \\ &\quad + a_4d(x_{2k+1}, x_{2k+2}) + a_5 \times 0 \\ &\quad + a_6\{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})\} \end{aligned}$$

$$\Rightarrow (1 - a_4 - a_6)d(x_{2k+1}, x_{2k+2}) \leq (a_1 + a_2 + a_6)d(x_{2k}, x_{2k+1}) + a_3d(y_{2k}, y_{2k+1})$$

$$\Rightarrow d(x_{2k+1}, x_{2k+2}) \leq \frac{a_1 + a_2 + a_6}{1 - a_4 - a_6}d(x_{2k}, x_{2k+1}) + \frac{a_3}{1 - a_4 - a_6}d(y_{2k}, y_{2k+1}).$$

Similarly

$$\begin{aligned} d(y_{2k+1}, y_{2k+2}) &\leq d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) \\ &\leq a_1d(y_{2k}, y_{2k+1}) + a_2d(S(y_{2k}, x_{2k}), y_{2k}) + a_3d(x_{2k}, x_{2k+1}) \\ &\quad + a_4d(T(y_{2k+1}, x_{2k+1}), y_{2k+1}) + a_5d(S(y_{2k}, x_{2k}), y_{2k+1}) \\ &\quad + a_6d(T(y_{2k+1}, x_{2k+1}), y_{2k}) \\ d(y_{2k+1}, y_{2k+2}) &\leq \frac{a_1 + a_2 + a_6}{1 - a_4 - a_6}d(y_{2k}, y_{2k+1}) + \frac{a_3}{1 - a_4 - a_6}d(x_{2k}, x_{2k+1}) \end{aligned}$$

Adding we have

$$\begin{aligned}
 d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) &\leq \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_6} d(x_{2k}, x_{2k+1}) \\
 &\quad + \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_6} d(y_{2k}, y_{2k+1}) \\
 &\leq \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_6} \\
 &\quad \times [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \\
 &= h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})],
 \end{aligned}$$

where

$$0 < h = \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_6} < 1.$$

Also, we have

$$d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) = h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].$$

Therefore

$$\begin{aligned}
 d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\leq h[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \\
 &\leq \dots \leq h^n[d(x_0, x_1) + d(y_0, y_1)]
 \end{aligned}$$

Now if $d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \delta_0$ then

$$\delta_n \leq h\delta_{n-1} \leq \dots \leq h^n\delta_n.$$

For $m > n$

$$\begin{aligned}
 d(x_n, x_m) + d(y_n, y_m) &\leq \delta_{m-1} + \delta_{m-2} + \dots + \delta_n \\
 &\leq (h^{m-1} + h^{m-2} + \dots + h^n)\delta_0 \\
 &= h^n(1 + h + \dots + h^{m-n-1})\delta_0 \\
 &= \frac{h^n(1 - h^{m-n})}{1 - h}\delta_0 \\
 &= \frac{h^n - h^m}{1 - h}\delta_0 \\
 &\leq \frac{h^n}{1 - h}\delta_0 \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

From (e) it follow that for $0 \ll c$ and for large n , we have $\frac{h^n}{1-h}\delta_0 \ll c$.

Thus, according to (d), $d(x_n, x_m) + d(y_n, y_m) \ll c$. Hence by definition 1.2(iii) $\{d(x_n, x_m) + d(y_n, y_m)\}$ is a Cauchy sequence. Since

$$d(x_n, x_m) \ll d(x_n, x_m) + d(y_n, y_m)$$

and

$$d(y_n, y_m) \ll d(x_n, x_m) + d(y_n, y_m).$$

Again by (d) $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X , so there exists x and y in X such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$.

Now, we show that $x = S(x, y)$ and $y = S(y, x)$. On the contrary, let us assume that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that $d(x, S(x, y)) = k > 0$ and $d(y, S(y, x)) = l > 0$.

Consider

$$\begin{aligned} l_1 = d(x, S(x, y)) &\leq d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y)) \\ &= d(x, x_{2k+2}) + d(T(x_{2k+1}, y_{2k+1}), S(x, y)) \\ &\leq a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x_{2k+1}) + a_3 d(y, y_{2k+1}) \\ &\quad + a_4 d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) \\ &\quad + a_6 d(T(x_{2k+1}, y_{2k+1}), x) \\ &= a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x_{2k+1}) + a_3 d(y, y_{2k+1}) \\ &\quad + a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) + a_6 d(x_{2k+2}, x) \end{aligned}$$

(taking $k \rightarrow \infty$ we receive)

$$\begin{aligned} &= a_1 \cdot 0 + a_2 d(S(x, y), x_{2k+1}) + a_3 \cdot 0 + a_4 \cdot 0 \\ &\quad + a_5 d(S(x, y), x_{2k+1}) + a_6 \cdot 0 \\ &\Rightarrow (1 - a_2 - a_5) d(x, S(x, y)) \leq 0 \\ &\Rightarrow d(x, S(x, y)) \leq 0 \\ &\Rightarrow l_1 \leq 0, \end{aligned}$$

which is a contradiction.

Therefore,

$$d(x, S(x, y)) = 0.$$

So

$$x = S(x, y).$$

Similarly, we can prove that

$$y = S((y, x)).$$

It follows, similarly, that

$$x = T(x, y) \text{ and } y = T(y, x).$$

So we have proved that (x, y) is a common coupled fixed point of S and T .

In order to prove the uniqueness let $(x', y') \in X \times X$ be another common coupled fixed point of S and T .

Then

$$\begin{aligned} d(x, x') &= d(S(x, y), T(x', y')) \\ &\leq a_1 d(x, x') + a_2 d(S(x, y), x) + a_3 d(y, y') \\ &\quad + a_4 d(T(x', y'), x') \\ &= a_5 d(S(x, y), x') + a_6 d(T(x', y'), x) \\ &= a_1 d(x, x') + a_2 d(x, x) + a_3 d(y, y') \\ &\quad + a_4 d(x', x') + a_5 d(x, x') \\ &\quad + a_6 d(x', x) \\ &= (a_1 + a_5 + a_6) d(x', x) + a_3 d(y, y') \\ \Rightarrow (1 - a_1 - a_5 - a_6) d(x, x') &\leq a_3 d(y, y') \\ \Rightarrow d(x, x') &\leq \frac{a_3}{1 - a_1 - a_5 - a_6} d(y, y') \end{aligned}$$

Similarly, one can prove that

$$d(y, y') \leq \frac{a_3}{1 - a_1 - a_5 - a_6} d(x, x')$$

Adding we get

$$\begin{aligned} d(x, x') + d(y, y') &\leq \frac{a_3}{1 - a_1 - a_5 - a_6} \\ &\quad \times [d(x, x') + d(y, y')] \\ \Rightarrow (1 - \frac{a_3}{1 - a_1 - a_5 - a_6}) [d(x, x') + d(y, y')] &\leq 0 \\ \Rightarrow d(x, x') + d(y, y') &= 0 \\ \Rightarrow x = x' \text{ and } y = y'. \end{aligned}$$

□

Corollary 2.2. Let (X, d) be a complete cone metric space with a cone P having non-empty interior and let $T : X \times X \rightarrow X$ satisfying

$$\begin{aligned} d(T(x, y), T(u, v)) \leq & a_1 d(x, u) + a_2 d(T(x, y), x) + a_3 d(y, v) + a_4 d(T(u, v), u) \\ & + a_5 d(T(x, y), u) + a_6 d(T(u, v), x), \end{aligned}$$

for all $x, y, u, v \in X$ where $a_i, i = 1, 2, \dots, 6$ are non-negative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1$. Then T has a unique coupled fixed point in X .

Corollary 2.3. Let (X, d) be a complete cone metric space with a cone P having non-empty interior and let $S, T : X \times X \rightarrow X$ satisfying

$$\begin{aligned} d(S(x, y), T(u, v)) \leq & ad(x, u) + bd(y, v) + c[d(S(x, y), x) + d(T(u, v), u)] \\ & + e[d(S(x, y), u) + d(T(u, v), x)], \end{aligned}$$

for all $x, y, u, v \in X$ where a, b, c, e are non-negative real numbers such that $a + b + 2c + 2e < 1$. Then S and T have a unique common coupled fixed point in X .

Corollary 2.4. Let (X, d) be a complete cone metric space with a cone P having non-empty interior and let $T : X \times X \rightarrow X$ satisfying

$$\begin{aligned} d(T(x, y), T(u, v)) \leq & ad(x, u) + bd(y, v) + c[d(T(x, y), x) + d(T(u, v), u)] \\ & + e[d(T(x, y), u) + d(T(u, v), x)], \end{aligned}$$

for all $x, y, u, v \in X$ where a, b, c, e are non-negative real numbers such that $a + b + 2c + 2e < 1$. Then T has a unique coupled fixed point in X .

Example 2.5. Let $X = \{0, 1\}$ and $E = \mathbb{R}$. Define $d : X \times X \rightarrow E$ by $d(x, y) = \frac{2}{3}(x - y)^2$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define $S, T : X \times X \rightarrow X$ as follows

$$\begin{aligned} S(x, y) &= \frac{xy}{4}, \\ T(x, y) &= \frac{xy}{3}. \end{aligned}$$

Take $a_1 = \frac{1}{2}, a_2 = 1 = a_5, a_3 = 2, a_4 = 0 = a_6$. We can show that S and T satisfy the generalized contractive condition of Theorem 2.1. We got the point $(0, 0)$ is a unique common fixed point of S and T .

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