COMMON COUPLED FIXED POINT THEOREMS OF TWO MAPPINGS SATISFYING GENERALIZED CONTRACTIVE CONDITION IN CONE METRIC SPACE

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Abstract: In this paper, we prove the existence and uniqueness of a common coupled fixed point theorem in complete cone metric space for a pair of mappings satisfying a generalised contractive condition. Some results are also given in the form of corollaries. The results are also verified with the help of example.

AMS Subject Classification: 47H10, 54H25
Key Words: cone metric space, mixed monotone property, coupled fixed point, fixed point theorems

1. Introduction

Dajun Guo and V. Lakshmikantham [1] gave the existence theorems of the coupled fixed points for both continuous and discontinuous operators and gave applications to the initial value problems of ordinary differential equations with discontinuous right-hand sides. Bhaskar and Lakshmikantham [2] proved the
existence of coupled fixed point theorem for a mixed monotone mapping in a metric space with the help of partial order, using a weak contractivity type of assumption. Since then this new concept is extended and used in various directions. This concept is extended to tripled fixed point by Berinde and Borcut [3] to quadrupled fixed point by Karapinar [4]. Coupled fixed point is also extended in various spaces like metric space, G-metric space, b-metric space, partially ordered metric space, fuzzy metric space, cone metric space etc.

The concept of cone metric space introduced by Huang and Zang [5] in 2007 as generalizations of metric space. They generalised metric space by replacing the set of real numbers with an ordering Banach space. Thus, cone naturally induces a partial order in Banach spaces. In recent years many authors established various coupled fixed point theorems in cone metric space (see [6-17] and references there in).

Let $E$ be a real Banach Space. A subset $P$ of $E$ is called a cone if

1. $P$ is closed, non-empty and $P \neq 0$
2. $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$ we define the partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in \text{int}.P$(interior of $P$).

There are two kinds of cone. They are normal cone and non-normal cones. A cone $P \subset E$ is normal if there is a number $K > 0$ such that for all $x, y \in P$, $0 \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$. In other words if $x_n \leq y_n \leq z_n$ and $\lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = x$ imply $\lim_{n \to \infty} y_n = x$. Also, a cone $P \subset E$ is regular if every increasing sequence which is bounded above is convergent.

The aim of the present work is to prove the existence and uniqueness of a common coupled fixed point theorem satisfying a generalised contractive condition. Some other results are also given in the form of corollaries.

**Definition 1.1.** ([5]) Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:

1. $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.
Definition 1.2. ([5]) Let \((X, d)\) be a cone metric space (CMS), \(x_n\) a sequence in \(X\) and \(x \in X\). For every \(c \in E\) with \(0 \ll c\), we say that \(x_n\) is

1. a Cauchy sequence if there is some \(k \in \mathbb{N}\) such that, for all \(n, m \geq k\), \(d(x_n, x_m) \ll c\);

2. a convergent sequence if there is some \(k \in \mathbb{N}\) such that, \(n \geq k\), \(d(x_n, x) \ll c\). Then \(x\) is called limit of the sequence \(x_n\).

3. \((X, d)\) is a complete cone metric space if every Cauchy sequence in \(X\) is convergent in \(X\).

Let \((X, d)\) be a cone metric space (CMS). Then the following properties are often used:

1. If \(E\) is a real Banach space with a cone \(P\) and if \(a \leq ha\) where \(a \in P\) and \(h \in [0, 1)\), then \(a = 0\).

2. If \(\theta \leq u \ll c\) for each \(\theta \ll c\), then \(u = \theta\).

3. If \(a \leq b + c\) for each \(\theta \ll c\), then \(a \leq b\).

4. If \(u \leq v\) and \(v \leq w\), then \(u \ll w\).

5. If \(c \in \text{int.} P, 0 \leq a_n\) and \(a_n \rightarrow \theta\), then there exists an \(k\) such that for all \(n > k\) we have \(a_n \ll c\).

It follows from (e) that the sequence \(x_n\) converges to \(x \in X\) if \(d(x_n, x) \rightarrow \theta\) as \(n \rightarrow \infty\) and \(x_n\) is a Cauchy sequence if \(d(x_n, x_m) \rightarrow \theta\) as \(s, m \rightarrow \infty\). In the case when the cone is not necessarily normal, then \(d(x_n, y_n) \rightarrow d(x, y)\) if \(x_n \rightarrow x\) and \(y_n \rightarrow y\) is not applicable.

Definition 1.3. ([2]) An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \(F : X \times X \rightarrow X\) if \(x = F(x, y)\) and \(y = F(y, x)\).

2. Main Results

We prove the following theorem:

Theorem 2.1. Let \((X, d)\) be a complete cone metric space with a cone \(P\) having non-empty interior and let \(S, T : X \times X \rightarrow X\) satisfying

\[d(S(x, y), T(u, v)) \leq a_1d(x, u) + a_2d(S(x, y), x) + a_3d((y, v))\]
\[ +a_4d(T(u, v), u) + a_5d(S(x, y), u) + a_6d(T(u, v), x), \]

for all \( x, y, u, v \in X \), where \( a_i, i = 1, 2, \ldots, 6 \) are non-negative real numbers such that \( a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1 \).

Then \( S \) and \( T \) have a unique common coupled fixed point in \( X \).

**Proof.** Let \( x_0 \) and \( y_0 \) be arbitrary points in \( X \). Let

\[ x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k}) \]

and

\[ x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1}) \]

for \( k = 0, 1, 2, \ldots \).

Then

\[
\begin{align*}
&d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\
&\leq a_1d(x_{2k}, x_{2k+1}) + a_2d(S(x_{2k}, y_{2k}), x_{2k}) + a_3d(y_{2k}, y_{2k+1}) \\
&+ a_4d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5d(S(x_{2k}, y_{2k}), x_{2k+1}) \\
&+ a_6d(T(x_{2k+1}, y_{2k+1}), x_{2k}) \\
&= a_1d(x_{2k}, x_{2k+1}) + a_2d(x_{2k+1}, x_{2k}) + a_3d(y_{2k}, y_{2k+1}) \\
&+ a_4d(x_{2k+2}, x_{2k+1}) + a_5d(x_{2k+1}, x_{2k+1}) + a_6d(x_{2k+2}, x_{2k}) \\
&= a_1d(x_{2k}, x_{2k+1}) + a_2d(x_{2k+1}, x_{2k}) + a_3d(y_{2k}, y_{2k+1}) \\
&+ a_4d(x_{2k+1}, x_{2k+2}) + a_5 \times 0 \\
&+ a_6\{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})\}
\end{align*}
\]

\[
\Rightarrow (1 - a_4 - a_6)d(x_{2k+1}, x_{2k+2}) \leq (a_1 + a_2 + a_6)d(x_{2k}, x_{2k+1}) + a_3d(y_{2k}, y_{2k+1})
\]

\[
\Rightarrow d(x_{2k+1}, x_{2k+2}) \leq \frac{a_1 + a_2 + a_6}{1 - a_4 - a_6}d(x_{2k}, x_{2k+1}) + \frac{a_3}{1 - a_4 - a_6}d(y_{2k}, y_{2k+1}).
\]

Similarly

\[
\begin{align*}
d(y_{2k+1}, y_{2k+2}) &\leq d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) \\
&\leq a_1d(y_{2k}, y_{2k+1}) + a_2d(S(y_{2k}, x_{2k}), y_{2k}) + a_3d(x_{2k}, x_{2k+1}) \\
&+ a_4d(T(y_{2k+1}, x_{2k+1}), y_{2k+1}) + a_5d(S(y_{2k}, x_{2k}), y_{2k+1}) \\
&+ a_6d(T(y_{2k+1}, x_{2k+1}), y_{2k}) \\
&= a_1d(y_{2k}, y_{2k+1}) + a_2d(y_{2k+1}, y_{2k}) + a_3d(x_{2k}, x_{2k+1}) \\
&+ a_4d(y_{2k+2}, y_{2k+1}) + a_5d(y_{2k+1}, y_{2k+1}) + a_6d(y_{2k+2}, y_{2k}) \\
&= a_1d(y_{2k}, y_{2k+1}) + a_2d(y_{2k+1}, y_{2k}) + a_3d(x_{2k}, x_{2k+1}) \\
&+ a_4d(y_{2k+1}, y_{2k+2}) + a_5 \times 0 \\
&+ a_6\{d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2})\}
\end{align*}
\]

\[
\Rightarrow (1 - a_4 - a_6)d(y_{2k+1}, y_{2k+2}) \leq (a_1 + a_2 + a_6)d(y_{2k}, y_{2k+1}) + a_3d(x_{2k}, x_{2k+1})
\]

\[
\Rightarrow d(y_{2k+1}, y_{2k+2}) \leq \frac{a_1 + a_2 + a_6}{1 - a_4 - a_6}d(y_{2k}, y_{2k+1}) + \frac{a_3}{1 - a_4 - a_6}d(x_{2k}, x_{2k+1}).
\]
Adding we have
\[
\begin{align*}
    d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) &\leq \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4} d(x_{2k}, x_{2k+1}) \\
    &\quad + \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4} d(y_{2k}, y_{2k+1}) \\
    &\leq \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4} \times [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \\
    &= h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})],
\end{align*}
\]
where
\[
0 < h = \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4} < 1.
\]

Also, we have
\[
\begin{align*}
    d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) &= h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].
\end{align*}
\]
Therefore
\[
\begin{align*}
    d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\leq h[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \\
    &\leq \cdots \leq h^n[d(x_0, x_1) + d(y_0, y_1)]
\end{align*}
\]
Now if \(d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \delta_0\) then
\[
\delta_n \leq h\delta_{n-1} \leq \cdots \leq h^n\delta_n.
\]

For \(m > n\)
\[
\begin{align*}
    d(x_n, x_m) + d(y_n, y_m) &\leq \delta_{m-1} + \delta_{m-2} + \cdots + \delta_n \\
    &\leq (h^{m-1} + h^{m-2} + \cdots + h^n)\delta_0 \\
    &= h^n(1 + h + \cdots + h^{m-n-1})\delta_0 \\
    &= \frac{h^n(1 - h^{m-n})}{1 - h}\delta_0 \\
    &= \frac{h^n - h^m}{1 - h}\delta_0 \\
    &\leq \frac{h^n}{1 - h}\delta_0 \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
\]
From \((e)\) it follow that for \(0 \ll c\) and for large \(n\), we have \(\frac{h^n}{1 - h}\delta_0 \ll c\).
Thus, according to $(d)$, $d(x_n, x_m) + d(y_n, y_m) \ll c$. Hence by definition 1.2(iii) $\{d(x_n, x_m) + d(y_n, y_m)\}$ is a Cauchy sequence. Since
\[ d(x_n, x_m) \ll d(x_n, x_m) + d(y_n, y_m) \]
and
\[ d(y_n, y_m) \ll d(x_n, x_m) + d(y_n, y_m). \]
Again by $(d)$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$, so there exists $x$ and $y$ in $X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now, we show that $x = S(x, y)$ and $y = S(y, x)$. On the contrary, let us assume that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that $d(x, S(x, y)) = k > 0$ and $d(y, S(y, x)) = l > 0$.

Consider
\[
\begin{align*}
l_1 &= d(x, S(x, y)) \\
&= d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y)) \\
&= d(x, x_{2k+2}) + d(T(x_{2k+1}, y_{2k+1}), S(x, y)) \\
&\leq a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x_{2k+1}) + a_3 d(y, y_{2k+1}) \\
&\quad + a_4 d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) \\
&\quad + a_6 d(T(x_{2k+1}, y_{2k+1}), x) \\
&= a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x_{2k+1}) + a_3 d(y, y_{2k+1}) \\
&\quad + a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) + a_6 d(x_{2k+2}, x)
\end{align*}
\]
(taking $k \to \infty$ we receive)
\[
\begin{align*}
&= a_1.0 + a_2 d(S(x, y), x_{2k+1}) + a_3.0 + a_4.0 \\
&\quad + a_5 d(S(x, y), x_{2k+1}) + a_6.0 \\
&\Rightarrow (1 - a_2 - a_5)d(x, S(x, y)) \leq 0 \\
&\Rightarrow d(x, S(x, y)) \leq 0 \\
&\Rightarrow l_1 \leq 0,
\end{align*}
\]
which is a contradiction.

Therefore,
\[ d(x, S(x, y)) = 0. \]

So
\[ x = S(x, y). \]

Similarly, we can prove that
\[ y = S((y, x)) \]

It follows, similarly, that

\[ x = T(x, y) \text{ and } y = T(y, x) \]

So we have proved that \((x, y)\) is a common coupled fixed point of \(S\) and \(T\).

In order to prove the uniqueness let \((x', y') \in X \times X\) be another common coupled fixed point of \(S\) and \(T\).

Then

\[
\begin{align*}
    d(x, x') &= d(S(x, y), T(x', y')) \\
    &\leq a_1 d(x, x') + a_2 d(S(x, y), x) + a_3 d(y, y') \\
    &\quad + a_4 d(T(x', y', x')) \\
    &= a_5 d(S(x, y), x') + a_6 d(T(x', y'), x) \\
    &= a_1 d(x, x') + a_2 d(x, x) + a_3 d(y, y') \\
    &\quad + a_4 d(x', x') + a_5 d(x, x') \\
    &\quad + a_6 d(x', x) \\
    \Rightarrow (1 - a_1 - a_5 - a_6)d(x, x') &\leq a_3 d(y', y') \\
    \Rightarrow d(x, x') &\leq \frac{a_3}{1 - a_1 - a_5 - a_6} d(y', y')
\end{align*}
\]

Similarly, one can prove that

\[
    d(y, y') \leq \frac{a_3}{1 - a_1 - a_5 - a_6} d(x, x')
\]

Adding we get

\[
    d(x, x') + d(y, y') \leq \frac{a_3}{1 - a_1 - a_5 - a_6} \times [d(x, x') + d(y, y')] \\
\Rightarrow (1 - \frac{a_3}{1 - a_1 - a_5 - a_6})[d(x, x') + d(y, y')] \leq 0 \\
\Rightarrow d(x, x') + d(y, y') = 0 \\
\Rightarrow x = x' \text{ and } y = y'.
\]

\[\square\]
Corollary 2.2. Let $(X, d)$ be a complete cone metric space with a cone $P$ having non-empty interior and let $T : X \times X \to X$ satisfying

$$d(T(x, y), T(u, v)) \leq a_1 d(x, u) + a_2 d(T(x, y), x) + a_3 d(y, v) + a_4 d(T(u, v), u) + a_5 d(T(x, y), u) + a_6 d(T(u, v), x),$$

for all $x, y, u, v \in X$ where $a_i, i = 1, 2, \ldots, 6$ are non-negative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1$. Then $T$ has a unique coupled fixed point in $X$.

Corollary 2.3. Let $(X, d)$ be a complete cone metric space with a cone $P$ having non-empty interior and let $S, T : X \times X \to X$ satisfying

$$d(S(x, y), T(u, v)) \leq ad(x, u) + bd(y, v) + c[d(S(x, y), x) + d(T(u, v), u)] + e[d(S(x, y), u) + d(T(u, v), x)],$$

for all $x, y, u, v \in X$ where $a, b, c, e$ are non-negative real numbers such that $a + b + 2c + 2e < 1$. Then $S$ and $T$ have a unique common coupled fixed point in $X$.

Corollary 2.4. Let $(X, d)$ be a complete cone metric space with a cone $P$ having non-empty interior and let $T : X \times X \to X$ satisfying

$$d(T(x, y), T(u, v)) \leq ad(x, u) + bd(y, v) + c[d(T(x, y), x) + d(T(u, v), u)] + e[d(T(x, y), u) + d(T(u, v), x)],$$

for all $x, y, u, v \in X$ where $a, b, c, e$ are non-negative real numbers such that $a + b + 2c + 2e < 1$. Then $T$ has a unique coupled fixed point in $X$.

Example 2.5. Let $X = \{0, 1\}$ and $E = \mathbb{R}$. Define $d : X \times X \to E$ by $d(x, y) = \frac{2}{3}(x - y)^2$ for all $x, y \in X$. Then $(X, d)$ is a cone metric space. Define $S, T : X \times X \to X$ as follows

$$S(x, y) = \frac{xy}{4},$$
$$T(x, y) = \frac{xy}{3}.$$

Take $a_1 = \frac{1}{2}, a_2 = 1 = a_5, a_3 = 2, a_4 = 0 = a_6$. We can show that $S$ and $T$ satisfy the generalized contractive condition of Theorem 2.1. We got the point $(0, 0)$ is a unique common fixed point of $S$ and $T$. 
References


