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UNIVERSALITY AND TRANSITIVITY OF SEMIGROUPS OF OPERATORS

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Abstract: Let $B(X)$ denote the algebra of all bounded linear operators on a infinite-dimensional separable complex Banach space X and M a nonzero subspace of X. In this paper we study the notion of disjoint or diagonally subspace universal respect to M (in short $d-M$ universal) and the notion of d-M topologically transitive for the sequence

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}, (N \geq 2)$$

of a $C0$-semigroups of operators on X. Also, we give a necessary and sufficient condition for which this sequence is be d-M topologically transitive.

AMS Subject Classification: 47A16, 47D06, 47D03

Key Words: Banach space operators, C_0 semigroups, diagonally subspace universal, topologically transitive

1. Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on a infinite-dimensional separable complex Banach space X. For $x \in X$, the orbit of x under T is the set $Orb(T,x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called a hypercyclic vector for T if $Orb(T,x)$ is dense in X and the operator T is said to

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be hypercyclic if there is some vector $x \in X$ which is hypercyclic. More general, a sequence $(T_n)_{n\geq 0}$ of operators in $B(X)$ is called hypercyclic or universal if $\{T_n(x), n \geq 0\}$ is dense in X for some $x \in X$, in this case x is called universal for the family $(T_n)_{n\geq 0}$ see [9].

In 2007, L. Bernal-González in [3] and J. P. Bès and A. Peris in [4] introduced independently the definition of disjoint hypercyclic for tuple of linear operators. They introduced the concept of diagonally-universality for a tuple of sequences in $B(X)$. They also gave the definition of diagonally universal for a tuple of sequences in $B(X)$.

On the other hand, recall that the family $(T(t))_{t\geq 0}$ of operators on X is called a strongly continuous semigroup(C_0-semigroup) of operators if:

1. $T(0) = I$,
2. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$
3. $\lim\limits_{t\downarrow 0} T(t)x := x$ for every $x \in X$.

The linear operator A defined in

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exist } \}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt} \,|_{t=0} \text{ for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A, see[10]. A C_0- semigroup $\tau = (T_t)_{t\geq 0}$ of operators in $B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit of τ, $Orb(\tau, x) = \{T_t x : t \geq 0\}$ is dense in X. In this case x is called the hypercyclic vector of τ [9].

Definition 1.1. Let $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ be a $N \geq 2$ sequences in $B(X)$ and let M be a nonzero subspace of X. We say that the N sequences of operators $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are disjoint or diagonally subspace universal respect to M (in short $d - M$ universal), if there exist a vector $(x, x, ..., x)$ in the diagonal of X^N, such that $\{(T_{1,j}x, T_{2,j}x, ..., T_{N,j}x,), j \in \mathbb{N}\} \cap M^N$ is dense in M^N. We call x a $d - M$ universal vector. We denote by

$$dU((T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}, M)$$

the set of all $d - M$ universal vectors of the sequences $(T_{1,j})_{j=1}^{\infty}$, $(T_{2,j})_{j=1}^{\infty}$,..., $(T_{N,j})_{j=1}^{\infty}$.

Definition 1.2. We say that the $N \geq 2$ sequences of a C_0-Semigroup

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$$

are d-M topologically transitive if for any non-empty open subsets $V_0, V_1, ..., V_N$ in M, there exists $t \geq 0$ so that

$$V_0 \cap T_{1,t}^{-1}(V_1) \cap T_{2,t}^{-1}(V_2) \cap \cdots \cap T_{N,t}^{-1}(V_N)$$

contains a non-empty open set of M.

Let M a nonzero subspace of X. In the following, we will study the notion diagonally subspace universal respect to M (in short $d-M$ universal) and the notion of d-M topologically transitive for the sequence

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}, (N \geq 2)$$

of a $C0$-semigroups of operators on X. We will prove that, if

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$$

is a sequence of C_0-semigroup with generators $A_1, A_2, ..., A_N$ and if there exists $t_0 > 0$ such that $T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$ are surjective and d-universal, then

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0} \text{ are } d-D(A_j) \text{ universal for all } j = 1, 2, ..., N.$$

Also, we give necessary and sufficient condition for which a sequence

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0} \text{ with } (N \geq 2)$$

of $C0$-semigroup be d-M topologically transitive.

2. Main results

We begin with the following lemma.

Lemma 2.1. *Let $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ be a sequences of C_0-semigroup with $N \geq 2$ and $M^N = M \times M \times ... \times M$ where M is a non zero subspace of X. Then*

$$dU(((T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}), M)$$

$$= \bigcap_{(i_1,i_2,...,i_N)\in\mathbb{N}^N} \bigcup_{t\geq 0} \bigcap_{j=1}^{N} (T_{j,t}^{-1}(A_{i_j} \bigcap M)).$$

Where $(A_j)_{j\in\mathbb{N}}$ is a countable open basis for the topology of X^N.

Proof. $x \in dU((T_{1,t}, T_{2,t}, ..., T_{N,t})_{t\geq 0}, M)$ if and only if

$$(\{(T_{1,t}x, T_{2,t}x, ..., T_{N,t}x)_{t\geq 0}\} \bigcap M^N)$$

is dens in M^N this is equivalent to the fact that for all $I = (i_1, i_2, ..., i_N) \in \mathbb{N}^N$; $\prod_{k=1}^{N} A_{i_k} \bigcap \{(T_{1,t}x, T_{2,t}x, ..., T_{N,t}x)_{t\geq 0}\} \bigcap M^N) \neq \emptyset$. Equivalently, for all $I = (i_1, i_2, ..., i_N) \in \mathbb{N}^N$, there exists $t_0 > 0$ such that

$$(T_{1,t_0}x, T_{2,t_0}x, ..., T_{N,t_0}x) \in \prod_{k=1}^{N} A_{i_k} \bigcap M^N.$$

This is equivalent to $x \in \bigcap_{(i_1,i_2,...,i_N)\in\mathbb{N}^N} \bigcup_{t\geq 0} \bigcap_{j=1}^{N} (T_{j,t}^{-1}(A_{i_j} \bigcap M))$. □

Proposition 2.1. *Let $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ be a sequences of C_0-semigroup with $N \geq 2$ and $M^N = M \times M \times ... \times M$, where M a nonzero subspace of X. If there exists $t_0 \geq 0$ such that $T_{1,t_0}, T_{2,t_0}...., T_{N,t_0}$ is M-universal, then*

$$dU(T_{1,t_0}, T_{2,t_0}...., T_{N,t_0}, M) = dU(T_{1,t}, T_{2,t}...., T_{N,t}, M)_{t\geq 0}.$$

Proof. $x \in dU(T_{1,t_0}, T_{2,t_0}...., T_{N,t_0}, M)$ if and only if

$$x \in \bigcap_{(i_1,i_2,...,i_N)\in\mathbb{N}^N} \bigcup_{n\in\mathbb{N}} (T_{1,t_0}^{-n}(A_{i_1} \bigcap M) \bigcap T_{2,t_0}^{-n}(A_{i_2} \bigcap M) ... \bigcap T_{N,t_0}^{-n}(A_{i_N} \bigcap M)).$$

Where $\{A_i\}_{i\in\mathbb{N}}$ is a countable open basis for the topology of X^N. This is equivalent to the fact that for all $I = (i_1, i_2, ..., i_N) \in \mathbb{N}^N$,$t_0$ fixed there exist $n \in \mathbb{N}$ such that

$$x \in (T_{1,t_0}^{-n}(A_{i_1} \bigcap M) \bigcap T_{2,t_0}^{-n}(A_{i_2} \bigcap M) ... \bigcap T_{N,t_0}^{-n}(A_{i_N} \bigcap M))$$

equivalent to

$$x \in (T_{1,nt_0}^{-1}(A_{i_1} \bigcap M) \bigcap T_{2,nt_0}^{-1}(A_{i_2} \bigcap M) ... \bigcap T_{N,nt_0}^{-1}(A_{i_N} \bigcap M)).$$

We put $t = nt_0 > 0$, for all $I = (i_1, i_2, ..., i_N) \in \mathbb{N}^N$, there exist $t \geq 0$ such that

$$x \in (T_{1,t}^{-1}(A_{i_1} \bigcap M) \bigcap T_{2,t}^{-1}(A_{i_2} \bigcap M) ... \bigcap T_{N,t}^{-1}(A_{i_N} \bigcap M))$$

if and only if

$$x \in \bigcap_{(i_1,i_2,...,i_N)\in\mathbb{N}^N} \bigcup_{t\geq 0} (T_{1,t}^{-1}(A_{i_1} \bigcap M) \bigcap T_{2,t}^{-1}(A_{i_2} \bigcap M) ... \bigcap T_{N,t}^{-1}(A_{i_N} \bigcap M)).$$

This is equivalent to the fact that

$$x \in dU(T_{1,t}, T_{2,t}, ..., T_{N,t})_{t\geq 0}.$$

□

Corollairy 2.1. *If there exists $t_0 > 0$ such that $T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$ are d-universality, then for all $t \geq 0$, then $T_{1,t}, T_{2,t}, ..., T_{N,t}$ are d-universality.*

Proposition 2.2. *Let $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ be N sequences of operators in $B(X)$ with $N \geq 2$ and M be a nonzero subspace of X.*

If $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ are d-M topologically transitive,

then $dU((T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}, M)$ is dens in M^N.

Proof. Let $I = (i_1, i_2, ..., i_N) \in \mathbb{N}^N$ and $G_I = \bigcup_{n\in\mathbb{N}} \bigcap_{k=1}^{N} T_{k,n}^{-1}(A_{n_k} \bigcap M)$ with $(A_n)_{n\geq 0}$ is countable open basis of X. Since $T_{k,n}$ are continuous and $(A_{n_k} \bigcap M)$ is open set in X for all $k = 1, 2, ..., N$, then $(G_I)_{I\in\mathbb{N}}$ is a sequences of open set in M. Since $(T_{1,j}), (T_{2,j}), ..., (T_{N,j})$ are d-M-topologically transitive then there exist $n \geq 0$ such that $\bigcap_{k=1}^{N} T_{k,n}^{-1}(A_{n_k} \bigcap M) \bigcap A_m \bigcap M \neq \emptyset$ for each i_k and $m \geq 0$, consequently $G_I \bigcap (A_m \bigcap M) \neq \emptyset$, it follows that G_I intersects each element of basis for the topology of M. This implies that (G_I) is dens open sequence in M, by the Baire Category Theorem

$$dU((T_{1,t})_{j\geq 0}, (T_{2,t})_{j\geq 0}, ..., (T_{N,t})_{j\geq 0}, M) = \bigcap_{I\in\mathbb{N}} G_I$$

is dense in M^N. □

Proposition 2.3. *If $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ in $B(X)$ are surjective and d-universal, then $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are d-M universal for all subset M dens in X*

Proof. Let M be subset dense in X and $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are continuous and surjective, therefor $\{(T_{1,j}(M) \times T_{2,j}(M) \times ... \times T_{N,j}(M)), j \in \mathbb{N}\}$ is dense in X^N and $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are d-universal, then there is some $x \in X$ vector a d-universal such that $\{(T_{1,j}(x), T_{2,j}(x), ..., T_{N,j}(x)), j \in \mathbb{N}\}$ is dense in X^N, hence $\{(T_{1,j}(x), T_{2,j}(x), ..., T_{N,j}(x)), j \in \mathbb{N}\} \bigcap M^N$ is dense in X^N, since $\{(T_{1,j}(x), T_{2,j}(x), ..., T_{N,j}(x)), j \in \mathbb{N}\} \bigcap M^N \subset M^N$ and hence $\{(T_{1,j}(x), T_{2,j}(x), ..., T_{N,j}(x)), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N. □

Theorem 2.1. *Let $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ be a sequence of C_0-semigroup with generators $A_1, A_2, ..., A_N$. If there exists $t_0 > 0$ such that $T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$ are surjective and d-universal, then $(T_{1,t})_{t\geq 0}$, $(T_{2,t})_{t\geq 0}$,..., $(T_{N,t})_{t\geq 0}$ are $d - D(A_j)$ universal for all $j = 1, 2, ..., N$.*

Proof. We have for all $j \in \{1, 2..., N\}$ the subspace $D(A_j)$ is dense in X. Since

$$T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$$

are surjective and d-universal, then by Proposition 2.3, $T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$ are $d - D(A_j)$ universal and according to Proposition 2.1; $(T_{1,t})_{t\geq 0}$, $(T_{2,t})_{t\geq 0}$,..., $(T_{N,t})_{t\geq 0}$ are $d - D(A_j)$ universal. □

Theorem 2.2. *Let $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ with $(N \geq 2)$ be a sequence of $C0$-semigroup and M be a nonzero subspace of X. Then the following conditions are equivalent:*

1. *the sequence $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ is d-M topologically transitive.*
2. *For every non-empty open subsets $V_0, V_1, ...V_N$ of M, there exists $t_0 \geq 0$ such that $V_0 \bigcap (T_{1,t_0}^{-1}(V_1) \bigcap T_{2,t_0}^{-1}(V_2), ... \bigcap T_{N,t_0}^{-1}(V_N))$ is a non-empty open set of M.*
3. *For every non-empty open subsets $V_0, V_1, ...V_N$ of M, there exists $t_0 \geq 0$ such that $V_0 \bigcap (T_{1,t_0}^{-1}(V_1) \bigcap T_{2,t_0}^{-1}(V_2), ... \bigcap T_{N,t_0}^{-1}(V_N))$ is non-empty and*

$$(T_{1,t_0}(M) \times T_{2,t_0}(M) \times ... \times T_{N,t_0}(M)) \subseteq M^N.$$

Proof. $1 \Leftrightarrow 2$ is clear. For $3 \Rightarrow 2$, let $V_0, V_1, ..., V_N$ are nonempty open subsets of M by the statement (3), there exists $t_0 \geq 0$ such that

$$V_0 \bigcap (T_{1,t_0}^{-1}(V_1) \bigcap T_{2,t_0}^{-1}(V_2), ... \bigcap T_{N,t_0}^{-1}(V_N))$$

is non-empty and

$$(T_{1,t_0}(M) \times T_{2,t_0}(M) \times ... \times T_{N,t_0}(M)) \subseteq M^N.$$

Since for all $i \in \{1, 2, ...N\}$

$$T_{i,t_0} \mid M : M \quad \to \quad M$$

are continuous, then $T^{-1}_{i,t_0}(V_i)$ for all $i = 1, 2, ...N$, therefore

$$V_0 \bigcap (T^{-1}_{1,t_0}(V_1) \bigcap T^{-1}_{2,t_0}(V_2), ... \bigcap T^{-1}_{N,t_0}(V_N))$$

is nonempty open subset of M. Now, to prove $1 \Rightarrow 3$, Let $V_0, V_1, ..., V_N$ are nonempty open subsets of M and the sequence $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ is d-M topologically transitive. There exists $t_0 \geq 0$ such that

$$(T^{-1}_{1,t_0}(V_1) \bigcap T^{-1}_{2,t_0}(V_2), ... \bigcap T^{-1}_{N,t_0}(V_N))$$

contains a nonempty open V_k of M, it follows that

$$V_k \subset T^{-1}_{1,t_0}(V_1) \bigcap T^{-1}_{2,t_0}(V_2), ... \bigcap T^{-1}_{N,t_0}(V_N)$$

and $T^{-1}_{1,t_0}(V_1) \bigcap T^{-1}_{2,t_0}(V_2), ... \bigcap T^{-1}_{N,t_0}(V_N)$ nonempty.

We prove that $(T_{1,t_0}(M) \times T_{2,t_0}(M) \times ... \times T_{N,t_0}(M)) \subseteq M^N$.

Let $x \in M$, we have $V_k \subset (T^{-1}_{1,t_0}(V_1) \bigcap T^{-1}_{2,t_0}(V_2), ... \bigcap T^{-1}_{N,t_0}(V_N))$, then for all $i = 1, 2, ...N$, $V_k \subset (T^{-1}_{i,t_0}(V_i)$ this implies that $T_{i,t_0}(V_k) \subset V_i \subset M$ for each $i \ from \ 1 \ to \ N$.

Let $x_0 \in V_k$, since V_k is open set of M, then for all r small enough we have $x_0 + rx \in V_k$ therefore for all $i = 1, 2, ...N$, $T_{i,t_0}(x_0 + rx) = T_{i,t_0}(x_0) + rT_{i,t_0}(x) \in T_{i,t_0}(V_k) \subset M$. As $T_{i,t_0}(x_0) \in M$ for all $i = 1, 2, ...N$, then $T_{i,t_0}(x) \in M$ for all $i = 1, 2, ...N$. We then conclude that $T_{i,t_0}(M) \in M$, for all $i = 1, 2, ...N$ and hence $(T_{1,t_0}(M) \times T_{2,t_0}(M) \times ... \times T_{N,t_0}(M)) \subseteq M^N$. □

References

[1] S. I. Ansari, *Existance of hypercyclic operatos on topological vector space,* J. Funct. Anal. 148 (1997), 384-390.

[2] F. Bayart, E. Matheron, *Dynamics of Linear Operators,* Cambridge Tracts in Mathematics 179, Cambridge University Press, 2009.

[3] L. Bernal-González, *Disjoint hypercyclic operators,* Studia Math, 182 (2) (2007), 113-130.

[4] J. P. Bès and A. Peris, *Disjointness in hypercyclicity.* J. Math. Anal. Appl, 336 (2007) 297-315.

[5] J. P. Bès, Ö. Martin and R. Sanders, *Weighted Shifts And Disjoint Hypercyclicity,* J. Operator. Theory,(72) (2014),15-40.

[6] B. F. Madore and R. A. Matínez-Avendaño , *Subspace Hypercyclicity,* J. Math. Anal. Appl, (2011),502-511.

[7] G. Costakis and A. Peris, *Hypercyclic semigroups and somewhere dense orbits,* C. R.Math. Acad. Sci. Paris 335 (2002), 895-898.

[8] K-J. Engel and R. Nagel , *One-Parameter Semigroups for Linear Evolution Equations,* Springer-Verlag New York, 2000.

[9] K. G. Erdmann and A. Persi , *Linear Chaos* , Universitext. Springer, London, 2011.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag New York, (1983).