A NOTE ON RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH ANTI-PERIODIC BOUNDARY CONDITIONS

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Abstract: In this work, we approach an anti-periodic boundary value problem for a class of retarded functional differential equations. We employ the method of lower and upper solutions to prove a uniqueness result for such problem.

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1. Introduction

The ordinary differential equations are powerful theoretical representations of processes of evolution in which the variation rate of process state at each time depends on the state of the process at this time. However, there is a lot of real phenomena in which the variation rate of state in each moment depends not only on the state of the process at this moment, but also the history of phenomena states. Thus, for such cases, it makes convenient the use of other

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theoretical tools that describe such phenomena more appropriately. Such tools are the retarded functional differential equations. See [4], [5], [7], [8], [9] and [14].

This note deals with an anti-periodic boundary value problem for a class of retarded functional differential equations and it was inspired by the works [1], [2], [3], [6], [11] and [12]. The functional differential inequalities generated by this problem are considered and a uniqueness result is established by the method of lower and upper solutions. The method of lower and upper solutions together with the monotone iterative technique has been used in the treatment of nonlinear differential equations, see [10], [11] and [13].

2. The Problem and the Obtained Results

Let $a, b \in \mathbb{R}$, $a < b$. We denote by $C([a, b], \mathbb{R})$ the space of the continuous functions $\phi : [a, b] \rightarrow \mathbb{R}$ endowed with the norm $\|\phi\|_\infty = \sup_{\tau \in [a, b]} |\phi(\tau)|$.

Let us consider the retarded functional differential equation

$$x'(t) = f(t, x_t), \quad t \in [0, T], \quad (2.1)$$

subject to the anti-periodic boundary condition

$$x(0) = -x(T), \quad (2.2)$$

where

- ★ f is a continuous real function defined on $[0, T] \times C([-r, 0], \mathbb{R})$;
- ★ $r \geq 0$ and $T > 0$;
- ★ x_t represents the mapping $x_t : [-r, 0] \rightarrow \mathbb{R}$ defined by $x_t(\tau) = x(t + \tau)$ for $\tau \in [-r, 0]$, where $t \in [0, T]$.

We denote by $C^1([0, T], \mathbb{R})$ the set of the functions $x \in C([0, T], \mathbb{R})$ that are continuously differentiable on $[0, T]$.

Definition 2.1. A real function x defined on $[-r, T]$ is said to be a solution of Problem (2.1)-(2) on $[0, T]$ if $x|_{[0, T]} \in C^1([0, T], \mathbb{R})$ and it satisfies (2.1)-(2).

Now, we present the concept of lower and upper solutions for Problem (2.1)-(2).

Definition 2.2. Functions $u : [-r, T] \rightarrow \mathbb{R}$ and $v : [-r, T] \rightarrow \mathbb{R}$ are said to be a pair of lower and upper related solutions for Problem (2.1)-(2) if they satisfy

$$\begin{aligned} u|_{[0,T]} &\in C^1([0, T], \mathbb{R}), \quad v|_{[0,T]} \in C^1([0, T], \mathbb{R}) \\ u'(t) &\leq f(t, u_t), \quad v'(t) \geq f(t, v_t), \quad \text{for all } t \in [0, T]; \\ u(0) + v(T) &\leq 0 \quad \text{and} \quad v(0) + u(T) \geq 0. \end{aligned}$$

The following assumption will be fundamental to obtain our results:

- (A) For each $t \in [0, T]$, $f(t, \varphi) < f(t, \psi)$ whenever $\varphi, \psi \in C([-r, 0], \mathbb{R})$, $\varphi > \psi$, (that is, $\varphi(s) > \psi(s)$ for $s \in [-r, 0]$).

The next result establishes a relation between the lower and upper solutions of Problem (2.1)-(2).

Theorem 2.1. Let u and v be a lower solution and a upper solution of Problem (2.1)-(2), respectively. Assume that Condition (A) is fulfilled. Then,

$$u(t) \leq v(t), \quad \text{for all } t \in [0, T]. \quad (2.3)$$

Proof. Suppose the statement (2.3) is not true. Then $\Lambda = \sup\{u(t) - v(t); t \in [0, T]\} > 0$. Since $u - v$ is a continuous function on the compact set $[0, T]$, there exists $\eta \in [0, T]$ such that

$$u(\eta) - v(\eta) = \Lambda > 0. \quad (2.4)$$

Furthermore,

$$u(t) - v(t) \leq \Lambda, \quad \text{for all } t \in [0, T]. \quad (2.5)$$

Let us consider the following three cases:

Case 1: $\eta \in (0, T)$.

Since $u - v$ is continuously differentiable and reaches its maximum at η , it follows that $(u - v)'(\eta) = 0$. On the other hand, from the definition of the functions u and v , we have

$$0 = (u - v)'(\eta) \leq f(\eta, u_\eta) - f(\eta, v_\eta),$$

that is,

$$f(\eta, u_\eta) \geq f(\eta, v_\eta). \quad (2.6)$$

But, if (2.6) holds, we can infer that $u_\eta \leq v_\eta$ on $[-r, 0]$ and, in particular, $u(\eta) \leq v(\eta)$, which contradicts (2.4).

Case 2: $\eta = 0$.

Then, $u(0) - v(0) = \Lambda$ and, by (2.5), $u(T) - v(T) \leq \Lambda$. However, from the definition of the functions u and v , we obtain

$$\Lambda = u(0) - v(0) \leq -v(T) + u(T) \leq \Lambda.$$

Thus $u(T) - v(T) = \Lambda > 0$. Thereby, we can construct a sequence $\{t_k\} \subset (0, T)$ such that $t_k < t_{k+1}$, $\lim_{k \rightarrow \infty} t_k = T$ and $D_- \xi(t_k) = \liminf_{h \rightarrow 0^-} \frac{\xi(t_k + h) - \xi(t_k)}{h} \geq 0$, where $\xi(t) = u(t) - v(t)$ for $t \in [0, T]$.

Let K_0 be a positive integer such that, for $k \geq K_0$, one has $u(t_k) - v(t_k) > 0$ (such K_0 exists by virtue of the continuity of $u - v$). Therefore, from the definition of the functions u and v , we conclude that

$$0 \leq D_- \xi(t_k) \leq u'(t_k) - v'(t_k) \leq f(t_k, u_{t_k}) - f(t_k, v_{t_k}).$$

This yields $u(t_k) - v(t_k) \leq 0$, which does not occur.

Case 3: $\eta = T$.

Here we can use an analogous argument to that used in the previous case.

The contradictions pointed in the three cases guarantee us that the assertion of the theorem is valid. $\square$

By employing Theorem 2.1, we obtain a uniqueness criteria for Problem (2.1)-(2).

Theorem 2.2. *Assume that Condition (A) is fulfilled. Then, Problem (2.1)-(2) has at most one solution.*

Proof. Suppose x_1, x_2 are two distinct solutions of (2.1)-(2). From anti-periodic boundary condition, we know that

$$x_1(0) + x_1(T) = 0 \quad \text{and} \quad x_2(0) + x_2(T) = 0.$$

Then

$$x_1(0) + x_1(T) + x_2(0) + x_2(T) = 0,$$

that is,

$$x_1(0) + x_2(T) = -(x_1(T) + x_2(0)).$$

We have two possibilities: either $x_1(0) + x_2(T) < 0$ or $x_1(0) + x_2(T) \geq 0$. Let us assume, without loss of generality, that $x_1(0) + x_2(T) \geq 0$ and, accordingly, $x_1(T) + x_2(0) \leq 0$.

We have

$$\begin{aligned}x_1'(t) &= f(t, x_{1_t}) \\x_1(0) + x_2(T) &\geq 0\end{aligned}$$

and

$$\begin{aligned}x_2'(t) &= f(t, x_{2_t}) \\x_2(0) + x_1(T) &\leq 0.\end{aligned}$$

By Theorem 2.1 we obtain

$$x_2(t) \leq x_1(t), \quad t \in [0, T].$$

In particular, $x_2(0) \leq x_1(0)$ and $x_2(T) \leq x_1(T)$. This together with anti-periodic boundary condition implies

$$0 \leq x_1(0) - x_2(0) = -x_1(T) + x_2(T) \leq 0,$$

that is, $x_1(0) = x_2(0)$ and $x_1(T) = x_2(T)$. Consequently, the functions x_1 and x_2 satisfy the following systems:

$$\begin{aligned}x_1'(t) &= f(t, x_{1_t}) \\x_1(0) + x_2(T) &= 0\end{aligned}$$

and

$$\begin{aligned}x_2'(t) &= f(t, x_{2_t}) \\x_2(0) + x_1(T) &= 0.\end{aligned}$$

By applying Theorem 2.1 again, we conclude that $x_1(t) \leq x_2(t)$ and $x_2(t) \leq x_1(t)$, for all $t \in [0, T]$, which proves the result. $\square$

The last theorem shows a relation between the solution of Problem (2.1)-(2) and the lower and upper solutions of the same problem when Condition (A) is satisfied.

Theorem 2.3. *Let u and v be a lower solution and a upper solution of Problem (2.1)-(2), respectively, and x be a solution of the same problem. Assume that (A) holds. Then,*

$$u(t) \leq x(t) \leq v(t), \quad \text{for all } t \in [0, T]. \quad (2.7)$$

Proof. At first, we will see that $u(t) \leq x(t)$ for all $t \in [0, T]$. Assume that there is $\tau \in [0, T]$ such that $u(\tau) > x(\tau)$ and denote by $\mu = \sup\{(u - x)(t); t \in [0, T]\} > 0$. Since $u - x$ is a continuous function on the compact set $[0, T]$, there is $\beta \in [0, T]$ such that

$$u(\beta) - x(\beta) = \mu > 0.$$

We consider the following three cases: $\beta \in (0, T)$, $\beta = 0$ and $\beta = T$. Analogously to the proof of Theorem ??lem1, we obtain contradictions that ensure the validity of the inequality $u(t) \leq x(t)$ for all $t \in [0, T]$.

The proof of the inequality $x(t) \leq v(t)$, for $t \in [0, T]$, is similar. □

To finish this note, we exhibit a simple example.

Example 2.1. Let $T = 2$ and $f : [0, T] \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ the function defined by $f(t, \varphi) = -(\varphi(0))^7$ for $(t, \varphi) \in [0, T] \times C([-r, 0], \mathbb{R})$. By Theorem 2.2, the zero solution is the unique anti-periodic solution of the problem:

$$x'(t) = f(t, x_t)$$

$$x(0) + x(2) = 0.$$

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