STRONGLY NILPOTENT RADICALS IN THE PROJECTIVE PRODUCT OF GAMMA RINGS AND THEIR INTRINSIC PROPERTIES

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Abstract: The object of this paper is to study strongly nilpotent radical in the projective product of gamma rings and their deep properties. This also includes some profound results on semi-prime Gamma-ring/prime Gamma-ring/semi-simple Gamma-ring. It is also shown that if the projective product of two Gamma-rings is semi-simple then the component Gamma-rings are also semi-simple and the converse of this result is also investigated.

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1. Introduction

In study of Gamma-ring theory which was introduced by Nobusawa [7] and later re-defined by Barnes [10], different kinds of radicals play an important role. There is a very strong theory of various radicals on general rings and Banach algebras [1,6]. Many prominent mathematicians have extended fruitfully many significant technical results on radicals of general ring to the radicals of Gamma-ring, see [2,3,4,5,8,9,11].

2. Fundamental Definitions

The following concepts are useful for our purpose:

Definition 2.1. A gamma ring $\Gamma(X)$ in the sense of Nabusawa is said to be **simple** if for any two nonzero elements $xy \in X$, there exist $\Gamma\gamma \in$ such that $x\gamma y \neq 0$

Definition 2.2. A gamma ring $\Gamma(X)$ in the sense of Nabusawa is said to be **semi-simple** if for any nonzero elements $x \in X$, there exist $\Gamma\gamma \in$ such that $x\gamma x \neq 0$

Definition 2.3. If I is an additive subgroup of a gamma ring $\Gamma(X)$ and $\Gamma XI \subseteq I$ (or $\Gamma IX \subseteq I$), then I is called a left (or right) gamma ideal of X . If I is both left and right gamma ideal then it is said to be a gamma ideal of $\Gamma(X)$ or simply an ideal.

Definition 2.4. An ideal I of a gamma ring $\Gamma(X)$ is said to be **prime** if for any two ideals A and B of X , $\Gamma AB \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Definition 2.5. An ideal I of the gamma ring $\Gamma(X)$ is said to be **semi-prime** if for any ideal U of X , $\Gamma UU \subseteq I \Rightarrow U \subseteq I$

Definition 2.6. An element a of a gamma ring $\Gamma(X)$ is strongly nilpotent if there exist a positive integer n such that $\Gamma^n(a)a = (aaa \dots a)a = 0$. A subset S of X is strongly nil if each of its elements is strongly nilpotent. S is strongly nilpotent if there exist a positive integer n such that $\Gamma(S)^n S = 0$. Clearly a strongly nilpotent set is also strongly nil.

Definition 2.7. The strongly nilpotent radical, denoted by $S(X)$ of a gamma ring $\Gamma(X)$ is defined as the sum of all strongly nilpotent ideals of $\Gamma(X)$

Definition 2.8. Let $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ be two gamma rings. Let $X = X_1 \times X_2$ and

$\Gamma\Gamma = {}_1 \times {}_2$. Then defining addition and multiplication on X and Γ by,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \quad ,$$

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$\text{and } (x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2) = (x_1\alpha_1y_1, x_2\alpha_2y_2)$$

for every $(x_1, x_2) (y_1, y_2) \in X$ and $\Gamma(\alpha_1, \alpha_2) (\beta_1, \beta_2) \in \Gamma(X)$ is a gamma ring. We call this gamma ring as the **Projective product of gamma rings**.

3. Main Results

Theorem 3.1. *If U is an ideal of $\Gamma(X)$, then ΓUU is also an ideal of $\Gamma(X)$.*

Proof. Since U is an ideal of $\Gamma(X)$ so $\Gamma XU \subseteq U$ and $\Gamma UX \subseteq U$
 Let $\Gamma V = UU$. Then $\Gamma \Gamma \Gamma UU \subseteq XU \subseteq U \subseteq X \Rightarrow UU \subseteq X \Rightarrow V \subseteq X$
 Now, $\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma X V = X(UU) = (XU)U \subseteq UU = V \Rightarrow XV \subseteq V$
 Similarly, $\Gamma V X \subseteq V$ So V is an ideal of $\Gamma(X)$

Theorem 3.2. *If $\Gamma(X)$ is a prime gamma ring, then there does not exist any nonzero nilpotent ideal of X .*

Proof. Let $\Gamma(X)$ be a prime gamma ring and U be a strongly nilpotent ideal of $\Gamma(X)$

Then there exist a positive integer n such that

$$\Gamma(U)^n U = 0$$

$$\Gamma \Gamma \Gamma \Rightarrow (UU \dots U)U = 0$$

$$\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow (UU)(UU) \dots (UU) = 0$$

$$\Gamma \Gamma \Gamma \Gamma \Rightarrow VV \dots V = 0, \text{ where } V = UU \text{ is an ideal of } X$$

$$\Gamma \Gamma \Gamma \Rightarrow V(V \dots V) = 0$$

$$\Gamma \Gamma \Gamma \Rightarrow VW = 0, \text{ where } W = V \dots V \text{ is an ideal of } X$$

$\Rightarrow V = 0$ or $W = 0$ [Since $\Gamma(X)$ is prime]

If $W = 0$, then by repeated application of prime-ness of $\Gamma(X)$, we get $V = 0$

Thus in either case, $V = 0$ which implies $U + U = 0 \Rightarrow U = 0$, since $\Gamma(X)$ is prime.

We assumed U to be a strongly nilpotent ideal and obtained $U = 0$ which implies that there does not exist any non zero strongly nilpotent ideal of a prime gamma ring.

Theorem 3.3 If there does not exist any non zero strongly nilpotent ideal of a gamma ring $\Gamma(X)$, then it is a semi-prime gamma ring.

Proof. Let $\Gamma(X)$ be gamma ring in which there does not exist any non zero strongly nilpotent ideal and let U be an ideal of $\Gamma(X)$ such that,

$$\Gamma U U = 0 \Rightarrow (U)^n U = 0, n = 1$$

$\Rightarrow U$ is a strongly nilpotent ideal $\Rightarrow U = 0$

Thus we get, $\Gamma U U = 0 \Rightarrow U$. Hence $\Gamma(X)$ is a semi-prime gamma ring.

Theorem 3.4 Every prime gamma ring is semi-simple.

Proof. Let $\Gamma(X,)$ be a prime gamma ring.

Since a prime gamma ring does not contain any non zero strongly nilpotent ideal so the only strongly nilpotent ideal is the zero ideal. $S(X)$ being the sum of all strongly nilpotent ideals of $\Gamma(X)$ is equal to zero. i.e $S(X) = 0$

Again $S(X) = 0$ if and only if X is semi-simple [11]

Hence $\Gamma(X,)$ is semi-simple.

Theorem 3.5 Every simple gamma ring is a prime gamma ring.

Proof. Let $\Gamma(X,)$ be a simple gamma ring. Then for any two nonzero elements $xy \in X$, there exist $\Gamma\gamma \in$ such that $x\gamma y \neq 0$

Let U, V be two ideals of X such that $\Gamma UV = 0$. We show $U = 0$ or $V = 0$

If possible let $U \neq 0$ and $V \neq 0$. Then there exist $0 \neq x \in U$ and $0 \neq y \in V$

Since $\Gamma(X,)$ is simple so there exist $\Gamma\gamma \in$ such that $x\gamma y \neq 0$

Now $\Gamma x\gamma y \in UV = 0 \Rightarrow x\gamma y = 0$, which is a contradiction. So our supposition is wrong.

So we must have $U = 0$ or $V = 0$. Hence $\Gamma(X,)$ is a prime gamma ring.

Theorem 3.6 If $\Gamma(X,)$ be a gamma ring then $S(X) = 0$ implies X is semi-prime.

Proof. If $S(X) = 0$ then X is semi-simple [11]. Then for any nonzero element $x \in X$, there exist $\Gamma\gamma \in$ such that $x\gamma x \neq 0$. We show X is semi-prime.

Let U be an ideal of X such that $\Gamma U U = 0$. We show $U = 0$

If possible let $U \neq 0$. Then there exist $0 \neq x \in U$. Since $\Gamma(X,)$ is semi-simple so there exist $\Gamma\gamma \in$ such that $x\gamma x \neq 0$

But $\Gamma x\gamma x \in U U = 0 \Rightarrow x\gamma x = 0$, which is a contradiction. So our supposition is wrong.

So we must have $U = 0$. Hence $\Gamma(X,)$ is a semi-prime gamma ring and the result.

Theorem 3.7 Let $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ be two gamma rings and $\Gamma(X)$ be their Projective product with the right operator rings R_1, R_2 and R respectively. Then $R = R_1 \times R_2$, where $R_1 \times R_2$ is defined as, for any $t = (t_1, t_2) \in R_1 \times R_2$ and $m = (x, y) \in X$, $mt = (x, y)(t_1, t_2) = (xt_1, yt_2)$

Proof. Since R_1, R_2 are rings so with this definition R is also a ring.

We have, $\Gamma R = \{\sum_i [\gamma_i, z_i] : \gamma_i \in, z_i \in X, \Gamma R_1 = \{\sum_i [\alpha_i, x_i] : \alpha_i \in_1, x_i \in X_1$ and

$$\Gamma R_2 = \{\sum_i [\beta_i, y_i] : \beta_i \in_2, y_i \in X_2$$

Let $t = \sum_i [\gamma_i, z_i] \in R$ and $m = (x, y) \in X$ be any elements, where $\Gamma \gamma_i = (\alpha_i, \beta_i) \in$ and $z_i = (x_i y_i) \in X$

$$\text{Now, } mt = m \sum_i [\gamma_i, z_i] = \sum_i m \gamma_i z_i$$

$$= \sum_i (x, y) (\alpha_i, \beta_i) (x_i, y_i) = \sum_i (x \alpha_i x_i, y \beta_i y_i) = (\sum_i x \alpha_i x_i, \sum_i y \beta_i y_i)$$

$$= (x \sum_i [\alpha_i, x_i], y \sum_i [\beta_i, y_i]) = (xy) (\sum_i [\alpha_i, x_i], \sum_i [\beta_i, y_i])$$

$$\Rightarrow mt = m (\sum_i [\alpha_i, x_i], \sum_i [\beta_i, y_i])$$

So $R = R_1 \times R_2$, which is the right operator ring $\Gamma(X)$

Theorem 3.8 Let $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ be two gamma rings and $\Gamma(X)$ be their Projective product. Then every strongly nilpotent ideal of $\Gamma(X)$ gives rise to two strongly nilpotent ideals of $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ and vice versa.

Proof. Let U be a strongly nilpotent ideal of $\Gamma(X)$. Since U is an ideal X so $U = A \times B$, where A, B are ideals of X_1 and X_2 respectively.

Then there exist a positive integer n such that $\Gamma(U)^n U = 0$

$$\Gamma \Gamma \Gamma \Rightarrow (UU \dots U)U = 0$$

$$\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow ((A \times B)(1 \times 2)(A \times B)(1 \times 2) \dots (A \times B)(1 \times 2))(A \times B) = 0$$

$$\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow (A \times B)(1 \times 2)(A \times B)(1 \times 2) \dots (A \times B)(1 \times 2) = 0$$

$$\begin{aligned}
\Gamma\Gamma\Gamma\Gamma\Gamma &=> (A_1 A_1 \dots A_1) \times (B_2 B_2 \dots B_2) = 0 \\
\Gamma\Gamma\Gamma\Gamma\Gamma &=> ((A_1 A_1 \dots A_1) A) \times ((B_2 B_2 \dots B_2) B) = 0 \\
\Gamma\Gamma &=> ((A_1)^n A) \times ((B_2)^n B) = 0 \\
\Gamma\Gamma &=> (A_1)^n A = 0 \text{ and } (B_2)^n B = 0
\end{aligned}$$

$\Rightarrow A$ and B are strongly nilpotent ideals of X_1 and X_2 respectively.

Conversely, let A and B be two strongly nilpotent ideals of X_1 and X_2 respectively.

So $U = A \times B$ is an ideal of $\Gamma(X)$.

Since A and B are strongly nilpotent ideals so there exist positive integers m, n such that $\Gamma\Gamma(A_1)^m A = 0$ and $(B_2)^n B = 0 \dots \dots (1)$

Without the loss of generality let, $m \leq n$ Then we have,

$$\begin{aligned}
\Gamma\Gamma\Gamma(U)^n U &= ((A_1)^n A) \times ((B_2)^n B) \\
\Gamma\Gamma\Gamma &= ((A_1)^m (A_1)^{n-m} A) \times ((B_2)^n B) \\
\Gamma\Gamma\Gamma &= ((A_1)^m A) ((A_1)^{n-m}) \times ((B_2)^n B) \\
&= 0 \times 0 = 0 \text{ [Using (1)]}
\end{aligned}$$

Thus there exists a positive integer n such that $\Gamma(U)^n U = 0$. So U is also strongly nilpotent and hence the theorem.

Theorem 3.9 Let $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ be two gamma rings and $\Gamma(X)$ be their Projective product. Then $S(X) = S(X_1) \times S(X_2)$

Proof. $S(X) =$ Sum of all strongly nilpotent ideals of X

$$\Rightarrow S(X) = \sum_{\substack{P \text{ is a strongly} \\ \text{nilpotent ideal of } X}} P$$

$$\Rightarrow S(X) = \sum_{\substack{A \text{ and } B \text{ are strongly} \\ \text{nilpotent ideals of } X_1 \text{ and } X_2 \text{ respectively}}} (A \times B) \text{ [By theorem}$$

3.8]

$$\Rightarrow S(X) = \left(\sum_{\substack{A \text{ is a strongly} \\ \text{nilpotent ideal of } X_1}} A \right) \times \left(\sum_{\substack{B \text{ is a strongly} \\ \text{nilpotent ideal of } X_2}} B \right)$$

$\Rightarrow S(X) = S(X_1) \times S(X_2)$. Hence the result.

Theorem 3.10 If the projective product of two gamma rings is semi-simple then the component gamma rings are also semi-simple and vice versa.

Proof. Let $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ be two gamma rings and $\Gamma(X)$ be their Projective product. Suppose $\Gamma(X)$ is semi-simple. We show $\Gamma(X_{1,1})$ and $\Gamma(X_{2,2})$ are also semi-simple.

Since $\Gamma(X)$ is semi-simple, so the strongly nilpotent radical is zero i.e $S(X) = 0$

By our theorem 3.9, $S(X) = S(X_1) \times S(X_2)$

So we have, $S(X_1) \times S(X_2) = 0$

$\Rightarrow S(X_1) = 0$ and $S(X_2) = 0$ [Since $A \times B = 0 \Rightarrow A = 0, B = 0$]

$\Rightarrow$ Both X_1 and X_2 are semi-simple.

Conversely, let Both X_1 and X_2 be semi-simple. We show X is also semi-simple.

For this, let, $z = (x, y) \in X$ be any element such that $\Gamma z \gamma z = 0 \forall \gamma \in$, where $\gamma = (\alpha \beta)$

$$\Gamma z \gamma z = 0 \forall \gamma \in$$

$$\Gamma \Gamma \Rightarrow (x \alpha x, y \beta y) = 0 \forall \alpha \in {}_1, \beta \in {}_2$$

$$\Gamma \Gamma \Rightarrow x \alpha x = 0 \text{ and } y \beta y = 0 \forall \alpha \in {}_1, \beta \in {}_2$$

$\Rightarrow x = 0$ and $y = 0$ [X_1 and X_2 are semi-simple]

$$\Rightarrow z = (x, y) = (0, 0) = 0$$

So, X is also semi-simple and hence the result.

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