STRONGLY NILPOTENT RADICALS IN THE PROJECTIVE PRODUCT OF GAMMA RINGS AND THEIR INTRINSIC PROPERTIES

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Abstract: The object of this paper is to study strongly nilpotent radical in the projective product of gamma rings and their deep properties. This also includes some profound results on semi-prime Gamma-ring/prime Gamma-ring/semi-simple Gamma-ring. It is also shown that if the projective product of two Gamma-rings is semi-simple then the component Gamma-rings are also semi-simple and the converse of this result is also investigated.

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1. Introduction

In study of Gamma-ring theory which was introduced by Nobusawa [7] and later re-defined by Barnes [10], different kinds of radicals play an important role. There is a very strong theory of various radicals on general rings and Banach algebras [1,6]. Many prominent mathematicians have extended fruitfully many significant technical results on radicals of general ring to the radicals of Gamma-ring, see [2,3,4,5,8,9,11].
2. Fundamental Definitions

The following concepts are useful for our purpose:

**Definition 2.1.** A gamma ring $\Gamma(X)$ in the sense of Nabusawa is said to be **simple** if for any two nonzero elements $xy \in X$, there exist $\Gamma \gamma \in$ such that $x\gamma y \neq 0$

**Definition 2.2.** A gamma ring $\Gamma(X)$ in the sense of Nabusawa is said to be **semi-simple** if for any nonzero elements $x \in X$, there exist $\Gamma \gamma \in$ such that $x\gamma x \neq 0$

**Definition 2.3.** If $I$ is an additive subgroup of a gamma ring $\Gamma(X)$ and $\Gamma XI \subseteq I$ (or $\Gamma IX \subseteq I$), then $I$ is called a left (or right) gamma ideal of $X$. If $I$ is both left and right gamma ideal then it is said to be a gamma ideal of $\Gamma(X)$ or simply an ideal.

**Definition 2.4.** An ideal $I$ of a gamma ring $\Gamma(X)$ is said to be **prime** if for any two ideals $A$ and $B$ of $X$, $\Gamma AB \subseteq I => A \subseteq I or B \subseteq I$..

**Definition 2.5.** An ideal $I$ of the gamma ring $\Gamma(X)$ is said to be **semi-prime** if for any ideal $U$ of $X$, $\Gamma UU \subseteq I => U \subseteq I$

**Definition 2.6.** An element $a$ of a gamma ring $\Gamma(X)$ is strongly nilpotent if there exist a positive integer $n$ such that $\Gamma a^n a = (aaa \ldots a)a = 0$. A subset $S$ of $X$ is strongly nil if each of its elements is strongly nilpotent. $S$ is strongly nilpotent if there exist a positive integer $n$ such that $\Gamma S^n S = 0$. Clearly a strongly nilpotent set is also strongly nil.

**Definition 2.7.** The strongly nilpotent radical, denoted by $S(X)$ of a gamma ring $\Gamma(X)$ is defined as the sum of all strongly nilpotent ideals of $\Gamma(X)$

**Definition 2.8.** Let $\Gamma (X_1,1)$ and $\Gamma (X_2,2)$ be two gamma rings. Let $X = X_1 \times X_2$ and $\Gamma \Gamma = 1 \times 2$. Then defining addition and multiplication on $X$ and $\Gamma$ by,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 x_2 + y_2)$$,

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1 \alpha_2 + \beta_2)$$

and $$(x_1, x_2) (\alpha_1, \alpha_2) (y_1, y_2) = (x_1 \alpha_1 y_1 x_2 \alpha_2 y_2)$$
for every \((x_1, x_2)(y_1, y_2) \in X\) and \(\Gamma(\alpha_1, \alpha_2)(\beta_1, \beta_2) \in \Gamma(X)\) is a gamma ring. We call this gamma ring as the **Projective product of gamma rings**.

### 3. Main Results

**Theorem 3.1.** If \(U\) is an ideal of \(\Gamma(X)\), then \(\Gamma U U\) is also an ideal of \(\Gamma(X)\).

**Proof.** Since \(U\) is an ideal of \(\Gamma(X)\) so \(\Gamma X U \subseteq U\) and \(\Gamma U X \subseteq U\)

Let \(\Gamma V = U U\). Then \(\Gamma \Gamma \Gamma \Gamma U U \subseteq X U \subseteq U \subseteq X \Rightarrow U U \subseteq X \Rightarrow V \subseteq X\)

Now, \(\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma X V = X (U U) = (X U) U \subseteq U U = V \Rightarrow X V \subseteq V\)

Similarly, \(\Gamma V X \subseteq V\) So \(V\) is an ideal of \(\Gamma(X)\)

**Theorem 3.2.** If \(\Gamma(X)\) is a prime gamma ring, then there does not exist any nonzero nilpotent ideal of \(X\).

**Proof.** Let \(\Gamma(X)\) be a prime gamma ring and \(U\) be a strongly nilpotent ideal of \(\Gamma(X)\)

Then there exist a positive integer \(n\) such that

\[
\Gamma(U)^n U = 0
\]

\[
\Gamma \Gamma \Gamma \Rightarrow (U U \ldots U)U = 0
\]

\[
\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow (U U) (U U) \ldots (U U) = 0
\]

\[
\Gamma \Gamma \Gamma \Rightarrow V V \ldots V = 0, \text{ where } V = U U \text{ is an ideal of } X
\]

\[
\Gamma \Gamma \Gamma \Rightarrow V (V \ldots V) = 0
\]

\[
\Gamma \Gamma \Gamma \Rightarrow V W = 0, \text{ where } W = V \ldots V \text{ is an ideal of } X
\]

\[
\Rightarrow V = 0 \text{ or } W = 0 \text{ [Since } \Gamma(X) \text{ is prime]}
\]

If \(W = 0\), then by repeated application of prime-ness of \(\Gamma(X)\), we get \(V = 0\)

Thus in either case, \(V = 0\) which implies \(U + U = 0 \Rightarrow U = 0\), since \(\Gamma(X)\) is prime.

We assumed \(U\) to be a strongly nilpotent ideal and obtained \(U = 0\) which implies that there does not exist any non zero strongly nilpotent ideal of a prime gamma ring.
Theorem 3.3 If there does not exist any non zero strongly nilpotent ideal of a gamma ring \( \Gamma(X) \), then it is a semi-prime gamma ring.

Proof. Let \( \Gamma(X) \) be gamma ring in which there does not exist any non zero strongly nilpotent ideal and let \( U \) be an ideal of \( \Gamma(X) \) such that,

\[
\Gamma U U = 0 \implies (U)^n U = 0, \quad n = 1
\]

\( \Rightarrow U \) is a strongly nilpotent ideal\( \Rightarrow U = 0 \)
Thus we get, \( \Gamma U U = 0 \Rightarrow U \). Hence \( \Gamma(X) \) is a semi-prime gamma ring.

Theorem 3.4 Every prime gamma ring is semi-simple.

Proof. Let \( \Gamma (X, \gamma) \) be a prime gamma ring. 
Since a prime gamma ring does not contain any non zero strongly nilpotent ideal so the only strongly nilpotent ideal is the zero ideal. \( S(X) \) being the sum of all strongly nilpotent ideals of \( \Gamma(X) \) is equal to zero. i.e \( S(X) = 0 \)
Again \( S(X) = 0 \) if and only if \( X \) is semi-simple [11] 
Hence \( \Gamma (X, \gamma) \) is semi-simple.

Theorem 3.5 Every simple gamma ring is a prime gamma ring.

Proof. Let \( \Gamma (X, \gamma) \) be a simple gamma ring. Then for any two nonzero elements \( xy \in X \), there exist \( \Gamma \gamma \in \) such that \( x\gamma y \neq 0 \)
Let \( U, V \) be two ideals of \( X \) such that \( \Gamma U V = 0 \). We show \( U = 0 \) or \( V = 0 \)
If possible let \( U \neq 0 \) and \( V \neq 0 \). Then there exist \( 0 \neq x \in U \) and \( 0 \neq x \in V \)
Since \( \Gamma (X, \gamma) \) is simple so there exist \( \Gamma \gamma \in \) such that \( x\gamma y \neq 0 \)
Now \( \Gamma x \gamma y \in UV = 0 \Rightarrow x\gamma y = 0 \), which is a contradiction. So our supposition is wrong.
So we must have \( U = 0 \) or \( V = 0 \). Hence \( \Gamma (X, \gamma) \) is a prime gamma ring.

Theorem 3.6 If \( \Gamma (X, \gamma) \) be a gamma ring then \( S(X) = 0 \) implies \( X \) is semi-prime.

Proof. If \( S(X) = 0 \) then \( X \) is semi-simple [11]. Then for any nonzero element \( x \in X \), there exist \( \Gamma \gamma \in \) such that \( x\gamma x \neq 0 \). We show \( X \) is semi-prime.
Let \( U \) be an ideal of \( X \) such that \( \Gamma U U = 0 \). We show \( U = 0 \)
If possible let \( U \neq 0 \). Then there exist \( 0 \neq x \in U \). Since \( \Gamma (X, \gamma) \) is semi-simple so there exist \( \Gamma \gamma \in \) such that \( x\gamma x \neq 0 \)
But \( \Gamma x \gamma x \in U U = 0 \Rightarrow x\gamma x = 0 \), which is a contradiction. So our supposition is wrong.
So we must have $U = 0$. Hence $\Gamma(X)$ is a semi-prime gamma ring and the result.

**Theorem 3.7** Let $\Gamma(X_1, 1)$ and $\Gamma(X_2, 2)$ be two gamma rings and $\Gamma(X)$ be their Projective product with the right operator rings $R_1, R_2$ and $R$ respectively. Then $R = R_1 \times R_2$, where $R_1 \times R_2$ is defined as, for any $t = (t_1, t_2) \in R_1 \times R_2$ and $m = (x, y) \in X$, $mt = (x, y)(t_1, t_2) = (xt_1, yt_2)$

**Proof.** Since $R_1, R_2$ are rings so with this definition $R$ is also a ring.

We have, $\Gamma R_1 = \{ \sum \gamma_i : \gamma_i \in X_1, \gamma_i \in X, \Gamma R_1 = \{ \sum [\alpha_i, x_i] : \alpha_i \in 1, x_i \in X_1$ and

$$\Gamma R_2 = \{ \sum [\beta_i, y_i] : \beta_i \in 2, y_i \in X_2$$

Let $t = \sum [\gamma_i, z_i] \in R$ and $m = (x, y) \in X$ be any elements, where $\Gamma \gamma_i = (\alpha_i, \beta_i) \in$ and $z_i = (x_i y_i) \in X$

Now, $mt = m \sum [\gamma_i, z_i] = \sum m \gamma_i z_i$

$$= \sum_i (x, y)(\alpha_i, \beta_i)(x_i, y_i) = \sum_i (x\alpha_i x_i, y\beta_i y_i) = \sum_i x\alpha_i x_i \sum y\beta_i y_i)$$

$$= (x \sum [\alpha_i, x_i], y \sum [\beta_i, y_i]) = (xy) (\sum [\alpha_i, x_i] \sum [\beta_i, y_i])$$

$$\Rightarrow mt = m (\sum [\alpha_i, x_i] \sum [\beta_i, y_i])$$

So $R = R_1 \times R_2$, which is the right operator ring $\Gamma(X)$

**Theorem 3.8** Let $\Gamma(X_1, 1)$ and $\Gamma(X_2, 2)$ be two gamma rings and $\Gamma(X)$ be their Projective product. Then every strongly nilpotent ideal of $\Gamma(X)$ gives rise to two strongly nilpotent ideals of $\Gamma(X_1, 1)$ and $\Gamma(X_2, 2)$ and vice versa.

**Proof.** Let $U$ be a strongly nilpotent ideal of $\Gamma(X)$. Since $U$ is an ideal $X$ so $U = A \times B$, where $A, B$ are ideals of $X_1$ and $X_2$ respectively.

Then there exist a positive integer $n$ such that $\Gamma(U)^n U = 0$

$$\Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow (UU \ldots U) U = 0$$

$$\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow (A \times B)(1 \times 2)(A \times B)(1 \times 2) \ldots (A \times B)(1 \times 2) (A \times B) = 0$$

$$\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma \Rightarrow (A \times B)(1 \times 2)(A \times B)(1 \times 2) \ldots (A \times B)(1 \times 2) = 0$$
\[
\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma = (A_1 A_{1...1} A) \times (B_2 B_{2...2} B) = 0
\]
\[
\Gamma \Gamma \Gamma \Gamma \Gamma \Gamma = ((A_1 A_{1...1} A) \times ((B_2 B_{2...2} B) B) = 0
\]
\[
\Gamma \Gamma = ((A_1)^n A) \times ((B_2)^n B) = 0
\]
\[
\Gamma \Gamma = (A_1)^n A = 0 \text{ and } (B_2)^n B = 0
\]

\[\Rightarrow A \text{ and } B \text{ are strongly nilpotent ideals of } X_1 \text{ and } X_2 \text{ respectively.}\]

Conversely, let \( A \) and \( B \) be two strongly nilpotent ideals of \( X_1 \) and \( X_2 \) respectively.

So \( U = A \times B \) is an ideal of \( \Gamma(X) \).

Since \( A \) and \( B \) are strongly nilpotent ideals so there exist positive integers \( mn \) such that \( \Gamma \Gamma (A_1)^m A = 0 \text{ and } (B_2)^n B = 0 \) \ldots \ldots (1)

Without the loss of generality let, \( m \leq n \) Then we have,

\[
\Gamma \Gamma(U)^n U = ((A_1)^n A) \times ((B_2)^n B)
\]
\[
\Gamma \Gamma = ((A_1)^m (A_1)^{n-m} A) \times ((B_2)^n B)
\]
\[
\Gamma \Gamma = ((A_1)^m A) ((1A)^{n-m}) \times ((B_2)^n B)
\]
\[= 0 \times 0 = 0 \text{ [Using (1)]}\]

Thus there exists a positive integer \( n \) such that \( \Gamma(U)^n U = 0 \). So \( U \) is also strongly nilpotent and hence the theorem.

**Theorem 3.9** Let \( \Gamma(X_{1,1}) \) and \( \Gamma(X_{2,2}) \) be two gamma rings and \( \Gamma(X) \) be their Projective product. Then \( S(X) = S(X_1) \times S(X_2) \)

**Proof.** \( S(X) = \sum \) of all strongly nilpotent ideals of \( X \)

\[\Rightarrow S(X) = \sum_{P \text{ is a strongly nilpotent ideal of } X} P\]

\[\Rightarrow S(X) = \sum_{A \text{ and } B \text{ are strongly nilpotent ideals of } X_1 \text{ and } X_2 \text{ respectively}} (A \times B) \text{ [By theorem 3.8]}\]

\[\Rightarrow S(X) = \left( \sum_{A \text{ is a strongly nilpotent ideal of } X_1} A \right) \times \left( \sum_{B \text{ is a strongly nilpotent ideal of } X_2} B \right)\]
Theorem 3.10 If the projective product of two gamma rings is semi-simple then the component gamma rings are also semi-simple and vice versa.

Proof. Let \( \Gamma(X, 1) \) and \( \Gamma(X, 2) \) be two gamma rings and \( \Gamma(X) \) be their Projective product. Suppose \( \Gamma(X) \) is semi-simple. We show \( \Gamma(X, 1) \) and \( \Gamma(X, 2) \) are also semi-simple.

Since \( \Gamma(X) \) is semi-simple, so the strongly nilpotent radical is zero i.e \( S(X) = 0 \)

By our theorem 3.9, \( S(X) = S(X, 1) \times S(X, 2) \)

\[ \Rightarrow S(X, 1) = 0 \text{ and } S(X, 2) = 0 \]

\[ \Rightarrow \text{Both } X_1 \text{ and } X_2 \text{ are semi-simple.} \]

Conversely, let \( X_1 \) and \( X_2 \) be semi-simple. We show \( X \) is also semi-simple.

For this, let, \( z = (x, y) \in X \) be any element such that \( \Gamma z \gamma z = 0 \forall \gamma \in \Gamma \)

where \( \gamma = (\alpha \beta) \)

\[ \Gamma z \gamma z = 0 \forall \gamma \in \Gamma \]

\[ \Gamma \gamma \Rightarrow (x \alpha x, y \beta y) = 0 \forall \alpha \in 1, \beta \in 2 \]

\[ \Gamma \gamma \Rightarrow x \alpha x = 0 \text{ and } y \beta y = 0 \forall \alpha \in 1, \beta \in 2 \]

\[ \Rightarrow x = 0 \text{ and } y = 0 \]

\[ \Rightarrow z = (x, y) = (0, 0) = 0 \]

So, \( X \) is also semi-simple and hence the result.

References


