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HIDDEN VARIABLE BIVARIATE FRACTAL INTERPOLATION SURFACE WITH FUNCTION VERTICAL SCALING FACTOR

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Abstract: In this work, the method of constructing hidden variable fractal interpolation surfaces with function vertical scaling factors has been introduced. In Particular, the fractal interpolation surfaces with two types of function vertical scaling factors has been generated and results are compared.

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Key Words: fractal interpolation, fractal surfaces, hidden variable fractal surfaces, function vertical scaling factor, sine vertical scaling factor

1. Introduction

Fractal geometry deals with the study of irregular and self similar objects. The Natural world can be modeled with the help of Fractal geometry. Euclidean geometry does not provide all the tools to model such fractal objects arising in the real world. Fractal geometry was first studied by Mandelbrot (1982). Digital surface representation is a thrust area in computer graphics.Digital surface representation is also applied in diverse areas such as applied science, geology, geography, meteorology, medicine etc. M. F. Barnsley is the pathfinder of frac-

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tal interpolation [4, 5, 6]. Fractal surfaces were firstly addressed by Massopust [7]. Iterated function systems (IFS) are carrying on an important role in fractal interpolation scheme. Many authors dealt with the construction of fractal interpolation surfaces.H. Xie and H. Sun [2]introduced the notion of bivariate fractal interpolation surfaces (FIS) on arbitrary rectangular lattices with arbitrarily selected vertical scaling factors. But graphs of their attractors are not continuous functions. Leoni dalla [3]corrected and improved the result of H. Xie, et al. Robert Malysz [8] discussed a algorithm which employing arbitrarily selected interpolation points. But it uses same vertical factor for all the members of the IFS. In the paper [1], A.K.B. Chand et al constructed hidden variable bivariate fractal interpolation surfaces (HFIS). Their construction generates both self similar and non self similar FIS. Recently Zhigang Feng et al [9], exploited the method of constructing fractal interpolation surfaces with function vertical scaling factor. They have taken function vertical scaling factor as a special class of continuous bivariate function (PVF). With this construction, they relaxed the constraint that the interpolation nodes on the boundary are collinear and the vertical scaling factors are equal.

In this paper, hidden variable bivariate FIS with function vertical scaling factors has been constructed. Under some conditions, the IFS convergence to its attractor. A class of function vertical scaling factor using the sine function is also inaugurated.Further, we have generated FIS using sine function and the one introduced by Zhigang Feng [9]. On comparing the roughness and Root mean square roughness values of the generated surfaces, it is found that sine function vertical scaling factor (SVF) gives better approximations than the one held by using polynomial vertical scaling factor.

2. Construction of IFS

Let $G = I \times J$, where I=[a,b] and J=[c,d], are the intervals and their partitions are $a = x_0 < x_1 < ..... < x_m = b$, $c = y_0 < y_1 < ...... < y_n = d$. Denote the sub intervals of I *and* J as $I_i = [x_{i-1}, x_i]$; $J_j = [y_{j-1}, y_j]$ and let $G_{ij} = I_i \times J_j, for\ i = 1, 2, ...m, j = 1, 2, ...n$. Define the affine mappings $\varphi_i : I \to I_i$ as $\varphi_i(x) = a_i x + b_i$ and $\psi_j : J \to J_i$ as $\psi_j(y) = c_j y + d_j$ such that $\varphi_i(x_0) = x_{i-1}, \varphi_i(x_m) = x_i, \psi_j(y_0) = y_{j-1}, \psi_j(y_n) = y_j$.

Using (1) we can get, $a_i = \frac{x_i - x_{i-1}}{x_m - x_0}$,$b_i = x_{i-1} - \frac{x_i - x_{i-1}}{x_m - x_0} x_0$, $c_j = \frac{y_j - y_{j-1}}{y_n - y_0}$, $d_j = y_{j-1} - \frac{y_j - y_{j-1}}{y_n - y_0} y_0$. Take the interpolation knots as $\{(x_i, y_j, z_{i,j}) \in \Re^3 / i = 0, 1, 2, ...m, j = 0, 1, 2, ...n\}$. Then our procedure leads to find a function f

such that $f(x_i, y_j) = z_{ij}$ for all $i = 0, 1, 2, ...m,\ j = 0, 1, 2, ...n$. We generalize the interpolation knots by using hidden variables $\{t_{i,j}/i = 0, 1, 2, ...m, j = 0, 1, 2, ...n\}$. The generalized interpolation knots set is $\{(x_i, y_j, z_{i,j}, t_{i,j}) \in \Re^4/\ i = 0, 1, 2, ...m, j = 0, 1, 2, ...n\}$. Let $g_1 = \min_{i,j} z_{i,j}$; $g_2 = \max_{i,j} z_{i,j}$; $h_1 = \min_{i,j} t_{i,j}$; $h_2 = \max_{i,j} t_{i,j}$ and $\widetilde{G} = [g_1, g_2] \times [h_1, h_2]$ and $K = G \times \widetilde{G}$. Now define a bivariate continuous function $F_{i,j} : K \to \widetilde{G}$ as

$$F_{ij}(x, y, z, t) = (e_{ij}x + f_{ij}y + s_{ij}(x, y)z + \beta_{ij}t + g_{ij}xy + k_{ij}, \widetilde{e}_{ij}x + \widetilde{f}_{ij}y + \widetilde{\beta}_{ij}t + \widetilde{g}_{ij}xy + \widetilde{k}_{ij}),$$

satisfying the conditions

$$F_{ij}(x_0, y_0, z_{00}, t_{00}) = (z_{i-1,j-1}, t_{i-1,j-1}),$$

$$F_{ij}(x_m, y_0, z_{m0}, t_{m0}) = (z_{i,j-1}, t_{i,j-1}),$$

$$F_{ij}(x_0, y_n, z_{0n}, t_{0n}) = (z_{i-1,j}, t_{i-1,j}),$$

$$F_{ij}(x_m, y_n, z_{mn}, t_{mn}) = (z_{ij}, t_{ij}),$$

and

$$d(F_{ij}(x, y, z, t) - F_{ij}(x^*, y^*, z^*, t^*)) \le k\,|(z, t) - (z^*, t^*)|\,,$$

for $i \in 1, 2, ...m,\ j \in 1, 2, ...n,\ 0 \le k_3 < 1$ where $(x, y, z, t), (x^*, y^*, z^*, t^*) \in K$ and d is the sup. metric on K. $s_{ij}(x, y)$ is the function vertical scaling factor of the surface F_{ij} which is a bi variate continuous function satisfying the Lipschitz condition for x and y. Assume that the Lipschitz constants for x and y are $L_1$and L_2. Take $max\{|s_{ij}|\} = v$. Now we define our most important factor of fractal interpolation. i.e., Iterated function systems. Define the mappings $w_{ij} : K \to G_{ij}$ as

$$w_{i,j}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \varphi_i(x) \\ \psi_j(y) \\ F_{ij}(x, y, z, t) \end{pmatrix} = \begin{pmatrix} a_i x + b_i \\ c_j y + d_j \\ F_{ij}(x, y, z, t) \end{pmatrix}. \tag{1}$$

Then

$$\{K; w_{ij}/i = 1, 2, , ...m; j = 1, 2, ...n\} \tag{2}$$

is the required IFS.

Theorem 1. *Let the IFS defined above be associated with the sample data set*

$$\{(x_i, y_j, z_{i,j}, t_{i,j})/i = 0, 1, 2, ...m, j = 0, 1, 2, ...n\}.$$

If the function vertical scaling factor $s_{ij}(x, y)$ *satisfy* $0 < |s_{ij}(x, y)| < 1$*, for* $i = 1, 2, ...m, j = 1, 2, ...n$ *and if* $|\beta_{ij}| + |\widetilde{\beta_{ij}}| < 1$*, then there exists a metric* τ *on* $\Re^4$ *equivalent to the Euclidean metric such that the IFS is hyperbolic with respect to* τ*. And also there exists a unique non-empty compact set* $E \subseteq \Re^4$ *such that* $E = \bigcup_{j=1}^{n} \bigcup_{i=1}^{m} w_{ij}(G)$*.*

Proof. Let us start with the metric τ on $\Re^4$ as,

$$\tau\left((x_1, y_1, z_{11}, t_{11}), (x_2, y_2, z_{22}, t_{22})\right) \\ = |x_1 - x_2| + |y_1 - y_2| + \theta\left(|z_{11} - z_{22}| + |t_{11} - t_{22}|\right),$$

introduced by Barnsley. Where θ may be defined suitably later.

We prove that $w'_{ij}s$ are contraction with respect to τ. Now

$$\tau\Big(w_{ij}(x, y, z, t), w_{ij}(x^*, y^*, z^*, t^*)\Big) = \\ \tau\Bigg(\big(\varphi_i(x), \psi_j(y), e_{ij}x + f_{ij}y + s_{ij}(x, y)z + \beta_{ij}t + g_{ij}xy + k_{ij}, \widetilde{e}_{ij}x + \widetilde{f}_{ij}y + \widetilde{\beta}_{ij}t \\ + \widetilde{g}_{ij}xy + \widetilde{k}_{ij}\big), \\ \big(\varphi_i(x^*), \psi_j(y^*), e_{ij}x^* + f_{ij}y^* + s_{ij}(x^*, y^*)z^* + \beta_{ij}t^* + g_{ij}x^*y^* + k_{ij}, \widetilde{e}_{ij}x^* \\ + \widetilde{f}_{ij}y^* + \widetilde{\beta}_{ij}t^* + \widetilde{g}_{ij}x^*y^* + \widetilde{k}_{ij}\big)\Bigg) = \\ |\varphi_i(x) - \varphi_i(x^*)| + |\psi_j(y) - \psi_j(y^*)| \\ + \theta\Bigg[|e_{ij}x + f_{ij}y + s_{ij}(x, y)z + \beta_{ij}t + g_{ij}xy + k_{ij} - e_{ij}x^* - f_{ij}y^* - s_{ij}(x^*, y^*)z^* \\ -\beta_{ij}t^* - g_{ij}x^*y^* - k_{ij}| + \Big|\widetilde{e}_{ij}x + \widetilde{f}_{ij}y + \widetilde{\beta}_{ij}t + \widetilde{g}_{ij}xy + \widetilde{k}_{ij} \\ -\widetilde{e}_{ij}x^* - \widetilde{f}_{ij}y^* - \widetilde{\beta}_{ij}t^* - \widetilde{g}_{ij}x^*y^* - \widetilde{k}_{ij}\Big|\Bigg] \\ \le |a_i||x - x^*| + |c_j||y - y^*| + \theta\Bigg[|s_{ij}(x, y)z - s_{ij}(x^*, y^*)z^*| \\ + |e_{ij}||x - x^*| + |f_{ij}||y - y^*| + |g_{ij}||xy - x^*y^*| \\ + |\beta_{ij}||t - t^*| + |\widetilde{e}_{ij}||x - x^*| + |\widetilde{f}_{ij}||y - y^*| + |\widetilde{g}_{ij}||xy - x^*y^*| + |\widetilde{\beta}_{ij}||t - t^*|\Bigg]$$

$$\begin{aligned}\leq &|x-x^*|\left\{|a_i|+\theta[|e_{ij}|+|\widetilde{e}_{ij}|]\right\}+|y-y^*|\left\{|c_j|+\theta[|f_{ij}|+|\widetilde{f}_{ij}|]\right\}\\ &+\theta|xy-x^*y^*|\left\{|g_{ij}|+|\widetilde{g}_{ij}|\right\}\\ &+\theta|t-t^*|\left\{|\beta_{ij}|+|\widetilde{\beta}_{ij}|\right\}+\theta|s_{ij}(x,y)z-s_{ij}(x^*,y^*)z^*|.\end{aligned}$$

Since s_{ij} is a bivariate continuous function satisfying Lipschitz condition, we get

$$\begin{aligned}|s_{ij}(x,y)z-s_{ij}(x^*,y^*)z^*| &= |s_{ij}(x,y)z-s_{ij}(x^*,y)z+s_{ij}(x^*,y)z-s_{ij}(x^*,y^*)z^*|\\ &\leq |z||s_{ij}(x,y)-s_{ij}(x^*,y)|+|s_{ij}(x^*,y)z-s_{ij}(x^*,y^*)z^*|\\ &\leq |z|L_1|x-x^*|+|s_{ij}(x^*,y)z-s_{ij}(x^*,y)z^*-s_{ij}(x^*,y)z^*-s_{ij}(x^*,y^*)z^*|\\ &\leq |z|L_1|x-x^*|+|s_{ij}||z-z^*|+|z^*|L_2|y-y^*|\\ &\leq |g_2|\Big[L_1|x-x^X*|+L_2|y-y^*|\Big]+v|z-z^*|,\end{aligned}$$

and

$$\begin{aligned}|xy-x^*y^*| &= |xy-x^*y+x^*y-x^*y^*|\\ &= |y(x-x^*)+x^*(y-y^*)|\\ &\leq |y||x-x^*|+|x^*||y-y^*|\\ &\leq |d||x-x^*|+|b||y-y^*|\end{aligned}$$

Which implies that

$$\begin{aligned}\tau\Big(w_{ij}(x,y,z,t),w_{ij}(x^*,y^*,z^*,t^*)\Big) &\leq |x-x^*|\Bigg\{|a_i|+\theta\Bigg(|e_{ij}|+|\widetilde{e}_{ij}|\\ &\quad +|d|(|g_{ij}|+|\widetilde{g}_{ij}|)+|g_2|L_1\Bigg)\Bigg\}\\ &\quad +|y-y^*|\Bigg\{|c_j|+\theta\Bigg(|f_{ij}|+|\widetilde{f}_{ij}|+|b|(|g_{ij}|+|\widetilde{g}_{ij}|)+|g_2|L_2\Bigg)\Bigg\}\\ &\quad +\theta(|\beta_{ij}|+|\widetilde{\beta}_{ij}|)|t-t^*|+\theta v|z-z^*|.\end{aligned}$$

Now setting

$$\theta_1=\frac{\min_{1\leq i\leq m}(1-|a_i|)}{\max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} 2\left\{|e_{ij}|+|\widetilde{e}_{ij}|+|d|(|g_{ij}|+|\widetilde{g}_{ij}|)+|g_2|L_1\right\}},$$

$$\theta_2=\frac{\min_{1\leq j\leq n}(1-|c_j|)}{\max_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} 2\left\{|f_{ij}|+|\widetilde{f}_{ij}|+|b|(|g_{ij}|+|\widetilde{g}_{ij}|)+|g_2|L_2\right\}},$$

we receive

$$\tau\Big(w_{ij}(x,y,z,t), w_{ij}(x^*,y^*,z^*,t^*)\Big) \leq \frac{1+|a_i|}{2}|x-x^*|$$
$$+\frac{1+|c_j|}{2}|y-y^*| + \theta\delta[|t-t^*|+|z-z^*|]$$
$$< s\Big(|x-x^*|+|y-y^*|+\theta(|t-t^*|+|z-z^*|)\Big),$$

where

$$s = max\left\{\frac{1+|a_i|}{2},\ \frac{1+|c_j|}{2}, \delta\right\} < 1$$

and

$$\delta = \max_{1\leq i\leq m 1\leq j\leq n}\left\{v, |\beta_{ij}|+|\widetilde{\beta}_{ij}|\right\} < 1,$$

implies that w_{ij} 's are hyperbolic. Therefore there exists a unique non empty compact set $E \subset \Re^4$ such that $E = \cup\cup w_{ij}(E)$. □

Let us now generate fractal interpolation surfaces by taking the function vertical scaling factors in two different forms.As a first case,take the function vertical scaling factor introduced in [9].i.e.,$s_{ij} = \lambda_{ij}(x-a)(b-x)(y-c)(d-y), for$i=1,2,3...m;j=1,2,3,...n,$where\ \lambda_{ij}$ are constants. Obviously s_{ij} are bi variate continuous functions satisfying the Lipschitz conditions for $x\ and\ y$. Then the above IFS changed to the form $\hat{w}_{ij} : K \to K$ such that

$$\hat{w}_{i,j}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \varphi_i(x) \\ \psi_j(y) \\ \hat{F}_{ij}(x,y,z,t) \end{pmatrix}. \tag{3}$$

In the second case let us take $s_{ij} = sin(x-a)sin(b-x)sin(y-c)sin(d-y), for\ i = 1,2,3...m; j = 1,2,3,...n$. Then we have the following IFS.

$$\widetilde{w}_{i,j}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \varphi_i(x) \\ \psi_j(y) \\ \widetilde{F}_{ij}(x,y,z,t) \end{pmatrix}. \tag{4}$$

Using the end point conditions for $\hat{w}_{i,j}$ and $\widetilde{w}_{i,j}$ we can obtain the following for the first coordinate,

$$z_{i-1,j-1} = e_{ij}a + f_{ij}c + \beta_{ij}t_{00} + g_{ij}ac + k_{ij}$$

$$
\begin{aligned}
z_{i,j-1} &= e_{ij}b + f_{ij}c + \beta_{ij}t_{m0} + g_{ij}bc +_0 +k_{ij} \\
z_{i-1,j} &= e_{ij}a + f_{ij}d + \beta_{ij}t_{0n} + g_{ij}ad + k_{ij} \\
z_{i,j} &= e_{ij}b + f_{ij}d + \beta_{ij}t_{mn} + g_{ij}bd + k_{ij},
\end{aligned}
$$

choosing β_{ij} and $\widetilde{\beta_{ij}}$ as constrained free variables such that $|\beta_{ij}| + |\widetilde{\beta_{ij}}| < 1$. We get

$$
\begin{aligned}
g_{ij} &= \frac{z_{ij} - z_{i,j-1} - z_{i-1,j} + z_{i-1,j-1} - \beta_{i,j}\left[t_{mn} - t_{m0} - t_{0n} - t_{00}\right]}{(b-a)(d-c)} \\
e_{ij} &= \frac{z_{ij} - z_{i-1,j} - g_{ij}(bd - ac) - \beta_{i,j}\left[t_{mn} - t_{0n}\right]}{(b-a)} \\
f_{ij} &= \frac{z_{i-1,j} - z_{i-1,j-1} - g_{ij}(ad - ac) - \beta_{i,j}\left[t_{0n} - t_{00}\right]}{(d-c)} \\
k_{ij} &= z_{ij} - be_{i,j} - adg_{ij} - df_{i,j} - \beta_{ij}t_{mn}.
\end{aligned}
$$

For the second coordinate:

$$
\begin{aligned}
\widetilde{g_{ij}} &= \frac{\left[t_{ij} - t_{i-1,j} - t_{i,j-1} + t_{i-1,j-1}\right] - \widetilde{\beta_{i,j}}\left[t_{mn} - t_{m0} - t_{0n} + t_{00}\right]}{(b-a)(d-c)} \\
\widetilde{e_{ij}} &= \frac{t_{i,j-1} - t_{i-1,j-1} - \widetilde{g_{ij}}(bc - ac) - \widetilde{\beta_{i,j}}\left[t_{m0} - t_{00}\right]}{(b-a)} \\
\widetilde{f_{ij}} &= \frac{t_{i-1,j} - t_{i-1,j-1} - \widetilde{g_{ij}}(ad - ac) - \widetilde{\beta_{i,j}}\left[t_{0n} - t_{00}\right]}{(d-c)} \\
\widetilde{k_{ij}} &= t_{ij} - b\widetilde{e_{i,j}} - bd\widetilde{g_{ij}} - d\widetilde{f_{i,j}} - \widetilde{\beta_{ij}}t_{mn}.
\end{aligned}
$$

Now in the following theorem we give the conditions under which FIS can be generated using the IFS (3) & (4).

Theorem 2. *Let E be the attractor of the IFS stated in the equation (2). Let us take $s_{ij} = \lambda_{ij}(x-a)(b-x)(y-c)(d-y)$, where λ_{ij} are constants,for i=1,2,3...m; j=1,2,3,...n. in the definition of F_{ij}. If $\lambda_{ij} < \frac{16}{(b-a)^2(d-c)^2}$, $for\ i = 1,2,3...m; j = 1,2,3,...n.$ and $|\beta_{ij}| + |\widetilde{\beta_{ij}}| < 1$, then E is the graph of the continuous vector valued function $f : G \to \widetilde{G}$ such that $f(x_i, y_j) = (z_{ij}, t_{ij})\ for\ i = 1,2,3...m; j = 1,2,3,...n.$*

Proof. The condition $\lambda_{ij} < \frac{16}{(b-a)^2(d-c)^2}$ leads to $|s_{ij}| < 1$. In addition to that if $|\beta_{ij}| + |\widetilde{\beta_{ij}}| < 1$, then the IFS in (2) will have the attractor E.

Let $C(G)$ be the space of all continuous functions from $G\ to\ \widetilde{G}$. Then the space $(C(G), \|.\|_\infty)$ is a complete metric space, where the norm is defined as $\|f\|_\infty = max\left\{|f(x,y)| : (x,y) \in G\right\}$.

Form a subspace $C_0(G)$ of $C(G)$ by collecting the members of $C(G)$ which are satisfying the conditions, $f(x_0, y_0) = (z_{00}, t_{00})$, $f(x_m, y_0) = (z_{m0}, t_{m0})$, $f(x_0, y_n) = (z_{0n}, t_{0n})$ and $f(x_m, y_n) = (z_{mn}, t_{mn})$. Then $(C_0(G), \|.\|_\infty)$ is also a complete metric space. Now define a operator T on the space $C_0(G)$ as,

$$\begin{aligned}(Tf)(x,y) &= F_{ij}\left(\varphi_i^{-1}(x), \psi_j^{-1}(y), z(\varphi_i^{-1}(x), \psi_j^{-1}(y)), t(\varphi_i^{-1}(x), \psi_j^{-1}(y))\right)\\ &= F_{ij}\left(\varphi_i^{-1}(x), \psi_j^{-1}(y), f(\varphi_i^{-1}(x), \psi_j^{-1}(y))\right),\end{aligned}$$

for all $(x, y) \in G_{ij}$, $i = 1, 2, ..., m$; $j = 1, 2, ...n$.Let us attempt to establish that the operator T is well defined for all $(x, y) \in G_{ij}$. If (x, y) lies on the boundary of G_{ij}, $(i.e.\partial(G_{ij}))$ then following cases will arise.

(i) $(x, y) \in \partial(G_{ij}) \cap \partial(G_{i+1,j})$

(ii) $(x, y) \in \partial(G_{ij}) \cap \partial(G_{i,j+1})$

(iii) $(x, y) \in \partial(G_{ij}) \cap \partial(G_{i-1,j})$

(iv) $(x, y) \in \partial(G_{ij}) \cap \partial(G_{i,j-1})$

We prove that T is well defined for the first case and remaining can be proved similarly. When $(x, y) \in \partial(G_{ij}) \cap \partial(G_{i+1,j})$, (x, y) belongs to both $\partial(G_{ij})\ and \partial(G_{i+1,j})$. Let us consider first (x, y) as a point of $\partial(G_{ij})$. Then $\varphi_i^{-1}(x_i) = x_n\ and\ \psi_j^{-1}(y_j) = y_m$. Now,

$$\begin{aligned}(Tf)(x_i, y_j) =&F_{ij}\left(\varphi_i^{-1}(x_i), \psi_j^{-1}(y_j), z(\varphi_i^{-1}(x_i), \psi_j^{-1}(y_j)), t(\varphi_i^{-1}(x_i), \psi_j^{-1}(y_j))\right)\\ =&F_{ij}\left(x_n, y_m, z(x_n, y_m), t(x_n, y_m)\right)\\ =&(z_{ij}, t_{ij})\end{aligned}$$

Now we consider (x, y) as a point of $\partial(G_{i+1,j})$. Then $\varphi_{i+1}^{-1}(x_i) = x_0\ and\ \psi_j^{-1}(y_j) = y_m$.

$$\begin{aligned}(Tf)(x_i, y_j) = &F_{i+1,j}\Big(\varphi_{i+1}^{-1}(x_i), \psi_j^{-1}(y_j), z(\varphi_{i+1}^{-1}(x_i), \psi_j^{-1}(y_j)), t(\varphi_{i+1}^{-1}(x_i),\\ &\psi_j^{-1}(y_j))\Big)\\ =&F_{i+1,j}\left(x_0, y_m, z(x_0, y_m), t(x_0, y_m)\right)\\ =&(z_{ij}, t_{ij})\end{aligned}$$

Hence the operator is well defined. Now,

$$(Tf)(x_0, y_0) =F_{11}\left(\varphi_i^{-1}(x_0), \psi_j^{-1}(y_0), z(\varphi_i^{-1}(x_0), \psi_j^{-1}(y_0)), t(\varphi_i^{-1}(x_0), \psi_j^{-1}(y_0))\right)$$

$$=F_{11}\left(x_0, y_0, z(x_0, y_0), t(x_0, y_0)\right)$$
$$=(z_{00}, t_{00}), \quad since(x,y) \in G_{11},$$

$$(Tf)(x_m, y_0) = F_{m1}\left(\varphi_i^{-1}(x_m), \psi_j^{-1}(y_0), z(\varphi_i^{-1}(x_m), \psi_j^{-1}(y_0)), t(\varphi_i^{-1}(x_m), \psi_j^{-1}(y_0))\right)$$
$$=F_{m1}\left(x_m, y_0, z(x_m, y_0), t(x_m, y_0)\right)$$
$$=(z_{m0}, t_{m0}), \quad since(x,y) \in G_{m1}$$

Similarly we can prove

$$(Tf)(x_0, y_n) = (z_{0n}, t_{0n}) \text{ and } (Tf)(x_m, y_n) = (z_{mn}, t_{mn}).$$

If $f_1, f_2 \in (C_0(G))$, we can obtain the following result esily.

$$\|(Tf_1)(x,y) - (Tf_2)(x,y)\|_\infty \leq \delta \|f_1 - f_2\|_\infty$$

where $\delta = max\{s_{ij}, \beta_{ij}, \widetilde{\beta}_{ij}\} < 1$. From this we get that T is Lipschitz continuous and hence T is continuous and contraction. Since $(C_0(G))$ is a complete metric space, T possesses a fixed point f^*such that $(Tf^*)(x,y) = f^*(x,y) = (z_{f^*}(x,y), t_{f^*}(x,y))$. Let $\widetilde{E}$ be the graph of the function f^* then we have that $\widetilde{E} = \cup_{i=1}^m \cup_{j=1}^n \hat{w}_{i,j}(E)$. □

Corollary 3. *If we take the function vertical scaling factor in the definition of F_{ij}, then the theorem -(2) will also hold with the condition $|\beta_{ij}| + |\widetilde{\beta}_{ij}| < 1$.*

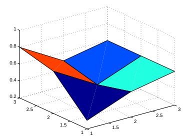
The proof is similar to theorem (2).

3. Examples and Results

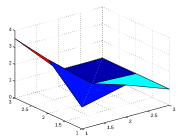
We consider the following sample data set

$$\{(1,1,0.3,0.3),(2,1,0.7,0.7),$$
$$(3,1,0.8,0.6),(1,2,0.5,0.4),(2,2,0.4,0.8),(3,2,0.5,0.8),(1,3,0.6,0.5),$$
$$(2,3,0.6,0.5),(3,3,0.6,0.9)\},$$

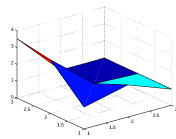
for our study. Fig.1. shows the corresponding linear surface for the sample data. Using the polynomial vertical scaling factor, the generalized hidden variable FIS's are shown in Fig.b and Fig.c for different values of β_{ij} *and* $\tilde{\beta_{ij}}$. In all



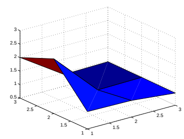
(a) Linear surface from the sample data



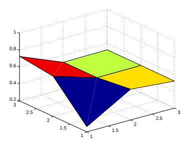
(b) HFIS(Poly) with $\beta_{ij} = 0.7,\ \tilde{\beta_{ij}} = 0.1$



(c) HFIS(Poly) with $\beta_{ij} = 0.1,\ \tilde{\beta_{ij}} = 0.1$



(d) HFIS (Trig)with $\beta_{ij} = 0.7,\ \tilde{\beta_{ij}} = 0.1$



(e) HFIS (Trig)with $\beta_{ij} = 0.1,\ \tilde{\beta_{ij}} = 0.1$

these cases we had fixed $\lambda's$ as $\lambda_{ij} = [1, 1; 1, 1]$. Fig.d and Fig.e depicts the generated hidden variable FIS's with the same set of β_{ij} *and* $\tilde{\beta_{ij}}$ values taken in the previous cases but with the trigonometric vertical scaling factor. Since $\lambda's$ are associated with only polynomial vertical scaling factor, we are free from $\lambda's$ in trigonometric vertical scaling factor. Hence the parameter $\lambda's$ is fixed and β_{ij} *and* $\tilde{\beta_{ij}}$ values alone is varied while generating surfaces using polynomial vertical scaling factor. Table-1 compares the average roughness(R_α), noise (N), variance (V) and root mean square roughness (R_q) of generated surfaces using polynomial vertical scaling factor and trigonometric vertical scaling factor.

The variation of the generated surfaces on changing the values of β_{ij} *and* $\tilde{\beta_{ij}}$ can be easily understood from table-1 and table -2. On comparison table-1 and table-2 we could see that table-2 roughness values and root mean square roughness values are nearer to the sample surface values rather than the table-1 roughness values. It concludes that hidden variable FIS with the trigonometric function vertical scaling factor give better approximations than HFIS with polynomial vertical scaling factor.

Table 1: Hidden variable FIS with polynomial vertical scaling factor

Details of figures	Average roughness (R_α)	Noise(N)	Variance of $Z_{i,j}(V)$	Root mean square roughness (R_q)
Fig.1 linear surface from interpolation data	0.5556	0	0.02278	0.0
Fig.2a.HFIS with $\beta_{ij} = 0.7, \tilde{\beta_{ij}} = 0.1$	1.6466	2.96	1.0376	4.1961
Fig.2b.HFIS with $\beta_{ij} = 0.1, \tilde{\beta_{ij}} = 0.1$	0.6685	1.20	0.0649	0.6065

Table 2: Hidden variable FIS with trigonometric vertical scaling factor

Details of figures	Average roughness (R_α)	Noise(N)	Variance of $Z_{i,j}(V)$	Root mean square roughness (R_q)
Fig.1 linear surface from interpolation data	0.5556	0	0.02278	0.0
Fig.3a.HFIS with $\beta_{ij} = 0.7, \tilde{\beta_{ij}} = 0.1$	1.2828	2.31	0.3342	2.5983
Fig.3b.HFIS with $\beta_{ij} = 0.1, \tilde{\beta_{ij}} = 0.1$	0.5362	0.97	0.0166	0.2102

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