International Journal of Pure and Applied Mathematics
Volume 107 No. 1 2016, 145-160
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i1.12
ijpam.eu

ON THE FRACTIONAL RICCATI DIFFERENTIAL EQUATION

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Abstract: In this paper, We tried to find an analytical solution of nonlinear Riccati conformable fractional differential equation. Fractional derivatives are described in the conformable derivative. The behavior of the solutions and the effects of different values of fractional order α are presented graphically and table. The results obtained by the CFD(conformable fractional derivative) are compared with homotopy perturbation method(HPM), fractional variational iteration method(FVIM).

Key Words: fractional Riccati differential equation, conformable fractional derivative, fractional calculus

1. Introduction

In recent years, it has worked that many phenomena in biology, chemistry, acoustics, control theory, psychology and other areas of science can be productively modeled by the use of fractional-order derivatives. The subject of

Received: October 24, 2015
Published: March 25, 2016

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fractional derivative is as old as calculus. In 1695, L'Hopital asked if the expression $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}$ f has any meaning. Since then, many researchers have been trying to generalize the concept of the usual derivative to fractional derivatives.

In mathematics, a Riccati equation is any first-order ordinary differential equation thatis quadratic in the unknown function. In other words, it is an equation of the form $\frac{dy(x)}{dx}=p\left(x\right)+Q\left(x\right)y\left(x\right)+R\left(x\right)y\left(x\right)^{2}$ where $P\left(x\right)\neq 0\,and\,R(x)\neq 0$. The equation is named after Jacopo Riccati (1676–1754)[1]. As it is well known, Riccati differential equations concerned with applications in pattern formation in dynamic games, linear systems with Markovian jumps, river flows, econometric models, stochastic control, theory, diffusion problems, and invariant embedding [2–9].Many studies have been conducted on solutions of the Riccati differential equations. Some of them, the approximate solution of ordinary Riccati differential equation obtained from homotopy perturbation method (HPM)[10–12], homotopy analysis method (HAM) [13], and variational iteration method proposed by He [14]. The He's homotopy perturbation method proposed by He [15–17] the variational iteration method [18] and Adomian decomposition method (ADM) [19]to solve quadratic Riccati differential equation of fractional order. The variational iteration method (VIM), which proposed by He [20-22], was successfully applied to autonomous ordinary and partial differential equations and other fields. Recently, the fractionel Riccati differential equation with modified Riemann-Liouville derivative is solved with help of fractional variational iteration method (FVIMM) [23].

Many researchers used an integral form for fractional derivative definition. The most popular definitions of fractional derivative are Liouville, Liouville left-sided derivative, Liouville right-sided derivative, Riemann-Liouville, Caputo and Grünwald-Letnikov definitions. For Riemann-Liouville, Caputo and other definitions and the characteristics of these definitions, we refer to reader to [24-32].

Liouville derivative:

$$D_x^{\alpha}\left(f\right)\left(x\right)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}f\left(\mu\right)\left(x-\mu\right)^{-\alpha}d\mu\ \ ,x\in\left(-\infty,\infty\right). \tag{1}$$

Liouville left-sided derivative:

$$D_{0^{+}}^{\alpha}\left(f\right)\left(x\right)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{0}^{x}f\left(\mu\right)\left(x-\mu\right)^{-\alpha+n-1}d\mu\ \ ,x>0. \tag{2}$$

Liouville fight-sided derivative:

$$D_{0^{-}}^{\alpha}\left(f\right)\left(x\right)=\frac{\left(-1\right)^{n}}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{x}^{\infty}f\left(\mu\right)\left(x-\mu\right)^{-\alpha+n-1}d\mu\ \ ,x<\infty. \tag{3}$$

Riemann-Liouville left-sided derivative:

$$ {}^{RL}D_{a^+}^{\alpha}\left(f\right)\left(x\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^{\infty} f\left(\mu\right)\left(x-\mu\right)^{-\alpha+n-1} d\mu \ , x \geq a. \tag{4} $$

Riemann-Liouville right-sided derivative:

$$ {}^{RL}D_{b^-}^{\alpha}\left(f\right)\left(x\right) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^{b} f\left(\mu\right)\left(x-\mu\right)^{-\alpha+n-1} d\mu \ , x \leq b. \tag{5} $$

Caputo left-sided derivative:

$$ *D_{a^+}^{\alpha}\left(f\right)\left(x\right) = \frac{1}{\Gamma(n-\alpha)}\int_a^{\infty} f^{(n)}\left(\mu\right)\left(x-\mu\right)^{-\alpha+n-1} d\mu \ , x \geq a. \tag{6} $$

Caputo right-sided derivative:

$$ *D_{b^-}^{\alpha}\left(f\right)\left(x\right) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^{b} f^{(n)}\left(\mu\right)\left(x-\mu\right)^{-\alpha+n-1} d\mu \ , x \leq b. \tag{7} $$

Grünwald-Letnikov left-sided derivative:

$$ {}^{GL}D_{a^+}^{\alpha} f\left(x\right) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma\left(\alpha+1\right)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, nh = x-a \tag{8} $$

Grünwald-Letnikov right-sided derivative:

$$ {}^{GL}D_{b^-}^{\alpha} f\left(x\right) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma\left(\alpha+1\right)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, nh = b-x \tag{9} $$

Recently, Khalil at al. give a new definition of fractional derivative and fractional integral [33]. This new definition benefit from a limit form as in usual derivatives. This new theory is improved by Abdeljawad[34]. This paper is planned as follows: In Section 2, we briefly give definitions related to the conformable fractional calculus theory. In Section 3, we define the fractional nonlinear Riccati differential equations with conformable derivative. We present the application of the fractional nonlinear Riccati differential equations and numerical results in Section 4. The conclusions are then given in the final Section 5.

2. Conformable Fractional Derivative

Here, some basic definitions and properties of the conformable fractional calculus theory which can be found in [33–37].

(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0\,(D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number[33].

(ii) All fractional derivatives do not satisfy the known product rule[33]:

$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f) \tag{10}$$

(iii)All fractional derivatives do not satisfy the known quotient rule[33]:

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{gD_a^\alpha - fD_a^\alpha}{g^2}. \tag{11}$$

(iv)All fractional derivatives do not satisfy the chain rule[33]:

$$D_a^\alpha(fog)(t) = f^{(\alpha)}(g(t))\,g^{(\alpha)}(t). \tag{12}$$

(v) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general.

(vi) Caputo definition assumes that the function f is differentiable.

Definition 2.1. Given a function $f : [0, \infty) \to R$. Then the "conformable fractional derivative "of f order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \tag{13}$$

For all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some (0, a), a>0, and $f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0) = f^{(\alpha)}(t)$[33].

One can easily show that T_α satisfies all the features in the following theorem[33].

Theorem 2.1. Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then

1. $T_\alpha(\mu f + \beta g) = \mu T_\alpha(f) + \beta T_\alpha(g)$ for all $a, b \in R$
2. $T_\alpha(t^p) = pt^{p-1}\varphi_\alpha(t)$ for all $p \in R$
3. $T_\alpha(\beta) = 0$, for all constant functions $f(t) = \beta$
4. $T_\alpha(fg) = gT_\alpha(f) + fT_\alpha(g)$
5. $\mathbf{T}_\alpha\left(\frac{\mathbf{f}}{\mathbf{g}}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{\mathbf{g^2}}$

6. If, in addition, f is differentiable, then $\mathbf{T}_\alpha(\mathbf{f})(\mathbf{t}) = t^{1-\alpha}\frac{df}{dt}(t)$.

Also:

1. $T_\alpha(1) = 0$.
2. $T_\alpha(e^{ax}) = ax^{1-\alpha}e^{ax}, \quad a \in R$
3. $T_\alpha(sinax) = ax^{1-\alpha}cosax, \quad a \in R$
4. $T_\alpha(cosax) = -ax^{1-\alpha}sinax, \quad a \in R$
5. $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$.
6. $T_\alpha\left(sin\left(\frac{t^\alpha}{\alpha}\right)\right) = cos\left(\frac{t^\alpha}{\alpha}\right)$,
7. $T_\alpha\left(cos\left(\frac{t^\alpha}{\alpha}\right)\right) = -sin\left(\frac{t^\alpha}{\alpha}\right)$
8. $T_\alpha\left(e^{\frac{t^\alpha}{\alpha}}\right) = e^{\frac{t^\alpha}{\alpha}}$

3. The Conformable Fractional Riccati Equation

In this paper, we have achieved the analytical solutions to conformable fractional Riccati differential equation[1]

$$\frac{d^\alpha y(x)}{dx^\alpha} = P(x) + Q(x)y(x) + R(x)y^2(x), \quad x \in R, \quad 0 < \alpha \leq 1, \quad t > 0, \tag{14}$$

subject to the initial conditions

$$y^{(k)}(0) = d_k, \quad k = 0, 1, 2, \ldots, n-1 \tag{15}$$

where α is fractional derivative order, n is an integer, $P(x)$, $Q(x)$ and $R(x)$ are known real functions, and d_k is a constant.

4. Applications

In this section, we present the solution of three examples of the Riccati differential equations as the applicability of conformable fractional differential equations.

Example 4.1. Let us consider the fractional Riccati differential equation: We get

$$\frac{d^{\alpha}y}{dx^{\alpha}} = -y^{2} + 1, \;\; 0 < \alpha \leq 1 \tag{16}$$

with initial conditions

$$y\left(0\right) = 0. \tag{17}$$

The same holds true for the Riccati equation. In fact, if one particular solution y_1 can be found, the general solution is obtained as

$$y = y_1 + \frac{1}{u} \tag{18}$$

substituting

A set of solutions to the Riccati equation is then given by

$$y = y_1 + \frac{1}{u}$$

where u is the general solution to the aforementioned linear equation.

For $T_{\alpha}\left(1\right) = 0,\, and\, y_1\left(x\right) = 1$ is particular solution.
$T_{\alpha}\left(f(x)\right) = x^{1-\alpha}\frac{df(x)}{dx}\;\, and\, T_{\alpha}\left(1+\frac{1}{u}\right) = -x^{1-\alpha}\frac{u^{'}}{u^{\mathbf{2}}}$

Substituting

$$1 + \frac{1}{u}$$

in the Riccati equation yields

$$-x^{1-\alpha}\frac{u^{'}}{u^{2}} = -\left(1+\frac{1}{u}\right)^{2} + 1$$

$$\Longrightarrow -x^{1-\alpha}\frac{u^{'}}{u^{2}} = -1 - \frac{2}{u} - \frac{1}{u^{2}} + 1$$

$$\Longrightarrow u^{'}x^{1-\alpha} = 2u + 1$$

Since $\frac{du}{2u+1} = x^{\alpha-1}dx$, integration gives $\frac{1}{2}\ln\left(2u+1\right) = \frac{x^{\alpha}}{\alpha} + lnc$ for some constant c .

$$\Longrightarrow 2u + 1 = e^{2\frac{x^{\alpha}}{\alpha}}c^{2}, \quad \left(c^{2} = c_1\right)$$

$$\Longrightarrow u = \frac{1}{2}(-1 + c_1 e^{2\frac{x^\alpha}{\alpha}})$$

Substituting

$$y = 1 + \frac{1}{u}$$

Now, the general solution is $y(x) = 1 + \frac{2}{-1+c_1 e^{2\frac{x^\alpha}{\alpha}}} = \frac{1+c_1 e^{2\frac{x^\alpha}{\alpha}}}{-1+c_1 e^{2\frac{x^\alpha}{\alpha}}}$, where $c_1$is constant. Finally, the initial condition $y(0) = 0$ implies that $c_1 = -1$. Thus, $y(x) = \frac{e^{2\frac{x^\alpha}{\alpha}}-1}{e^{2\frac{x^\alpha}{\alpha}}+1}$.for $\alpha = 1$, deterministic riccati differential equation solution is $y(x) = \frac{e^{2x}-1}{e^{2x}+1}$[38].

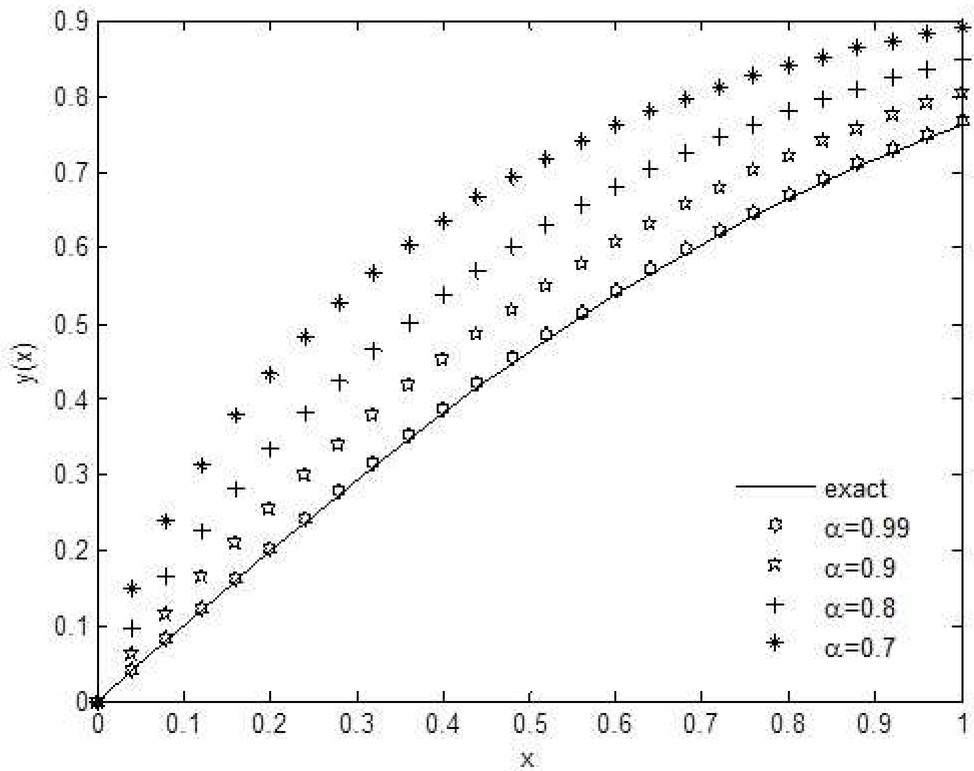

Figure 1: Plots of approx. solution $y(x)$ for different values $\alpha = 1, 0.99, 0.9, 0.8, 0.7$

Fig. 1 is plotted for approximate solution of time-fractional Riccati dif-

Table 1: Approximate solutions for (16). Values from the stated references are given in the table for the corresponding α values.

t	$[16]_{\alpha=0.5}$	$[23]_{\alpha=0.5}$	$y(x)_{\alpha=0.5}$	$[16]_{\alpha=0.75}$	$[23]_{\alpha=0.75}$	$y(x)_{\alpha=0.75}$	$y(x)_{\alpha=0.75}$	$y(x)_{exact}$
0	0	0	0	0	0	0	0	0
0.1	3.2738	0.0865	0.5597	0.1847	0.1901	0.2328	0.1030	0.0997
0.2	0.4541	0.1615	0.7136	0.3137	0.3100	0.3789	0.2025	0.1974
0.3	0.5739	0.2382	0.7989	0.4145	0.4050	0.4934	0.2947	0.2913
0.4	0.6444	0.3215	0.8524	0.4928	0.4834	0.5854	0.3866	0.3799
0.5	0.6741	0.4136	0.8884	0.4621	0.5504	0.6600	0.4688	0.4621
0.6	0.6719	0.5154	0.9137	0.5973	0.6103	0.7206	0.5435	0.5370
0.7	0.6480	0.6264	0.9320	0.6317	0.6669	0.7700	0.6104	0.6044
0.8	0.6133	0.7452	0.9456	0.6604	0.7237	0.8103	1.3649	0.6640
0.9	0.5796	0.8700	0.9560	0.6879	0.7836	0.8432	0.7212	0.7163
1	0.5585	0.9981	0.9640	0.7182	0.8487	0.8701	0.7658	0.7616

ferential equation for $\alpha = 1, 0.99, 0.9, 0.8, 0.7$. Eq.(16) are solved in using the homotopy perturbation method(HPM)[16] and fractional variational iteration method(FVIM)[23] . CFD solutions indicate that the present algorithm performs extremely efficiency, simplicity and reliability. The results obtained from CFD are fully compatible with the HPM and FVIM.

Table 1 shows the approximate solutions for Eq. (16) obtained for different values of a using the CFD[33],FVIM[16] and HPM[23].From the numerical results in Table 1, it is clear that the approximate solutions are in high agreement with the exact solutions, when $\alpha = 1$, and the solution continuously depends on the time-fractional derivative. Example 4.1 has been solved using HAM[13], ADM[19], VIM [14],FVIM[23] and HPM[11,16].

Example 4.2.

$$\frac{d^\alpha y}{dx^\alpha} = 2y - y^2 + 1,\ y(0) = 0,\ 0 < \alpha \leq 1 \tag{19}$$

The same holds true for the Riccati equation. In fact, if one particular solution y_1 can be found, the general solution is obtained as

$$y = y_1 + \frac{1}{u} \tag{20}$$

substituting

A set of solutions to the Riccati equation is then given by

$$y = y_1 + \frac{1}{u} \tag{21}$$

where u is the general solution to the aforementioned linear equation.

For $T_\alpha(1) = 0$, $and\, y_1(x) = 1 + \sqrt{2}$ is particular solution. $T_\alpha(f(x)) = x^{1-\alpha}\frac{df(x)}{dx}$ $and\, T_\alpha\left(1 + \sqrt{2} + \frac{1}{u}\right) = -x^{1-\alpha}\frac{u'}{u^2}$

Substituting

$$1 + \sqrt{2} + \frac{1}{u} \tag{22}$$

in the Riccati equation yields

$$-x^{1-\alpha}\frac{u'}{u^2} = 2(1 + \sqrt{2} + \frac{1}{u}) - \left(1 + \sqrt{2} + \frac{1}{u}\right)^2 + 1$$

$$\Longrightarrow -x^{1-\alpha}\frac{u'}{u^2} = 2 + 2\sqrt{2} - 1 - 2\sqrt{2} - 2 - \frac{2\sqrt{2}}{u} - \frac{1}{u^2} + 1$$

$$\Longrightarrow u'x^{1-\alpha} = 2\sqrt{2}u + 1$$

Since $\frac{du}{2\sqrt{2}u+1} = x^{\alpha-1}dx$, integration gives $\frac{1}{2\sqrt{2}}\ln\left(2\sqrt{2}u + 1\right) = \frac{x^\alpha}{\alpha} + lnc$ for some constant c .

$$\Longrightarrow 2\sqrt{2}u + 1 = e^{2\sqrt{2}\frac{x^\alpha}{\alpha}}c^2, \quad (c^{2\sqrt{2}} = c_2)$$

$$\Longrightarrow u = \frac{1}{2\sqrt{2}}(-1 + c_1 e^{2\sqrt{2}\frac{x^\alpha}{\alpha}})$$

Substituting

$$y = 1 + \sqrt{2} + \frac{1}{u}$$

Now, the general solution is $y(x) = 1+\sqrt{2}+\frac{2\sqrt{2}}{-1+c_2 e^{2\sqrt{2}\frac{x^\alpha}{\alpha}}} = \frac{-1+\sqrt{2}+c_2(1+\sqrt{2})e^{2\sqrt{2}\frac{x^\alpha}{\alpha}}}{-1+c_2 e^{2\sqrt{2}\frac{x^\alpha}{\alpha}}}$, where $c_1$is constant. Finally, the initial condition $y(0) = 0$ implies that $c_2 = 2\sqrt{2} - 3$ Thus, $y(x) = \frac{-1+\sqrt{2}+(1-\sqrt{2})e^{2\sqrt{2}\frac{x^\alpha}{\alpha}}}{-1+(2\sqrt{2}-3)e^{2\sqrt{2}\frac{x^\alpha}{\alpha}}}$.for $\alpha = 1$, deterministic riccati differential equation solution is $y(x) = \frac{-1+\sqrt{2}+(1-\sqrt{2})e^{2\sqrt{2}x}}{-1+(2\sqrt{2}-3)e^{2\sqrt{2}x}}$ or $y(x) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right))$[23,38].

Fig. 2 is plotted for approximate solution of time-fractional Riccati differential equation found in Example 4.2. In Fig. 2 , we have shown the graphic of approximate solution of Eq (19) for $\alpha = 1, 0.99, 0.9, 0.8, 0.7$. Figs. 1 and 2 show that a decrease in the fractional order α causes to an increase in the function.

Table 2 indicates the approximate solutions for Eq. (19) obtained for different values of a using the fractional variational iteration method[23], CFD[33]

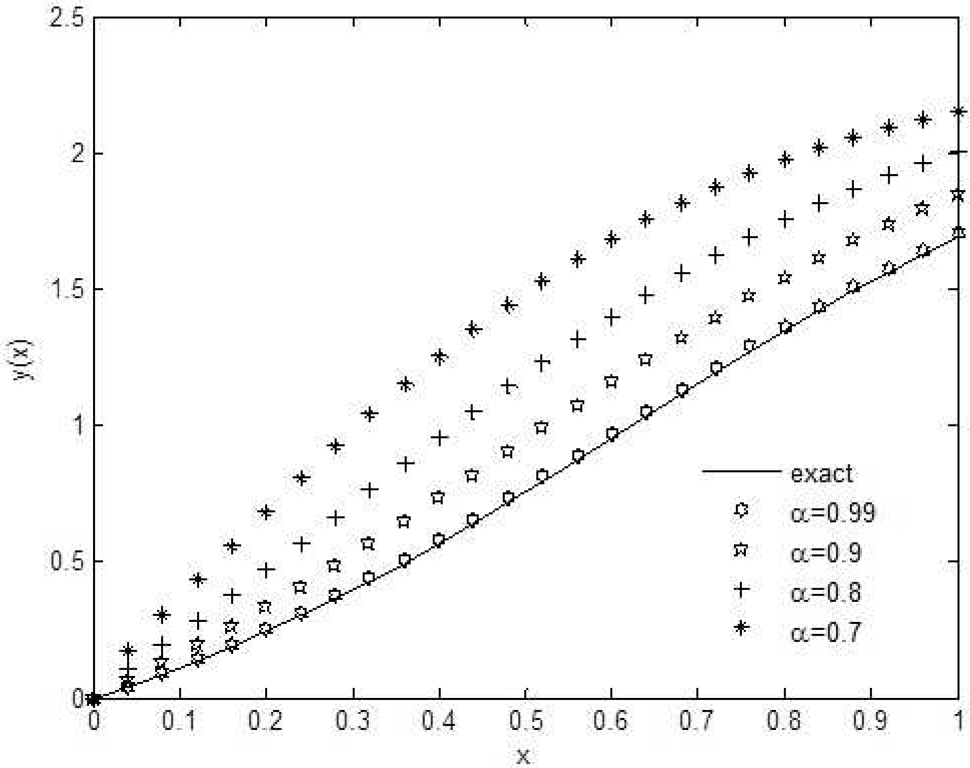

Figure 2: Plots of approx. solution $y(x)$ for different values of α

Table 2: Approximate solutions of (19). Values from the stated references are given in the table for the corresponding α values.

t	$[16]_{\alpha=0.5}$	$[23]_{\alpha=0.5}$	$y(x)_{\alpha=0.5}$	$[16]_{\alpha=0.75}$	$[23]_{\alpha=0.75}$	$y(x)_{\alpha=0.75}$	$y(x)_{\alpha=0.9}$	$y(x)_{exact}$
0	0	0	0	0	0	0	0	0
0.1	0.3217	0.5774	1.0185	0.2168	0.2444	0.2963	0.1144	0.1103
0.2	0.6296	0.9126	1.5173	0.4288	0.4697	0.5656	0.2496	0.2420
0.3	0.9409	1.1662	1.8253	0.6546	0.6987	0.8353	0.4061	0.3951
0.4	1.2507	1.3535	2.0181	0.8914	0.9243	1.0947	0.5819	0.5678
0.5	1.5494	1.4826	2.1414	1.1327	1.1379	1.3328	0.7727	0.7560
0.6	1.8254	1.5596	2.2221	1.3702	1.3314	1.5423	0.9719	0.9536
0.7	2.0665	1.5899	2.2762	1.5942	1.4976	1.7201	1.1719	1.1529
0.8	2.2606	1.5785	2.3133	1.7948	1.6302	1.8669	1.3649	1.3464
0.9	2.3968	1.5300	2.3393	1.9622	1.7244	1.9854	1.5441	1.5269
1	2.4660	1.4488	2.3578	2.0873	1.7765	2.0796	1.7048	1.6895

and HPM[16].From the numerical results in Table 2, it is clear that the approximate solutions are in high agreement with the exact solutions, when $\alpha = 1$, and the solution continuously depends on the time-fractional derivative.

Example 4.3.

$$\frac{d^{\alpha}y}{dx^{\alpha}} = -y + y^2,\ y\left(0\right) = \frac{1}{2},\ \ 0 < \alpha \leq 1 \tag{23}$$

The same holds true for the Riccati equation. In fact, if one particular solution y_1 can be found, the general solution is obtained as

$$y = y_1 + \frac{1}{u} \tag{24}$$

substituting

A set of solutions to the Riccati equation is then given by

$$y = y_1 + \frac{1}{u} \tag{25}$$

where u is the general solution to the aforementioned linear equation.

For $T_{\alpha}\left(1\right) = 0,\ and\, y_1\left(x\right) = 1$ is particular solution.

$T_{\alpha}\left(f(x)\right) = x^{1-\alpha}\frac{df(x)}{dx}\ and\, T_{\alpha}\left(1 + \frac{1}{u}\right) = -x^{1-\alpha}\frac{u'}{u^{2}}$

Substituting

$$1 + \frac{1}{u} \tag{26}$$

in the Riccati equation yields

$$-x^{1-\alpha}\frac{u^{'}}{u^2}=-\left(1+\frac{1}{u}\right)+\left(1+\frac{1}{u}\right)^2$$

$$\Longrightarrow -x^{1-\alpha}\frac{u^{'}}{u^2}=-1-\frac{1}{u}+1+\frac{2}{u}+\frac{1}{u^2}$$

$$\Longrightarrow u^{'}x^{1-\alpha}=-(u+1)$$

Since $\frac{du}{u+1}=-x^{\alpha-1}dx$, integration gives $\ln(u+1)=-\frac{x^\alpha}{\alpha}+lnc$ for some constant c .

$$\Longrightarrow u+1=e^{-\frac{x^\alpha}{\alpha}}c,$$

$$\Longrightarrow u=(-1+ce^{-\frac{x^\alpha}{\alpha}})$$

Substituting

$$y=1+\frac{1}{u}$$

Now, the general solution is $y(x)=1+\frac{1}{-1+ce^{-\frac{x^\alpha}{\alpha}}}=\frac{ce^{-\frac{x^\alpha}{\alpha}}}{-1+ce^{-\frac{x^\alpha}{\alpha}}}$, where c is constant. Finally, the initial condition $y(0)=\frac{1}{2}$ implies that $c=-1$. Thus, $y(x)=\frac{e^{-\frac{x^\alpha}{\alpha}}}{1+e^{-\frac{x^\alpha}{\alpha}}}$.for $\alpha=1$, deterministic riccati differential equation solution is $y(x)=\frac{e^{-x}}{1+e^{-x}}$[39,40].

Fig. 3 is plotted for approximate solution of time-fractional Riccati differential equation found in Example 4.3. In Fig. 3, we have shown the graphic of approximate solution of Eq (23) for α= 1, $0.99, 0.9, 0.8, 0.7$. Fig. 3 shows that a increase in the fractional order α causes to an increase in the function.

Table 3 indicates the approximate solutions for Eq. (23) obtained for different values of a using the CDF[33] and new homotopy perturbation method (NHPM)[38].From the numerical results in Table 3, it is clear that the approximate solutions are in substantially agreement with the exact solutions, when $\alpha=1$, and the solution continuously depends on the time-fractional derivative.

5. Conclusions

In this paper, analytical and numerical solutions of Riccati conformable fractional differential equation successfully obtained. It is also a promising method to solve other nonlinear equations. In this paper, we have discussed fractional Riccati equation having conformable fractional derivative(CFD) used for the

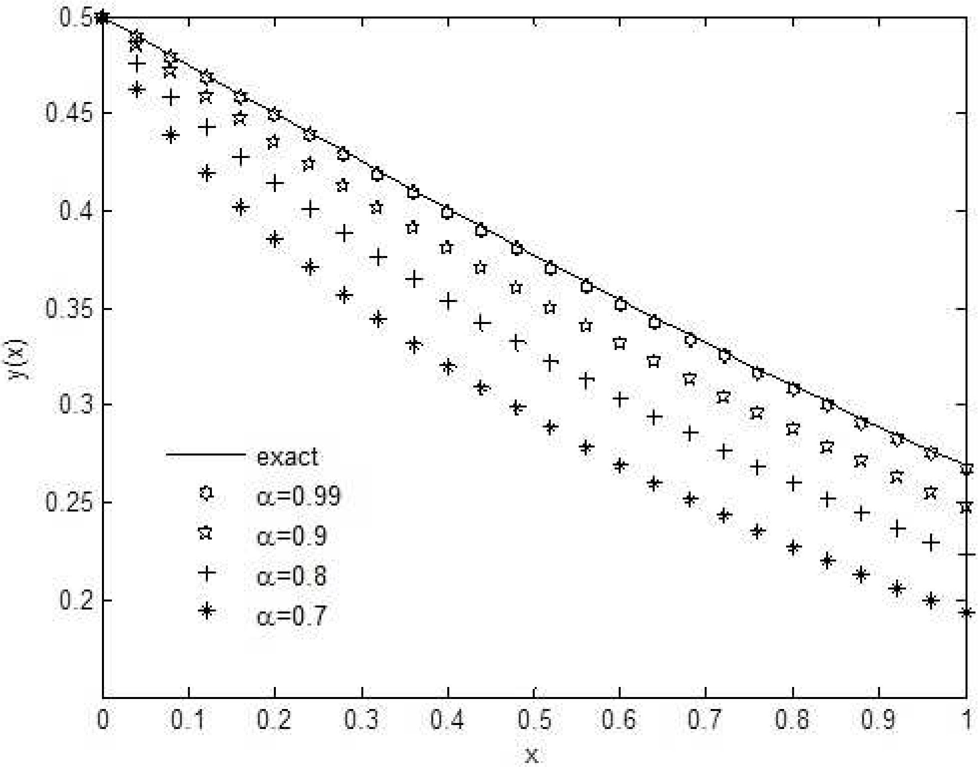

Figure 3: Plots of approx. solution $y(x)$ for different values of α

Table 3: Approximate results for (23). 5-term values from the stated references are given in the table for the corresponding α values.

t	$[38]_{\alpha=0.5}$	$[2]_{\alpha=1}$	$y(x)_{\alpha=0.5}$	$[38]_{\alpha=0.75}$	$y(x)_{\alpha=0.75}$	$y(x)_{\alpha=0.99}$	$y(x)_{\alpha exact}$
0	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.1	0.4304	0.4751	0.3470	0.4594	0.4410	0.4742	0.4750
0.2	0.3896	0.4368	0.2902	0.4190	0.4016	0.4489	0.4502
0.3	0.2830	0.2994	0.2506	0.2977	0.3681	0.4239	0.4256
0.4	-0.0218	-0.1664	0.2201	-0.0783	0.3384	0.3994	0.4013
0.5	-0.7399	-1.4063	0.1956	-1.0184	.0.3116	0.3755	0.3775
0.6	-2.1831	-4.1536	0.1752	-3.0014	0.2872	0.3522	0.3543
0.7	-4.7724	-9.5019	0.1580	-6.7066	0.2650	0.3297	0.3318
0.8	-9.0484	-18.9768	0.1432	-13.043	0.2446	0.3079	0.3100
0.9	-15.6809	-34.6077	0.1304	-23.180	0.2258	0.2870	0.2891
1	-25.4775	-59.0000	0.1192	-38.569	0.2086	0.2670	0.2689

first time by Khalil et al[33] . The obtained results indicate that this method is powerful and meaningful for solving the nonlinear fractional differential equations. Three examples indicate that the results of CFD are agreement with those obtained by HPM, ADM, HAM,FVIM which is available in the literature.

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