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ON THE PYTHAGOREAN HOLES OF CERTAIN GRAPHS

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Abstract: A *primitive hole* of a graph G is a cycle of length 3 in G. The number of primitive holes in a given graph G is called the primitive hole number of that graph G. The primitive degree of a vertex v of a given graph G is the number of primitive holes incident on the vertex v. In this paper, we introduce the notion of Pythagorean holes of graphs and initiate some interesting results on Pythagorean holes in general as well as results in respect of set-graphs and Jaco graphs.

1. Introduction

For general notations and concepts in graph theory, we refer to [1, 2, 6, 12]. All graphs mentioned in this paper are simple, connected undirected and finite, unless mentioned otherwise. Note that in the construction of a graph through steps and if the meaning of degree is clear at a step, the degree of a vertex v will be denoted $d(v)$. If we refer to degree in the context of a graph G, we will denote it as $d_G(v)$.

A *hole* of a simple connected graph G is a chordless cycle C_n, where $n \in N$, in G. A *primitive hole* of a graph G (see [7]) is a cycle of length 3 in G. The number of primitive holes in a given graph G is called the *primitive hole number* of G and is denoted by $h(G)$.

The *primitive degree* (see [7]) of a vertex v of a given graph G is the number of primitive holes incident on the vertex v and the primitive degree of the vertex v in the graph G is denoted by $d_G^p(v)$.

The notion of a set-graph has been introduced in [8] as follows.

Definition 1. [8] Let $A^{(n)} = \{a_1, a_2, a_3, \ldots, a_n\}, n \in \mathbb{N}$ be a non-empty set and the i-th s-element subset of $A^{(n)}$ be denoted by $A_{s,i}^{(n)}$. Now consider $\mathcal{S} = \{A_{s,i}^{(n)} : A_{s,i}^{(n)} \subseteq A^{(n)}, A_{s,i}^{(n)} \neq \emptyset\}$. The *set-graph* corresponding to set $A^{(n)}$, denoted $G_{A^{(n)}}$, is defined to be the graph with $V(G_{A^{(n)}}) = \{v_{s,i} : A_{s,i}^{(n)} \in \mathcal{S}\}$ and $E(G_{A^{(n)}}) = \{v_{s,i} v_{t,j} : A_{s,i}^{(n)} \cap A_{t,j}^{(n)} \neq \emptyset\}$, where $s \neq t$ or $i \neq j$.

The set $A^{(n)} \neq \emptyset$ and if $|A^{(n)}|$ is a singleton, then $G_{A^{(n)}}$ to be the trivial graph. Hence, we need to consider non-empty, non-singleton sets for the studies on set-graphs.

It is proved in [8] that any set-graph G has odd number of vertices. An important property on set-graphs is that the vertices of a set-graph G, corresponding to the sets of equal cardinality, have the same degree. Also, the vertices in a set-graph G, corresponding to the singleton subsets of $A^{(n)}$, are pairwise non-adjacent in G.

The following is a relevant result on the minimal and maximal degree of vertices in a set-graph and their relation.

Theorem 2. [8] *For any vertex v of a set-graph $G = G_{A^{(n)}}$, $2^{n-1} - 1 \leq d_G(v) \leq 2^n - 2$. That is, $\Delta(G) = 2\,\delta(G)$.*

It has also been proved in [8] that there exists a unique vertex v in a set-graph $G_{A^{(n)}}$ having the highest possible degree. Another relevant result reported in the Corrigendum to [8] is the following.

Theorem 3. *A set-graph $G_{A^{(n)}}, n \geq 2$ contains exactly $2n - 2$ largest complete subgraphs (cliques) $K_{2^{n-1}}$.*

The notion of the infinite Jaco graph (order 1) was introduced in [9] as follows.

Definition 4. [9] An *infinite Jaco graph (order 1)* is a directed graph, denoted by $J_\infty(1)$ whose vertex set is $V(J_\infty(1)) = \{v_i | i \in \mathbb{N}\}$ and the arc set is given by $E(J_\infty(1)) \subseteq \{(v_i, v_j) | i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \geq j$.

We denote a finite Jaco graph by $J_n(1)$ and its underlying graph by $J_n^*(1)$. In both instances we will refer to a Jaco graph and distinguish the context by the notation $J_n(1)$ or $J_n^*(1)$.

The set of vertices attaining degree $\Delta(J_n(1))$ is called the set of *Jaconian vertices* or the or the *Jaconian set* of the Jaco Graph $J_n(1)$ and is denoted by $\mathbb{J}(J_n(x))$ or simply by $\mathbb{J}_n(x)$ for brevity, if the context is clear. The lowest numbered (subscripted) Jaconian vertex is called the *prime Jaconian vertex* of a Jaco Graph.

If v_i is the prime Jaconian vertex of a Jaco Graph $J_n(1)$, the complete subgraph on vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ is called the *Hope subgraph* or *Hope graph* of a Jaco Graph and is denoted by $\mathbb{H}(J_n(x))$ or $\mathbb{H}_n(x)$ for brevity.

In this paper, we propose a new parameter called the number of Pythagorean holes of a graph. Further to some general results, we also discuss this parameter in respect of set-graphs and Jaco graphs.

2. Pythagorean Holes of Graphs

By a Pythagorean triple of positive integers, we mean an ordered triple (a, b, c), where $a < b < c$, such that $a^2 + b^2 = c^2$. Also, if (a, b, c) is a Pythagorean triple of positive integers, then for any (positive) integer k, the triple (ka, kb, kc) is also a Pythagorean triple. That is, we have $(ka)^2 + (kb)^2 = (kc)^2$.

Some results in this paper are Pythagorean triple specific and can be generalised to general ordered triples of positive integers. Some results could be generalised to other ordered triples satisfying other number theoretic conditions as well but for now, our interest lies in the notion of Pythagorean holes in respect of a Pythagorean triple. Using the concepts of Pythagorean triples, we now introduce the notion of Pythagorean holes of a given graph G as follows.

Definition 5. Let G be a non-empty finite graph and let the three vertices v_i, v_j, v_k in $V(G)$ induce primitive hole in G. This primitive hole is said to be a *Pythagorean hole* if $(d_G(v_i), d_G(v_j), d_G(v_k))$ are Pythagorean triple. That is, if $d_G^2(v_i) + d_G^2(v_j) = d_G^2(v_k)$. Let us denote the number of Pythagorean holes of a graph G by $h^p(G)$.

We can easily construct a graph with a Pythagorean hole as follows. Let (n_1, n_2, n_3) be a Pythagorean triple. Draw a triangle, say C_3, on the vertices v_1, v_2, v_3. We extend this triangle to a graph G where $d_G(v_i) = n_i$; $1 \leq i \leq 3$ as follows. Attach $n_1 - 2$ pendant vertex to the vertex v_1, attach $n_2 - 2$ pendant vertices to the vertex v_2 and add $n_3 - 2$ pendant vertices to v_3 to obtain a new graph G. Here, G is a unicyclic graph on $n_1 + n_2 + n_3 - 3$ vertices and edges each

and has one primitive hole. Therefore, the triangle $v_1 v_2 v_3 v_1$ is a Pythagorean hole in G.

By a *minimal graph* with respect to a given property, we mean a graph with minimum order and size satisfying that property. In view of this concept we introduce the following notion.

Definition 6. A *graphical embodiment* of a given Pythagorean triple is the minimal graph that consists of a Pythagorean hole with respect to that Pythagorean triple.

Clearly, the graph G mentioned above is not the graphical embodiment of the Pythagorean triple (n_1, n_2, n_3). Verifying the existence of a graphical embodiment to a given Pythagorean triple is an interesting question that leads to the following theorem.

Theorem 7. *There exists a unique graphical embodiment for every Pythagorean triple of positive integers.*

Proof. Let (n_1, n_2, n_3) be a Pythagorean triple of positive integers such that $n_1 < n_2 < n_3$. First, draw a triangle on vertices v_1, v_2, v_3. Now, plot $n_1 - 2$ vertices and attach them to the vertex v_1 so that $d(v_1) = n_1$. Now, attach the $n_1 - 2$ vertices to v_2 and v_3 also. At this step, $d(v_2) = d(v_3) = n_1$. Now, the $n_2 - n_1$ additional edges are required to be incident on the vertex v_2. Hence, plot new $n_2 - n_1$ vertices and attach them to v_2 and v_3. Now, $d(v_2) = n_2$, as required. But, here $d(v_3) = n_2$ and additionally $n_3 - n_2$ edges are to be incident on v_3. Hence, create new $n_3 - n_2$ vertices attach them to v_3 so that $d(v_3) = n_3$. In the resultant graph G, $d_G(v_1)^2 + d_G(v_2)^2 = d_G(v_3)^2$. Hence, the triangle $v_1 v_2 v_3 v_1$ is a Pythagorean hole in G. Clearly, this graph is the smallest graph with a Pythagorean hole corresponding to the given Pythagorean triple. Any graph other than G will have more vertices than G. Hence, G is a unique graphical embodiment of the given Pythagorean triple. □

The graph G in Figure 1 is an example for a graph containing a Pythagorean hole corresponding to a Pythagorean triple $(3, 4, 5)$. The graph G has the minimum number of vertices (that is, 6 vertices) required to contain a Pythagorean hole.

From figure 1, it can be noted that the graphical embodiment of the Pythagorean triple of $(3, 4, 5)$ has two Pythagorean holes. The uniqueness of the Pythagorean hole in a graphical embodiment of a given Pythagorean triple, other than triple $(3, 4, 5)$, is established in the following theorem.
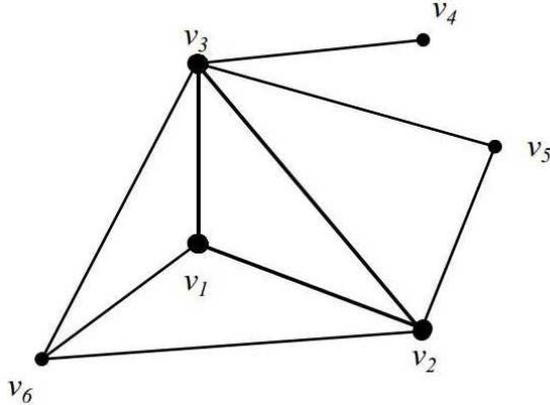
Figure 1: The graphical embodiment of the Pythagorean triple $(3, 4, 5)$.

Theorem 8. *The graphical embodiment of any Pythagorean triple of positive integers $(n_1, n_2, n_3) \neq (3, 4, 5)$, consists of a unique Pythagorean hole in it.*

Proof. Let $(n_1, n_2, n_3) \neq (3, 4, 5)$ be a Pythagorean triple of positive integers and let G be a graphical embodiment of this triple obtained as explained in Theorem 7. Let $v_1 v_2 v_3 v_1$ be a Pythagorean hole in G such that $d_G(v_1) = n_1, d_G(v_2) = n_2$ and $d_G(v_3) = n_3$.

It is to be noted that all new vertices that are adjacent to v_1 in G will be adjacent to both v_2 and v_3 also and hence are of degree 3. Similarly, all new vertices that are adjacent to v_2 is adjacent to v_3 also. Therefore, the degree of the vertices that are adjacent to v_2, but not to v_1, is 2 and the degree of the vertices that are adjacent only to v_3 is 1. Also, no two of these new vertices are mutually adjacent. Hence, for any three vertices v_i, v_j, v_k, other than v_1, v_2, v_3, do not satisfy the condition $d_G(v_i)^2 + d_G(v_j)^2 = d_G(v_k)^2$. Therefore, the triangle $v_1 v_2 v_3$ is the unique Pythagorean hole in G. $\qquad \square$

As stated earlier both Theorem 7 and Theorem 8 can be generalised to other triples with other number theoretic properties.The characteristics of the graphical embodiment of a Pythagorean triple seems to be much promising in this context. The size and order of the graphical embodiment are determined in the following result.

Theorem 9. *Let G be the graphical embodiment of a given Pythagorean triple (n_1, n_2, n_3), with usual notations. Then,*

(i) *the order (the number of vertices) of G is one greater than the highest number in the corresponding Pythagorean triple.*

(ii) *the size (the number of edges) of G is three less than the sum of numbers in the corresponding Pythagorean triple.*

Proof. Let G be a graphical embodiment of a Pythagorean triple (n_1, n_2, n_3). Let v_1, v_2, v_3 be the vertices with $d_G(v_1) = n_1, d_G(v_2) = n_2$, and $d_G(v_3) = n_3$. As explained in Theorem 7, $n_1 - 2$ vertices are attached to v_1, v_2, and v_3, further $n_2 - n_1$ vertices are attached to v_2 and v_3 and $n_3 - n_2$ pendant vertices are attached to v_3. Then,

(i) The number of vertices in G is $|V(G)| = 3 + n_1 - 2 + n_2 - n_1 + n_3 - n_2 = n_3 + 1$. That is, the order of the graphical embodiment is one greater than the highest number in the corresponding Pythagorean triple.

(ii) Let V_1 be the set of all newly introduced vertices which are adjacent to all three vertices v_1, v_2, and v_3. Therefore, for all vertices x in V_1, we have $d_G(x) = 3$. Therefore, $\sum_{x \in V_1} d_G(x) = 3(n_1 - 2)$. Similarly, let V_2 be the set of new vertices which are adjacent to v_2, and v_3. For all vertices y in V_2, we have $d_G(y) = 3$. Therefore, $\sum_{y \in V_2} d_G(y) = 2(n_2 - n_1)$. Also, let V_3 be the set of new vertices that are adjacent to v_3 only. Here, for all vertex z in V_3, we have $d_G(z) = 1$ and hence $\sum_{z \in V_3} d_G(z) = (n_3 - n_2)$.

Therefore, $\sum_{v \in V(G)} d_G(v) = n_1 + n_2 + n_3 + 3(n_1 - 2) + 2(n_2 - n_1) + (n_3 - n_2) = 2(n_1 + n_2 + n_3 - 3)$. Since for any connected graph G, $|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d_G(v)$, we have $|E(G)| = n_1 + n_2 + n_3 - 3$. That is, the size of G is three less than the sum of numbers in the Pythagorean triple.

This completes the proof.   □

Theorem 10.   *The primitive hole number of the graphical embodiment G of a Pythagorean triple is $h(G) = 2n_1 + n_2 - 5$.*

Proof. Let G be a graphical embodiment of a Pythagorean triple (n_1, n_2, n_3). Let v_1, v_2, v_3 be the vertices with $d_G(v_1) = n_1, d_G(v_2) = n_2$, and $d_G(v_3) = n_3$. As explained in Theorem 7, $n_1 - 2$ vertices are attached to all three vertices v_1, v_2 and v_3, $n_2 - n_1$ vertices are attached to both v_2 and v_3 and $n_3 - n_2$ pendant vertices are attached to the vertex v_3. Let V_1, V_2, V_3 be the set of vertices as explained in the previous theorem.

Then, every vertex in V_1 forms a triangle with any two vertices among v_1, v_2 and v_3. Hence each vertex in V_1 corresponds to three triangles in G. Therefore, the total number of such triangles is $3(n_1 - 2)$. Similarly, every vertex in V_2, being adjacent only to v_2 and v_3, forms a triangle in G and no vertex in V_3 is a part of a triangle in G, Therefore, the total number of triangles in G is $h(G) = 1 + 3(n_1 - 2) + (n_2 - n_1) = 2n_1 + n_2 - 5$. This completes the proof. $\square$

The following theorem discusses certain parameters of the graphical embodiments of the given Pythagorean triples.

Theorem 11. *Let G be the embodiment of a Pythagorean triple (n_1, n_2, n_3). Then we have*

(i) *the chromatic number of G is $\chi(G) = 4$.*

(ii) *the independence number of G is two less than the highest number in the Pythagorean triple. That is, $\alpha(G) = n_3 - 2$.*

(iii) *the covering number of G is $\beta(G) = 3$.*

(iv) *the domination number of G is $\gamma(G) = 1$.*

Proof. Let G be a graphical embodiment of a Pythagorean triple (n_1, n_2, n_3) in which v_1, v_2, v_3 are the vertices with $d_G(v_1) = n_1, d_G(v_2) = n_2,$ and $d_G(v_3) = n_3$. Let V_1 be the set of all new vertices that are attached to all three vertices v_1, v_2 and v_3, V_2 denote the set of all new vertices that are attached to both v_2 and v_3 and V_3 be the set of new vertices that are attached to v_3 alone. Then,

(i) Colour v_1 by the colour c_1, v_2 by the colour c_2 and v_3 by the colour c_3. Since all vertices in the set V_1 and are adjacent to the above three colors can not be used for coloring the vertices of V_1. Since no two vertices in V_1 are mutually adjacent in G, all vertices in V_1 can be colored using a fourth colour c_4. Since all vertices in V_2 are mutually non-adjacent in G and are not adjacent to the vertex v_1, we can use the colour c_1 for coloring all the vertices in V_2. Since all the vertices in V_3 are adjacent to v_3 only and are pairwise non-adjacent in G, we can colour them using either the colour c_1 or the colour c_2. Therefore, the minimal proper coloring of G contains four colours. That is, $\chi(G) = 4$.

(ii) All the newly introduced vertices in V_1, V_2 and V_3 are mutually non-adjacent and hence $V_1 \cup V_2 \cup V_3$ is the maximal independent set in G. Therefore, $\alpha(G) = n_1 - 2 + n_2 - n_1 + n_3 - n_2 = n_3 - 2$.

(iii) The relation between the independence number and the covering number of a connected graph is $\alpha(G) + \beta(G) = |V(G)|$. By Theorem 9, $|V(G)| =$

$n_3 + 1$ and by previous result, $\alpha(G) = n_3 - 2$. Therefore, the covering number of G is $n_3 + 1 - (n_3 - 2) = 3$.

(iv) Clearly, the vertex v_3 is adjacent to all other vertices in the graphical embodiment G, we have $\gamma(G) = 1$.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

It is noted from Theorem 11, the chromatic number, covering number and domination number of the graphical embodiment of any Pythagorean triple are always the same.

An immediate consequence of the general Pythagorean property is that the vertices of Pythagorean holes can be mapped onto interesting Euclidean geometric objects which we can construct along the sides of a right angled triangle or along the surfaces of the corresponding right angled prism.

Illustration 1. For real x, let

$$d_G(v_1) = +\sqrt{(\frac{a}{2})^2 - x^2}, \quad d_G(v_2) = +\sqrt{(\frac{b}{2})^2 - x^2} \text{ and } d_G(v_3) = +\sqrt{(\frac{c}{2})^2 - x^2},$$

where (a, b, c) is a Pythagorean triple of positive integers. Then,

$$d_G^2(v_1) = \int_{-\frac{a}{2}}^{\frac{a}{2}} \sqrt{(\frac{a}{2})^2 - x^2} dx = \frac{1}{2}\pi(\frac{a}{2})^2 = \frac{1}{8}\pi a^2$$

$$d_G^2(v_2) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sqrt{(\frac{b}{2})^2 - x^2} dx = \frac{1}{2}\pi(\frac{b}{2})^2 = \frac{1}{8}\pi b^2$$

$$d_G^2(v_3) = \int_{-\frac{c}{2}}^{\frac{c}{2}} \sqrt{(\frac{c}{2})^2 - x^2} dx = \frac{1}{2}\pi(\frac{c}{2})^2 = \frac{1}{8}\pi c^2.$$

Since $\frac{1}{8}\pi$ is a constant, we have

$$d_G^2(v_1) + d_G^2(v_2) = \frac{1}{8}\pi(a^2 + b^2) = \frac{1}{8}\pi c^2 = d_G^2(v_3).$$

That is, the Pythagorean property holds for the triple $(d_G(v_1), d_G(v_2), d_G(v_3))$. Geometrically it means that the area of the semi-circle with length of hypotenuse as its diameter is equal to the sum of the respective areas of the

semi-circles with lengths of the other two sides of a right angled triangle as the diameters.

Illustration 2.   For an arbitrary real number x, let $d_G(v_1) = x + bc$, $d_G(v_2) = x + ac$ and $d_G(v_3) = x + 2ab$, where (a, b, c) is a Pythagorean triple of positive integers. Then,

$$d_G^2(v_1) = \int_0^a (x + bc)dx = \frac{1}{2}a^2 + abc,$$

$$d_G^2(v_2) = \int_0^b (x + ac)dx = \frac{1}{2}b^2 + abc,$$

$$d_G^2(v_3) = \int_0^c (x + 2ab)dx = \frac{1}{2}c^2 + 2abc.$$

Since $\frac{1}{2}a^2 + abc + \frac{1}{2}b^2 + abc = \frac{1}{2}c^2 + 2abc$, it implies geometrically that the area under the straight line $f(x) = x + 2ab, x \in \mathbb{R}$ between the limits $x = 0$ and $x = c$ is equal to the sum of the respective areas under the straight lines $f(x) = x + bc, x \in \mathbb{R}$ between the limits $x = 0$ and $x = a$ and $f(x) = x + ac, x \in \mathbb{R}$ between the limits $x = 0$ and $x = b$ in respect of a right angled triangle.

Clearly, we can find many such mappings and it would be worthy to find the applications of these mappings.

3. On Pythagorean Holes of Set-Graphs

The following result is on the degree sequence of set-graphs.

Lemma 12.   Consider the degree sequence $(d_1, d_2, d_3, \ldots, d_r)$ of the set-graph $G_{A^{(n)}}$. Then, for any triple (d_i, d_j, d_k), where $d_i < d_j < d_k$, in that degree sequence, $d_i + d_j > d_k$.

Proof. Recall the theorem that for any set-graph $G = G_{A^{(n)}}$, $2\delta(G) = \Delta(G)$. Therefore, as $d_i + d_j > 2\delta(G) = \Delta(G) \geq d_k$, the result holds for the new vertices of degree $2^n + d_i$ or $2^n - 1$ as well. □

Theorem 13.   *A set-graph has no Pythagorean holes.*

Proof. Note that for any Pythagorean triple (a, b, c) with $a < b < c$, either a, b, c are all even or only one of a, b is even by taking modulo 4 (see [4]). But, for any set-graph $G = G_{A^{(n)}}$, we have $\Delta(G)$ is always even and the vertex of G having degree $\Delta(G)$ is the only vertex having even degree. Therefore, no set-graph can have a Pythagorean hole. □

4. Pythagorean Holes of Jaco Graphs

The primitive hole number of the underlying graph of a Jaco graph has been determined in the following theorem (see [7]).

Theorem 14. [7] Let $J_n^*(1)$ be the underlying graph of a finite Jaco Graph $J_n(1)$ with Jaconian vertex v_i, where n is a positive integer greater than or equal to 4. Then, $h(J_{n+1}^*(1)) = h(J_n^*(1)) + \sum_{j=1}^{(n-i)-1} (n-i) - j$.

It can be noted that the smallest Jaco graph having a Pythagorean hole is $J_8^*(1)$.

Theorem 15. For any primitive hole of the Jaco graph $J_n^*(1), n \in \mathbb{N}$ on the vertices v_i, v_j, v_k with $i < j < k$ we have a primitive hole on the vertices v_{li}, v_{lj}, v_{lk} in $J_{n \geq lk}^*(1), l \in \mathbb{N}$ if the edge $v_{li}v_{lk}$ exists.

Proof. For any primitive hole of the Jaco graph $J_n^*(1), n \in \mathbb{N}$ on the vertices v_i, v_j, v_k with $i < j < k$ the edge $v_i v_k$ exists hence $k \leq i + d_{J_n(1)}^+(v_i)$. So $lk \leq l(i + d_{J_n(1)}^+(v_i)) = li + ld_{J_n(1)}^+(v_i)$. Assume that the edge $v_{li}v_{lk}$ exists. Then, the subgraph induced by vertices $v_i, v_{i+1}, v_{i+2}, ...v_j, v_{j+1}, v_{j+2}, ..., v_k$ is a complete graph and the same holds for the vertex v_j. That is, edges $v_{li}v_{lj}$ and $v_{lj}v_{lk}$ exists. $\square$

It is known that there are 16 primitive Pythagorean triples (a, b, c) with $c \leq 100$. These, in ascending order of c are: $t_1 = (3, 4, 5)$; $(5, 12, 13)$; $(8, 15, 17)$; $(7, 24, 25)$; $t_2 = (20, 21, 29)$; $(12, 35, 37)$; $(9, 40, 41)$; $t_3 = (28, 45, 53)$; $(11, 60, 61)$; $(16, 63, 65)$; $(33, 56, 65)$; $t_4 = (48, 55, 73)$; $(13, 84, 85)$; $(36, 77, 85)$; $(39, 80, 89)$; $t_5 = (65, 72, 97)$.

From the definition of Jaco graphs, it easily follows that the Pythagorean triples labeled t_i; $1 \leq i \leq 5$ are applicable to the Pythagorean holes found in Jaco graphs. We shall refer to these Pythagorean triples as *type i*; $i \in \mathbb{N}$. With regard to Theorem 15, we shall refer to $lt_i = (a_i l, b_i l, c_i l)$ as *type i* as well. The number of Pythagorean holes of type i in a graph will be denoted $h_{t_i}^p(G)$.

Other Pythagorean triples generated by Euclid formula which are not of a specific primitive type offer some additional Pythagorean holes in Jaco graphs. Let us denote these additional types as e_i; $i \in \mathbb{N}$.
The following results will hold for type 1, $t_1 = (3, 4, 5)$.

Corollary 16. The Jaco graph $J_n^*(1)$, where $n = 5k + d_{J_\infty(1)}^+(v_{5k})$, has $h_{t_1}^p(J_n^*(1)) = k$ Pythagorean holes, if $3k + d_{J_\infty(1)}^+(v_{3k}) \geq 5k$.

Proof. The result follows from Theorem 15. □

Corollary 17. *The Jaco graphs $J_n^*(1), 8 \leq n \leq 15$ are the only Jaco graphs with a unique Pythagorean hole.*

Proof. Since $2 \times 5 = 10$ and $d_{J_{16}^*(1)}(v_{10}) = 10$ whilst $d_{J_n^*(1)}(v_{10}) < 10$, for $n \leq 15$, the Jaco graph $J_{16}^*(1)$ has two Pythagorean holes because, $d_{J_{16}^*(1)}^2(v_3) + d_{J_{16}^*(1)}^2(v_4) = d_{J_{16}^*(1)}^2(v_5)$ and $d_{J_{16}^*(1)}^2(v_6) + d_{J_{16}^*(1)}^2(v_8) = d_{J_{16}^*(1)}^2(v_{10})$. All other Jaco graphs $J_n^*(1)$, for $n \geq 16$, will have at least two Pythagorean holes. Since Jaco graphs $J_{n,1 \leq n \leq 7}^*(1)$ have no Pythagorean hole, the result follows. □

Theorem 18. *The Jaco graph $J_n^*(1), n \geq 8$ has $h_{t_1}^p(J_n^*(1)) = k$ Pythagorean holes for $5k + d_{J_\infty(1)}^+(v_{5k}) \leq n < 5(k+1) + d_{J_\infty(1)}^+(v_{5(k+1)})$. Alternatively, $h_{t_i}^p(J_n^*(1)) = \lfloor \frac{n}{8} \rfloor$.*

Proof. A direct consequence of Theorem 15 and Corollary 16. □

The adapted Fisher Table (see Table 1), $J_\infty(1), 35 \geq n \in \mathbb{N}$ depicts the values of $h(J_n^*(1))$ and $h_{t_1}^p(J_n^*(1))$.

It can be seen that Pythagorean holes are generally scarce within Jaco graphs compared to the number of primitive holes. But since the graph in figure 1 can be extended endlessly through edge-joints to produce a graph H with $h^p(H) = h(H)$, the inequality $h^p(G) \leq h(G)$ holds. To construct the adapted Fisher table an improvement to Theorem 14 is required.

Theorem 19. *For the underlying graph $J_n^*(1)$ of a finite Jaco Graph $J_n(1), n \in \mathbb{N}, n \geq 4$ we have the recursion formula $h(J_{n+1}^*(1)) = h(J_n^*(1)) + \sum\limits_{i=1}^{d_{J_\infty(1)}^-(v_{n+1})-1} i$.*

Proof. Consider the Jaco graph $J_n^*(1)$ with $h(J_n^*(1)) = k$. By extending from $J_n(1)$ to $J_{n+1}(1)$ a total of $d_{J_\infty(1)}^-(v_{n+1})$ arcs are added together with the vertex v_{n+1}. In the Jaco graph $J_{n+1}^*(1)$ vertex v_{n+1} is a vertex of the new underlying Hope graph, $\mathbb{H}(J_{n+1}^*(1))$. Hence, with v_{n+1} a common vertex to all, exactly $\sum\limits_{i=1}^{d_{J_\infty(1)}^-(v_{n+1})-1} i$, additional primitive holes are added. Hence, we have the result $h(J_{n+1}^*(1)) = h(J_n^*(1)) + \sum\limits_{i=1}^{d_{J_\infty(1)}^-(v_{n+1})-1} i$. □

The following results is a direct consequence of Theorem 15 and Theorem 18.

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$h(J_i^*(1))$	$h_{t_1}^p(J_n^*(1))$
$1=f_2$	0	1	0	0
$2=f_3$	1	1	0	0
$3=f_4$	1	2	0	0
4	1	3	0	0
$5=f_5$	2	3	1	0
6	2	4	2	0
7	3	4	5	0
$8=f_6$	3	5	8	1
9	3	6	11	1
10	4	6	17	1
11	4	7	23	1
12	4	8	29	1
$13=f_7$	5	8	39	1
14	5	9	49	1
15	6	9	64	1
16	6	10	79	2
17	6	11	94	2
18	7	11	115	2
19	7	12	136	2
20	8	12	164	2
$21=f_8$	8	13	192	2
22	8	14	220	2
23	9	14	256	2
24	9	15	292	3
25	9	16	328	3
26	10	16	373	3
27	10	17	418	3
28	11	17	473	3
29	11	18	528	3
30	11	19	583	3
31	12	19	649	3
32	12	20	715	4
33	12	21	781	4
$34=f_9$	13	21	859	4
35	13	22	937	4

Table 1

Theorem 20. *Consider the distinct Pythagorean triples $t_i, t_j, \ldots, t_k, e_r,$ $e_s, \ldots e_x$ where no triple is a multiple of another, with the form $(a_\alpha, b_\alpha, c_\alpha)$ and $(a_\beta, b_\beta, c_\beta)$ and $c = max\{c_i, ..., c_k, c_r, ..., c_x\}$. Then, $h^p(J_{c+d^+(v_c)}^*(1)) =$*

$$\sum_{t_l} h_{t_l}^p(J_{c+d^+(v_c)}^*(1)) + \sum_{e_m} h_{e_m}^p(J_{c+d^+(v_c)}^*(1)), \text{ for all } l = i, j, \ldots, k \text{ and } m = r, s \ldots, x.$$

The following results is a direct consequence of Theorem 20 and Corollary 16.

Corollary 21. *Let t_i (or e_m) denotes a Pythagorean triple $(a_\alpha, b_\alpha, c_\alpha)$, then $h_{t_i}^p(J_n^*(1)) = \lfloor \frac{n}{c_\alpha + d^+(v_{c_\alpha})} \rfloor$ or $h_{e_m}^p(J_n^*(1)) = \lfloor \frac{n}{c_\alpha + d^+(v_{c_\alpha})} \rfloor$.*

5. Conclusion and Scope for Further Studies

We have discussed particular types of holes called Pythagorean holes of given graphs and studied the existence thereof in certain graphs, particularly in set-graphs and Jaco graphs.

As all graphs do not contain Pythagorean holes, the study on the characteristics and structural properties of graphs containing Pythagorean holes arouses much interest. The number of Pythagorean holes in a given graph is a parameter which needs further investigation.

We proved that the graphical embodiment of all Pythagorean triples, except for $(3, 4, 5)$, contain exactly one Pythagorean hole. But, some graphs may contain more than one Pythagorean hole. The study on the graphs or graph classes containing more than one Pythagorean hole corresponding to one or more Pythagorean triples demands further investigation. Another interesting related area for further research is to construct the graphical embodiments of other triples and general n-tuples and study their characteristics.

The study seems to be promising as it can be extended to certain standard graph classes and certain graphs that are associated with the given graphs. More problems in this area are still open and hence there is a wide scope for further studies.

It was illustrated that the vertices of a Pythagorean hole can be mapped onto Euclidean geometric objects. An imaginative prospect is that these theoretical applications can perhaps find real application in nano technology. It is proposed that this avenue deserves further research.

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