ITERATIVE METHOD WITH HIGHER-ORDER CONVERGENCE FOR SCALAR EQUATIONS

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\textbf{Abstract:} We establish new higher-order iterative methods for the solution of scalar equations by using the decomposition technique mainly due to Adomian, see [2].

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1. Introduction

During last many years, much attention has been given to establish several iterative methods for solving nonlinear equations, see [1, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14] and the references therein. These methods can be classified as one-step, two-step and three-step methods. Two-step methods have been suggested by combining the well-known Newton method with other one-step im-
licit methods. In [5] Chun has proposed and studied several one-step and two-step iterative methods with higher-order convergence by using the decomposition technique of Adomian [2]. To overcome this drawback, Noor and Noor [12] suggested and analyzed a family of multi-step methods for solving nonlinear equations using a different type of decomposition technique mainly due to Daftardar-Gejji and Jafari [7], which does not involve the high-order differentials of the function.

Our problem, to recall, is solving equations in one variable. We are given a function \( f \), and would like to find at least one solution to the equation \( f(x) = 0 \). Note that, a priori, we do not put any restrictions on the function \( f \); we do need to be able to evaluate the function: otherwise, we cannot even check that a given solution \( x = \alpha \) is true, that is, \( f(\alpha) = 0 \). In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of “good behavior”. The more we assume, the more potential we have, on the one hand, to develop fast algorithms for finding the root. At the same time, the more we assume, the fewer functions are going to satisfy our assumptions! This is a fundamental paradigm in Numerical Analysis.

During the last many years, the numerical techniques for solving nonlinear equations has been successfully applied (see for example [2, 3, 4] and the references there in).

We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method (FPM) [10].

In order to use fixed point iteration method, we need the following information:

1. We need to know that there is a solution to the equation.
2. We need to know approximately where the solution is (that is, an approximation to the solution).

It is well known that the fixed point iteration method has the first order convergence.

We establish new higher-order iterative methods for the solution of scalar equations by using the decomposition technique mainly due to Adomian [2]. The methods are performing very well in comparison to the fixed point method and the methods discussed in [4].
2. Preliminaries

We need the following results: In the fixed point iteration method for solving the nonlinear equation \( f(x) = 0 \), the equation is usually rewritten as

\[
x = g(x),
\]

where

(i) there exists \([a, b]\) such that \( g(x) \in [a, b] \) for all \( x \in [a, b] \),

(ii) there exists \([a, b]\) such that \(|g'(x)| \leq L < 1 \) for all \( x \in [a, b] \).

Considering the following iteration scheme:

\[
x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots
\]

and starting with a suitable initial approximation \( x_0 \), we build up a sequence of approximations, say \( \{x_n\} \), for the solution of the nonlinear equation, say \( \alpha \).

The scheme will converge to the root \( \alpha \), provided that

(i) the initial approximation \( x_0 \) is chosen in the interval \([a, b]\),

(ii) \( g \) has a continuous derivative on \((a, b)\),

(iii) \(|g'(x)| < 1 \) for all \( x \in [a, b] \),

(iv) \( a \leq g(x) \leq b \) for all \( x \in [a, b] \) (see [10]).

The order of convergence for the sequence of approximations derived from an iteration method is defined in the literature, as

**Definition 2.1.** [16] Let \( \{x_n\} \) converge to \( \alpha \). If there exist an integer constant \( p \), and real positive constant \( C \) such that

\[
\lim_{n \to \infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C,
\]

then \( p \) is called the order and \( C \) the constant of convergence.

To determine the order of convergence of the sequence \( \{x_n\} \), let us consider the Taylor expansion of \( g(x_n) \)

\[
g(x_n) = g(x) + \frac{g'(x)}{1!}(x_n - x) + \frac{g''(x)}{2!}(x_n - x)^2 + \cdots
\]

\[
+ \frac{g^{(k)}(x)}{k!}(x_n - x)^k + \cdots
\]

Using (2.1) and (2.2) in (2.3) we have

\[
x_{n+1} - x = g'(x)(x_n - x) + \frac{g''(x)}{2!}(x_n - x)^2 + \cdots
\]

\[
+ \frac{g^{(k)}(x)}{k!}(x_n - x)^k + \cdots,
\]
and we can state the following result [10]:

**Lemma 2.2.** [3] Suppose that $g \in C^p[a,b]$. If $g^{(k)}(x) = 0$ for $k = 1,2,\ldots,p-1$ and $g^{(p)}(x) \neq 0$, then the sequence \{x_n\} is of order $p$.

### 3. New iterative methods

In this section we establish some new iterative methods for the solution of scalar equations by using the decomposition technique mainly due to Adomian [2].

Consider the nonlinear equation

$$f(x) = 0, \quad x \in \mathbb{R}. \quad (3.1)$$

We assume that $\alpha$ is a simple root of (3.1) and $\gamma$ is an initial guess sufficiently close to $\alpha$.

By using Taylor’s expansion, we get

$$x = g(x)$$

$$= g(\gamma) + (x - \gamma) g'(\gamma) + G(x), \quad (3.2)$$

where $G(x)$ is the auxiliary function is defined as

$$G(x) = g(x) - g(\gamma) - (x - \gamma) g'(\gamma). \quad (3.3)$$

Now (3.2) can be rewritten as:

$$x = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{G(x)}{1 - g'(\gamma)}$$

$$= c + N(x), \quad (3.4)$$

where

$$c = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)}, \quad (3.5)$$

and

$$N(x) = \frac{G(x)}{1 - g'(\gamma)}. \quad (3.6)$$

We can construct a sequence of higher-order iterative methods by applying the Adomain decomposition method [2].

Adomain decomposition method is based on looking for a solution having the infinite series

$$x = \sum_{n=0}^{\infty} x_i, \quad (3.7)$$
and the nonlinear function is decomposed as

$$N(x) = \sum_{n=0}^{\infty} A_n,$$  \hspace{1cm} (3.8)

where \(A_n\) are called as the Adomian polynomials depending on \(x_0, x_1, x_2, \ldots\) given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ N \left( \sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \hspace{1cm} n = 0, 1, 2, \ldots,$$  \hspace{1cm} (3.9)

or equivalently

$$A_0 = N(x_0),$$
$$A_1 = x_1 N'(x_0),$$
$$A_2 = x_2 N'(x_0) + \frac{1}{2} x_1^2 N''(x_0),$$
$$\ldots.$$  \hspace{1cm} (3.10)

From equations (3.4) and (3.8) we have

$$\sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n.$$  \hspace{1cm} (3.11)

Thus the iterative sequence is given by

$$x_0 = c,$$  \hspace{1cm} (3.12)

and

$$x_{n+1} = A_n, \hspace{0.5cm} n = 0, 1, 2, \ldots.$$  \hspace{1cm} (3.13)

An elementary calculations shows that

$$A_0 = N(x_0)$$
$$= \frac{G(x_0)}{1 - g'(\gamma)}$$
$$= \frac{g(x_0) - x_0}{1 - g'(\gamma)},$$  \hspace{1cm} (3.14)

and

$$N'(x) = \frac{G'(x)}{1 - g'(\gamma)}$$
$$= \frac{g'(x) - g'(\gamma)}{1 - g'(\gamma)}.$$  \hspace{1cm} (3.15)
We have
\[ N'(x_0) = \frac{g'(x_0) - g'(\gamma)}{1 - g'((\gamma)), \quad (3.16) \]
and
\[ N''(x_0) = \frac{g''(x_0)}{1 - g'(\gamma)}. \quad (3.17) \]
Consider
\[ A_1 = x_1 N'(x_0) = A_0 N'(x_0) \]
\[ = \frac{(g(x_0) - x_0)(g'(x_0) - g'(\gamma))}{(1 - g'((\gamma)))^2} \]
\[ \quad \text{and} \]
\[ A_2 = x_2 N'(x_0) + \frac{1}{2}x_1^2 N''(x_0) \]
\[ = A_1 N'(x_0) + \frac{1}{2}A_0^2 N''(x_0) \]
\[ = \frac{(g(x_0) - x_0)(g'(x_0) - g'(\gamma))^2}{(1 - g'((\gamma)))^3} + \frac{1}{2} \frac{(g(x_0) - x_0)^2 (g''(x_0))}{(1 - g'((\gamma)))^2}. \]
(3.19)
Note that \( x \) is approximated by
\[ X_m = x_0 + x_1 + \cdots + x_m = x_0 + A_0 + A_1 + \cdots + A_{m-1}, \]
(3.20)
where \( \lim_{m \to \infty} X_m = x \).
For \( m = 0 \),
\[ x = X_0 = x_0 = c = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)}. \]
(3.21)
This allows us to suggest the following one-step iterative method for solving the nonlinear equation (3.1).

**Algorithm 3.1.** For a given \( x_0 \) compute the approximate solution \( x_{n+1} \) by the iterative scheme
\[ x_{n+1} = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)}, \quad g'(x_n) \neq 1, \quad n = 0, 1, 2, \ldots, \]
which is mainly due to Kang et al. [15].
For $m = 1$, 
\[
x = X_1 = x_0 + x_1 = c + A_0 = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{g(x_0) - x_0}{1 - g'(\gamma)}.
\] (3.22)

This allows us to suggest the following one-step iterative method for solving the nonlinear equation (3.1).

**Algorithm 3.2.** For a given $x_0$ compute the approximate solution $x_{n+1}$ by the iterative scheme

**Predictor-step:**
\[
y_n = \frac{g(x_n) - x_ng'(x_n)}{1 - g'(x_n)},
\]

**Corrector-step:**
\[
x_{n+1} = \frac{g(x_n) - x_ng'(x_n)}{1 - g'(x_n)} + \frac{g(y_n) - y_n}{1 - g'(x_n)}.
\]

For $m = 2$, 
\[
x \approx X_2 = x_0 + x_1 + x_2 = c + A_0 + A_1 = \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{g(x_0) - x_0}{1 - g'(\gamma)} + \frac{(g(x_0) - x_0)(g'(x_0) - g'(\gamma))}{(1 - g'(\gamma))^2},
\] (3.23)

which provides the following iteration scheme:

**Algorithm 3.3.** For a given $x_0$ compute the approximate solution $x_{n+1}$ by the iterative scheme

**Predictor-step:**
\[
y_n = \frac{g(x_n) - x_ng'(x_n)}{1 - g'(x_n)},
\]

**Corrector-step:**
\[
x_{n+1} = \frac{g(x_n) - x_ng'(x_n)}{1 - g'(x_n)} + \frac{g(y_n) - y_n}{1 - g'(x_n)} + \frac{(g(y_n) - y_n)(g'(y_n) - g'(x_n))}{(1 - g'(x_n))^2}.
\]
For $m = 3$,

$$x = X_2 = x_0 + x_1 + x_2 + x_3$$

$$= c + A_0 + A_1 + A_2$$

$$= \frac{g(\gamma) - \gamma g'(\gamma)}{1 - g'(\gamma)} + \frac{g(x_0) - x_0}{1 - g'(\gamma)} + \frac{(g(x_0) - x_0)(g'(x_0) - g'(\gamma))}{(1 - g'(\gamma))^2}$$

$$+ \frac{(g(x_0) - x_0)(g'(x_0) - g'(\gamma))^2}{(1 - g'(\gamma))^3} + \frac{1}{2} \frac{(g(x_0) - x_0)^2 (g''(x_0))}{(1 - g'(\gamma))^2}. \quad (3.24)$$

This allows us to suggest the following one-step iterative method for solving the nonlinear equation (3.1).

**Algorithm 3.4.** For a given $x_0$ compute the approximate solution $x_{n+1}$ by the iterative scheme

- **Predictor-step:**
  $$y_n = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)},$$

- **Corrector-step:**
  $$x_{n+1} = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)} + \frac{g(y_n) - y_n}{1 - g'(x_n)} + \frac{(g(y_n) - y_n)(g'(y_n) - g'(x_n))}{(1 - g'(x_n))^2}$$

$$+ \frac{(g(y_n) - y_n)(g'(y_n) - g'(x_n))^2}{(1 - g'(x_n))^3} + \frac{1}{2} \frac{(g(y_n) - y_n)^2 (g''(y_n))}{(1 - g'(x_n))^2}.$$

4. Convergence analysis

In this section, we discuss the convergence analysis of Algorithms 3.2, 3.3 and 3.4.

**Theorem 4.1.** Let $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval $D$ and consider the nonlinear equation $f(x) = 0$ (or $x = g(x)$) has a simple root $\alpha \in D$, where $g(x) : D \subset \mathbb{R} \to \mathbb{R}$ be sufficiently smooth in the neighborhood of the root $\alpha$. Then the order of convergence of the Algorithm 3.2 is at least 3.

**Proof.** The functional equations of Algorithm 3.2 is given by

$$M(x) = \frac{-xg'(x) + g(x)}{1 - g'(x)} + \frac{g(-xg'(x) + g(x))}{1 - g'(x)} - \frac{xg'(x) + g(x)}{1 - g'(x)}.$$
However

\[ M'(x) = \frac{1}{(1 - g'(x))^2} \left[ g''(x) \left\{ -2xg'(x) + g(x) + (x)g'(x) \right\} \\
+ g' \left( \frac{-xg'(x) + g(x)}{1 - g'(x)} \right)x - g' \left( \frac{-xg'(x) + g(x)}{1 - g'(x)} \right)g(x) \\
- g \left( \frac{-xg'(x) + g(x)}{1 - g'(x)} \right) + g \left( \frac{-xg'(x) + g(x)}{1 - g'(x)} \right)g'(x) \right], \]

\[ M''(x) = \frac{1}{(-1 + g'(x))^5} \left[ 2g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^2 xg(x) \\
- 2g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) g'(x) \\
+ g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) [g'(x)]^2 \\
- 4g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) x [g''(x)]^2 g'(x) \\
+ 4g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^2 g(x) g'(x) \\
+ 2g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g'''(x) g(x) g'(x) \\
- g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g'''(x) g(x) [g'(x)]^2 \\
- 2x[g''(x)]^2 [g'(x)]^2 - 2x[g''(x)]^2 g'(x) - g'''(x) g(x) g'(x) \\
- 2xg'''(x) g'(x) + 4xg'''(x) [g'(x)]^2 - g''(x) g'(x) \\
+ 3g''(x) [g'(x)]^2 4 [g''(x)]^2 g(x) + g'''(x) g(x) - 3g''(x) [g'(x)]^3 \\
+ g'(x) \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g''(x) - 2 [g''(x)]^2 g \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) \\
- g'''(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) + g''(x) [g'(x)]^4 \\
- g''(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^2 x^2 \\
- g''(x) \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^2 g(x)^2 - 2 [g''(x)]^2 g(x) g'(x) \\
- g''(x) g(x) [g''(x)]^2 + 3g'(x) \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g''(x) [g'(x)]^2 \right]. \]
\[
\begin{align*}
- g'(x) & \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g''(x) [g''(x)]^3 \\
+ g'(x) & \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) \\
+ 4g'(x) & \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) x [g''(x)]^2 \\
- 4g(x) & \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^2 g(x) \\
- g'(x) & \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g'''(x) g(x) \\
- 2g''(x) & g(x) [g'(x)]^3 4x [g''(x)]^2 [g'(x)]^2 \\
- 2 & [g''(x)]^2 g(x) [g'(x)]^2 g'''(x) g(x) g''(x) \\
+ 4 & [g''(x)]^2 g \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g(x) \\
- 2 & [g''(x)]^2 g \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^2 \\
+ 3g'''(x) & g \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g'(x) \\
- 3g'''(x) & g \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^2 \\
+ g'''(x) & g \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^3 \\
- 3g' \left( \frac{xg(x) - g(x)}{-1 + g'(x)} \right) g''(x) g'(x),
\end{align*}
\]

\[
M'''(x) = \frac{1}{(-1 + g'(x))^7} \left[ 6g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) xg'''(x) g(x) [g''(x)]^2 \\
- 12g'' g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) xg'''(x) g(x) g'(x) - 3 [g''(x)]^2 \\
+ 6g''(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) x^2 g'''(x) g(x) g'(x) \\
+ 6g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) xg'''(x) g(x) g''(x) \\
+ 36g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) xg'''(x) g(x) [g(x)]^2 g(x)
\]
\[
- 3g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) xg'''(x) g(x) [g(x)]^2 g'(x) \\
- 3g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) x^2 g'''(x) [g'(x)]^2 \\
- 36g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) g'''(x) g(x) \\
+ 36g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) g''(x) [g(x)]^2 \\
+ 36g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) g'''(x) g(x) g'(x) \\
- 12g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) g''(x) [g'(x)]^3 \\
- 36g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) g'''(x) g(x) [g(x)]^2 \\
+ 12g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) g'''(x) g(x) [g'(x)]^3 \\
- 6g'''(x) g''(x) - 12xg''(x) g'''(x) + 9[g''(x)]^2 g'(x) \\
- 6g''(x) g(x) [g''(x)]^2 - 2g'''(x) g(x) g'(x) - 20g''(x) [g'(x)]^3 \\
+ 10g'''(x) [g'(x)]^2 + 18g''(x) [g(x)]^3 g(x) + g^{(4)}(x) g(x) \\
- 6[g''(x)]^2 [g'(x)]^3 + 9[g''(x)]^2 [g'(x)]^4 + 20g'''(x) [g'(x)]^4 \\
- 10g'''(x) [g'(x)]^5 + 2g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''' + 12x \\
+ 6 \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg'' - 6g'''(x) g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \\
- g^{(4)}(x) g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) + 2g'''(x) [g'(x)]^6 \\
- 3[g''(x)]^2 [g'(x)]^5 + 6xg'''(x) xg(x) g''(x) g'(x) \\
+ 18xg'''(x) xg''(x) [g'(x)]^2 - 30g''(x) g'''(x) g(x) g'(x) \\
- 3g'''(x) g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^3 x^2 g(x) \\
+ 3g'''(x) g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^3 x g(x) [g(x)]^2 \\
+ 18g''(x) g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^3 x g(x) \\
\]
\[-3g''(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) x^2 g(x) \]
\[-3g''(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g''(x) g'''(x) [g(x)]^2 \]
\[+ 9g''(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 xg'(x) \]
\[-9g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 x[g'(x)]^2 \]
\[-9g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 g(x) g(x) \]
\[+ 9g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 xg(x) g'(x) \]
\[+ 9g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 xg'(x) \]
\[+ 9g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 xg'(x) \]
\[-36g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) x[g'(x)]^3 g'(x) \]
\[+ 12g' \left( \frac{-xg'(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) g''(x) \]
\[+ 9g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 [g(x)]^3 g'(x) \]
\[-30xg'''(x) g''(x) [g'(x)]^3 + 18g''(x) g'''(x) g(x) [g'(x)]^2 \]
\[+ 6g''(x) g'''(x) g(x) [g'(x)]^3 + 12g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) g''(x) \]
\[-4g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg^4(x) g(x) g'(x) \]
\[+ 6g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg^4(x) g(x) [g'(x)]^2 \]
\[-4g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg^4(x) g(x) [g'(x)]^3 \]
\[+ 36g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) g[g'(x)]^3 g'(x) \]
\[+ 12g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg'''(x) g(x) g(x) \]
\[+ 4g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) xg^4(x) g(x) g'(x) \]
\[ -6g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) g^{(4)}(x) g(x) \left[ g(x) \right]^2 \]
\[ + 36g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) \left[ g'(x) \right]^3 g'(x) \]
\[ - 18g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) \left[ g'(x) \right]^3 g(x) \left[ g(x) \right]^2 \]
\[ + 18g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) xg''(x) \left[ g'(x) \right]^3 [g(x)]^2 \]
\[ + g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) xg^{(4)}(x) g(x) g(x) \left[ g(x) \right]^4 \]
\[ + 4g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) xg^{(4)}(x) g(x) \left[ g(x) \right]^3 \]
\[ - g' \left( \frac{gx'(x) - g(x)}{-1 + g'(x)} \right) xg^{(4)}(x) g(x) g(x) \left[ g(x) \right]^4 \]
\[ + 12xg'''(x) g(x) g''(x) \left[ g(x) \right]^4 - 6xg''(x) g'''(x) g(x) \left[ g(x) \right]^4 \]
\[ + 24xg''(x) g(x) g''(x) g(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g(x) \]
\[ - 36g''(x) g(x) g''(x) xg(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g(x) \right]^2 \]
\[ + 24g''(x) g(x) g''(x) g(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g'(x) \right]^3 \]
\[ - 6g''(x) g(x) g''(x) g(x) \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g'(x) \right]^4 \]
\[ - 2xg^{(4)}(x) g(x) + 8xg^{(4)}(x) \left[ g'(x) \right]^2 - 12xg^{(4)}(x) \left[ g'(x) \right]^3 \]
\[ + 12xg''(x) \left[ g(x) \right]^2 \left[ g(x) \right]^3 g'(x) + 12g''(x) g'''(x) g(x) \]
\[ - 3g^{(4)}(x) g(x) g'(x) + 2g'''(x) g(x) \left[ g(x) \right]^2 \]
\[ + g''' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^3 \left[ g(x) \right]^3 \]
\[ - g''' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^3 \left[ g(x) \right]^3 \]
\[ - 3g'' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) \left[ g''(x) \right]^2 \]
\[-9g''\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^3x^2\]

\[-9g''\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^2[g(x)]^2\]

\[+ 3g''\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^2g(x)\]

\[-30\left[g''(x)\right]^3g(x)g'(x) + 6\left[g''(x)\right]^3g(x)\left[g'(x)\right]^2\]

\[+ 12\left[g''(x)\right]^3[g(x)]^2 + 8g^{(4)}(x)\left[g(x)\right]^4 + 2g^{(4)}(x)g(x)\left[g'(x)\right]^3\]

\[-3\left[g^{(4)}(x)\right]^3g(x)\left[g'(x)\right]^4 - 24g''\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^2g'(x)\]

\[+ 36g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^2[g'(x)]^2\]

\[-10g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)g''(x)g'(x)\]

\[-20g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)g''(x)\left[g'(x)\right]^3\]

\[+ g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)g''(x)\left[g'(x)\right]^2\]

\[-18g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^3g(x)\]

\[-g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)g^{(4)}(x)g(x)\]

\[-24g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^2[g(x)]^3\]

\[+ 6g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\left[g''(x)\right]^2[g(x)]^2\left[g'(x)\right]^4\]

\[+ 10g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)g''(x)\left[g'(x)\right]^4\]

\[-2g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)g''(x)\left[g'(x)\right]^5 + g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)xg^{(4)}(x)\]

\[+ 18g'\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)xg''(x)\left[g(x)\right]^3 - 2xg^{(4)}(x)\left[g'(x)\right]^5\]

\[-12\left[g''(x)\right]^3\left[g'(x)\right]^3 - 6g''(x)g'''(x)g\left(\frac{xg'(x) - g(x)}{-1 + g'(x)}\right)\]

\[+ 6\left[g''(x)\right]^3g(x)\left[g'(x)\right]^3 + g^{(4)}(x)g(x)\left[g'(x)\right]^5\]
\[ + 18[g''(x)]^3 g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g'(x) \]
\[ - 18[g(x)]^3 g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^2 \]
\[ + 6[g''(x)]^3 g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^3 \]
\[ - 10g^{(4)}(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^2 \]
\[ + 10g^{(4)}(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^3 \]
\[ - 5g^{(4)}(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^4 \]
\[ + g^{(4)}(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g'(x)]^5 \]
\[ + 5g^{(4)}(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) g'(x) \]
\[ - 18g''(x) g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) [g''(x)]^3 xg(x) g'(x) \].

For the exact root \( \alpha \), we obtain \( M(\alpha) = \alpha \), \( M'(\alpha) = 0 = M''(\alpha) \), and \( M'''(\alpha) \neq 0 \). Hence the Algorithm 3.2 has third order convergence. \( \square \)

**Theorem 4.2.** Let \( f : D \subset \mathbb{R} \to \mathbb{R} \) for an open interval \( D \) and consider the nonlinear equation \( f(x) = 0 \) (or \( x = g(x) \)) has a simple root \( \alpha \in D \), where \( g(x) : D \subset \mathbb{R} \to \mathbb{R} \) be sufficiently smooth in the neighborhood of the root \( \alpha \). Then the order of convergence of the Algorithm 3.3 is at least 4.

**Proof.** The functional equations of Algorithm 3.3 is given by

\[
N(x) = \frac{xg'(x) - g(x)}{-1 + g'(x)} + \frac{g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) - xg'(x) - g(x)}{-1 + g'(x)} \]
\[ + \frac{1}{(-1 + g'(x))^2} \left\{ g \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) - xg'(x) - g(x) \right\} \]
\[ \times \left\{ g' \left( \frac{xg'(x) - g(x)}{-1 + g'(x)} \right) - g'(x) \right\} .
\]

For the exact root \( \alpha \), we obtain \( N(\alpha) = \alpha \), \( N'(\alpha) = 0 = N''(\alpha) = N'''(\alpha) \), and \( N''''(\alpha) \neq 0 \). Hence the Algorithm 3 has fourth order convergence. \( \square \)
Theorem 4.3. Let \( f : D \subset \mathbb{R} \to \mathbb{R} \) for an open interval \( D \) and consider the nonlinear equation \( f(x) = 0 \) (or \( x = g(x) \)) has a simple root \( \alpha \in D \), where \( g(x) : D \subset \mathbb{R} \to \mathbb{R} \) be sufficiently smooth in the neighborhood of the root \( \alpha \). Then the order of convergence of the Algorithm 3.4 is at least 4.

5. Applications

We present some examples to illustrate the efficiency of the developed methods, Algorithm 3.1, Algorithm 3.2, Algorithm 3.3 and Algorithm 3.4 and compare them with the fixed point method.

Example 5.1. Consider the equation \( x^3 + 4x^2 + 8x + 8 = 0 \). We have \( g(x) = -(1 + \frac{1}{2}x^2 + \frac{1}{8}x^3), \quad g'(x) = -(x + \frac{3}{8}x^2), \quad \text{and} \quad g''(x) = -(1 + \frac{3}{4}x). \) If we take \( x_0 = -1.9 \), then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>( x_n )</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>-1.9476</td>
<td>-2.0050</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>-1.9731</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>-1.9864</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>-1.9932</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>-1.9966</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>-1.9983</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>-1.9991</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_8 )</td>
<td>-1.9995</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_9 )</td>
<td>-1.9997</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_{10} )</td>
<td>-1.9998</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
<tr>
<td>( x_{11} )</td>
<td>-1.9999</td>
<td>-2</td>
<td>-2</td>
<td>-2.0001</td>
<td>-2</td>
</tr>
</tbody>
</table>

Example 5.2. Consider the equation \( x + \ln(x - 2) = 0 \). We have \( g(x) = 2 + e^{-x}, \quad g'(x) = -e^{-x}, \quad \text{and} \quad g''(x) = e^{-x}. \) If we take \( x_0 = 2.1 \), then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>( x_n )</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>2.1225</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>2.1197</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>2.1201</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
<td>2.12</td>
</tr>
</tbody>
</table>

Example 5.3. Consider the equation \( e^x - 3x^2 = 0 \). We have \( g(x) = \sqrt{\frac{x^3}{3}}, \quad g'(x) = \frac{\sqrt{x}}{2\sqrt{3}}, \quad \text{and} \quad g''(x) = \frac{\sqrt{x}}{4\sqrt{3}}. \) If we take \( x_0 = 0.8 \), then the comparison of the
methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.86131</td>
<td>0.90768</td>
<td>0.90991</td>
<td>0.91</td>
<td>0.91001</td>
</tr>
<tr>
<td>2</td>
<td>0.88812</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.9001</td>
<td>0.91001</td>
<td>0.91001</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.90551</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.90796</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.90908</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.90959</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.90982</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.90992</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.90997</td>
<td>0.91001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.90999</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.91000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.91</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 5.4.** Consider the equation $\sin x - 5x + 2 = 0$. We have $g(x) = \frac{1}{5} (\sin x + 2)$, $g'(x) = \frac{1}{5} \cos x$, and $g''(x) = -\frac{1}{5} \sin x$. If we take $x_0 = 0.5$, then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.49589</td>
<td>0.49501</td>
<td>0.49501</td>
<td>0.49501</td>
<td>0.49501</td>
</tr>
<tr>
<td>2</td>
<td>0.49516</td>
<td>0.49501</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>0.49503</td>
<td>0.49501</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.49501</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 5.5.** Consider the equation $x^3 - 5x^2 - 29 = 0$. We have $g(x) = 5 + \frac{29}{x^2}$, $g'(x) = -\frac{58}{x^3}$, and $g''(x) = \frac{174}{x^4}$. If we take $x_0 = 5$, then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.16</td>
<td>5.7923</td>
<td>5.8415</td>
<td>5.8471</td>
<td>5.8479</td>
</tr>
<tr>
<td>2</td>
<td>5.7643</td>
<td>5.8478</td>
<td>5.8480</td>
<td>5.8480</td>
<td>5.8480</td>
</tr>
<tr>
<td>3</td>
<td>5.8728</td>
<td>5.8480</td>
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</tr>
<tr>
<td>4</td>
<td>5.8408</td>
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</tr>
<tr>
<td>5</td>
<td>5.8501</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>5.8474</td>
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<td></td>
</tr>
<tr>
<td>7</td>
<td>5.8481</td>
<td></td>
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</tr>
<tr>
<td>8</td>
<td>5.8479</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>5.8480</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 5.6.** Consider the equation $x - \cos x = 0$. We have $g(x) = \cos x$, $g'(x) = -\sin x$, and $g''(x) = -\cos x$. If we take $x_0 = 0.6$, then the comparison
of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.82534</td>
<td>0.74402</td>
<td>0.73874</td>
<td>0.73912</td>
<td>0.73908</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.67831</td>
<td>0.73909</td>
<td>0.73909</td>
<td>0.73909</td>
<td>0.73909</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.77863</td>
<td>0.73874</td>
<td>0.73912</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.71188</td>
<td>0.73912</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_5$</td>
<td>0.75714</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_6$</td>
<td>0.7268</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_7$</td>
<td>0.7473</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_8$</td>
<td>0.73353</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_9$</td>
<td>0.74282</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>0.73656</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{11}$</td>
<td>0.74078</td>
<td>0.73909</td>
<td></td>
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</tr>
<tr>
<td>$x_{12}$</td>
<td>0.73794</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{13}$</td>
<td>0.73986</td>
<td>0.73909</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{14}$</td>
<td>0.73856</td>
<td>0.73909</td>
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<td></td>
</tr>
<tr>
<td>$x_{15}$</td>
<td>0.73944</td>
<td>0.73909</td>
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</tr>
<tr>
<td>$x_{16}$</td>
<td>0.73885</td>
<td>0.73909</td>
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<tr>
<td>$x_{17}$</td>
<td>0.73924</td>
<td>0.73909</td>
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<tr>
<td>$x_{18}$</td>
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<td>0.73909</td>
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</tr>
<tr>
<td>$x_{19}$</td>
<td>0.73916</td>
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<tr>
<td>$x_{20}$</td>
<td>0.73903</td>
<td>0.73909</td>
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</tr>
</tbody>
</table>

**Example 5.7.** Consider the equation $\cos x - 3x + 1 = 0$. We have $g(x) = \frac{1}{3} \cos x + \frac{1}{3}$, $g'(x) = -\frac{1}{3} \sin x$, and $g''(x) = -\frac{1}{3} \cos x$. If we take $x_0 = 0.5$, then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.62586</td>
<td>0.60852</td>
<td>0.60706</td>
<td>0.6071</td>
<td>0.6071</td>
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<tr>
<td>$x_2$</td>
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<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
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<tr>
<td>$x_3$</td>
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<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
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<tr>
<td>$x_4$</td>
<td>0.60697</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
<td>0.6071</td>
</tr>
</tbody>
</table>

**Example 5.8.** Consider the equation $xe^x = 1$. We have $g(x) = e^{-x}$, $g'(x) = -e^{-x}$, and $g''(x) = e^{-x}$. If we take $x_0 = 0.5$, then the comparison of
the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.60653</td>
<td>0.56631</td>
<td>0.56712</td>
<td>0.56714</td>
<td>0.56714</td>
</tr>
<tr>
<td>$x_2$</td>
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<td>0.56714</td>
<td>0.56714</td>
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<td>0.56714</td>
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<tr>
<td>$x_3$</td>
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<tr>
<td>$x_4$</td>
<td>0.56844</td>
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</tr>
<tr>
<td>$x_5$</td>
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<tr>
<td>$x_6$</td>
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<tr>
<td>$x_7$</td>
<td>0.56719</td>
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<tr>
<td>$x_8$</td>
<td>0.56716</td>
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<tr>
<td>$x_9$</td>
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<tr>
<td>$x_{10}$</td>
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</tr>
<tr>
<td>$x_{11}$</td>
<td>0.54524</td>
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</tr>
<tr>
<td>$x_{12}$</td>
<td>0.56007</td>
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</tr>
<tr>
<td>$x_{13}$</td>
<td>0.56486</td>
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<tr>
<td>$x_{14}$</td>
<td>0.56641</td>
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<tr>
<td>$x_{15}$</td>
<td>0.56691</td>
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<tr>
<td>$x_{16}$</td>
<td>0.56706</td>
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<tr>
<td>$x_{17}$</td>
<td>0.56712</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$x_{18}$</td>
<td>0.56713</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$x_{19}$</td>
<td>0.56714</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Example 5.9. Consider the equation $\cos x - 2x + 3 = 0$. We have $g(x) = \frac{1}{2} \cos x + \frac{3}{2}$, $g'(x) = -\frac{1}{2} \sin x$, and $g''(x) = -\frac{1}{2} \cos x$. If we take $x_0 = 1.5$, then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1.5354</td>
<td>1.5236</td>
<td>1.5236</td>
<td>1.5236</td>
<td>1.5236</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.5177</td>
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<tr>
<td>$x_3$</td>
<td>1.5265</td>
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<tr>
<td>$x_4$</td>
<td>1.5221</td>
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<tr>
<td>$x_5$</td>
<td>1.5243</td>
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<td>$x_6$</td>
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<td>$x_7$</td>
<td>1.5238</td>
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<td></td>
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</tr>
</tbody>
</table>

Example 5.10. Consider the equation $\sin x - 10x + 10 = 0$. We have $g(x) = 1 + \frac{\sin x}{10}$, $g'(x) = \frac{1}{10} \cos x$, and $g''(x) = -\frac{1}{10} \sin x$. If we take $x_0 = 1$, then the comparison of the methods is shown in the following table:

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>FPM</th>
<th>Algorithm 3.1</th>
<th>Algorithm 3.2</th>
<th>Algorithm 3.3</th>
<th>Algorithm 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1.0841</td>
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<td>1.0886</td>
<td>1.0886</td>
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<td>$x_2$</td>
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<td>1.0886</td>
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<tr>
<td>$x_3$</td>
<td>1.0886</td>
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<tr>
<td>$x_4$</td>
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</tbody>
</table>
6. Conclusions

New higher-order iterative methods for the solution of scalar equations by using the Adomian decomposition technique [2] are established.

Acknowledgement

This work was supported by the Dong-A University research fund.

References


