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SUBSPACE-HYPERCYCLIC TUPLES OF OPERATORS

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Abstract: In this paper we introduce subspace-hypercyclic tuples of operators and construct interesting examples of such operators. We state some sufficient conditions for n-tuples of operators to be subspace-hypercyclic. Surprisingly, we prove that subspace-hypercyclic tuples exist on finite-dimensional spaces.

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1. Introduction

Let X be a Banach space. An operator T on X is called hypercyclic, if there exists a vector $x \in X$ whose orbit under T, $orb(T,x) = \{x, Tx, T^2x, ...\}$, is dense in X. We call x a hypercyclic vector for T.

A good reference about hypercyclicity is the book of Grosse-Erdmann and Peris [5]. For more information about this concept one can also see [2] and [4].

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $T = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X and let

$$F = F_T = \{T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : k_i \geq 0, i = 1, ..., n\},$$

be the semigroup generated by T. For $x \in X$, the orbit of x under the tuple T

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is

$$orb(T, x) = \{Sx : S \in F\}.$$

A vector $x \in X$ is called a hypercyclic vector for T, if $orb(T, x)$ is dense in X. In this case the tuple T is called hypercyclic. The set of all hypercyclic vectors of T is denoted by $HC(T)$.

Feldmann in [3] proved some interesting properties of these operators. Also in [12] one can find more results about hypercyclic tuples. Recently, Madore and Martinez-Avendano in [7] introduced the concept of subspace-hypercyclicity for an operator as follows.

Definition 1.2. Let T be a bounded linear operator on X and let M be a nonzero closed subspace of X. We say that T is subspace-hypercyclic for M if there exists $x \in X$ such that $orb(T, x) \cap M$ is dense in M. We call x a subspace-hypercyclic vector for T.

For example let T be a bounded linear operator on X and let I be the identity operator on X. Then $T \oplus I : X \oplus X \to X \oplus X$ is subspace-hypercyclic for the subspace $M := X \oplus \{0\}$ with subspace-hypercyclic vector $x \oplus 0$. Clearly, $T \oplus I$ is not hypercyclic.

Rezai in [10] answered some questions of Madore and Avendano asked in [7]. One can also see [6], [8] and [9] for more information.

This paper is organized as follows. In Section 2 we introduce subspace-hypercyclic tuples of operators and we give some interesting examples of such operators. In Section 3 we show that surprisingly subspace-hypercyclic tuples exist on finite-dimensional spaces.

Throughout this paper, we assume that $B(X)$ is the space of bounded linear operators acting on a separable and infinite dimensional complex Banach space X. We also denote by M a nonzero closed subspace of X.

2. Definitions and Examples

We first introduce the notion of a subspace-hypercyclic tuple with respect to a non-zero closed subspace M of X as follows.

Definition 2.1. Let $T = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on X and let

$$F = F_T = \{T_1^{k_1} T_2^{k_2} ... T_n^{k_n} : k_i \geq 0, i = 1, ..., n\},$$

be the semigroup generated by T. For $x \in X$, the orbit of x under T is the set

$$orb(T, x) = \{Sx : S \in F\}.$$

A vector $x \in X$ is called an M-hypercyclic vector for T if $orb(T,x) \cap M$ is dense in M. In this case we say that T is a subspace-hypercyclic tuple with respect to M or an M-hypercyclic tuple.

The set of all M-hypercyclic vectors for T is denoted by $HC(T,M)$.

Theorem 2.2. *Let $T = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on X. If the semigroup F_T contains an M-hypercyclic operator, then T is a subspace-hypercyclic tuple with respect to M.*

Proof. Let $S \in F_T$ be an M-hypercyclic operator with a hypercyclic vector x. Then,

$$M = cl(orb(S,x) \cap M) \subseteq cl(orb(T,x) \cap M) \subseteq M.$$

Hence $cl(orb(T,x) \cap M) = M$ and T is an M-hypercyclic tuple. □

We first give a simple example of a subspace-hypercyclic tuple.

Example 2.3. Let A and B be hypercyclic operators on X and let I be the identity operator on X. Let $T_1 = A \oplus I$, $T_2 = I \oplus B$ and $T_3 = I \oplus C$ where C is an arbitrary operator on X. Then $T = (T_1, T_2, T_3)$ is a subspace-hypercyclic tuple with respect to $X \oplus \{0\}$ and $\{0\} \oplus X$, since T_1 is subspace-hypercyclic with respect to $X \oplus \{0\}$ and T_2 is subspace-hypercyclic with respect to $\{0\} \oplus X$.

Using Example 2.3, one can obtain different subspace-hypercyclic tuples. Recall that on l^2, the Hilbert space of all square summable complex sequences, the backward shift B is defined as

$$B(x_0, x_1, x_3, ...) = (x_1, x_2, ...).$$

It was shown in [11] that if λ is a scalar with $|\lambda| > 1$, then λB is a hypercyclic operator. By using this fact, we give an example of a subspace-hypercyclic tuple that is not a hypercyclic tuple as follows.

Example 2.4. Let $T = (I \oplus 2B, I \oplus 3B)$, where I is the identity operator and B is the backward shift on l^2. Let x be a hypercyclic vector for $2B$. Then $0 \oplus x$ is a subspace-hypercyclic vector for the tuple T with respect to subspace $M := \{0\} \oplus l^2$, since

$$cl(orb(T, (0 \oplus x)) \cap M) = \{0\} \oplus l^2.$$

Also it is clear that T is not a hypercyclic tuple.

Example 2.5. Let λ be a scalar with $|\lambda| > 1$ and let B be the backward shift on l^2. Let $m \in N$ and let

$$M := \{\{a_n\}_{n=0}^{\infty} : a_n = 0 \quad for \quad n < m\}.$$

Define $T = (I, \lambda B)$. Note that λB is an M-hypercyclic operator by Example 3.8 in [7]. So, T is an M-hypercyclic tuple by Theorem 2.2.

Example 2.6. Let B be the backward shift on l^2 and let λ be a scalar with $|\lambda| > 1$. By Example 3.7 in [7], λB is M-hypercyclic with respect to

$$M := \{\{a_n\}_{n=0}^{\infty} : a_{2k} = 0 \quad for \quad all \quad k\}.$$

Define $T = (\lambda B, \frac{1}{\lambda}B)$. Note that $\lambda B \in F_T$ and it is an M-hypercyclic operator. So T is an M-hypercyclic tuple by Theorem 2.2.

3. Finite Dimensions

Subspace-hypercyclic operators does not exist on finite dimensional spaces. Also it is proved in [7] that an operator can not be subspace-hypercyclic with respect to a finite-dimensional subspace . But surprisingly, we prove that subspace-hypercyclic tuples exist on finite-dimensional spaces.

Theorem 3.1. (see [7]) *Let H be a finite-dimensional Hilbert space. If $T \in B(H)$, then T is not subspace-hypercyclic for any closed nonzero subspace M of H.*

Theorem 3.2. (see [7]) *Let $T \in B(H)$ where H is a finite-dimensional Hilbert space. If T is subspace-hypercyclic for a subspace M, then M is not finite-dimensional.*

Theorem 3.3. (see [3]) *If $a, b > 1$ are relatively prime integers, then $\{\frac{a^n}{b^k} : n, k \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$, the positive real numbers.*

In the following example we show that subspace-hypercyclic tuples can exist on finite-dimensional spaces.

Example 3.4. Let I be the identity operator on the real Hilbert space $\mathbb{R}$ and let $T = (-2I \oplus I, \frac{1}{3}I \oplus I)$ be a tuple on finite-dimensional space $\mathbb{R} \oplus \mathbb{R}$. By Theorem 3.3, $cl(orb(T, (1 \oplus 0)) \cap M) = M$. Then T is a subspace-hypercyclic tuple with respect to finite-dimensional subspace $M := \mathbb{R} \oplus \{0\}$. Note that T is not a hypercyclic tuple on $\mathbb{R} \oplus \mathbb{R}$.

By generalizing previous example, we get the following theorem:

Theorem 3.5. *There are subspace-hypercyclic tuples of operators on $\mathbb{R}^n$ for any integer n greater than 1.*

Proof. Suppose that $a, b > 1$ are relatively prime integers. Fix a, b and consider $T = (-|a|I_{\mathbb{R}} \oplus I_{\mathbb{R}} \oplus ... \oplus I_{\mathbb{R}}, \frac{1}{|b|}I_{\mathbb{R}} \oplus I_{\mathbb{R}} \oplus ... \oplus I_{\mathbb{R}})$ as an n-tuple on $\mathbb{R}^n = \mathbb{R} \oplus ... \oplus \mathbb{R}$, n times. Then, T is a subspace-hypercyclic n-tuple with respect to $M := \mathbb{R} \oplus \{0\} ... \oplus \{0\}$, with M-hypercyclic vector $1 \oplus 0 \oplus ... \oplus 0$. □

Note that the above theorem shows that, unlike the subspace-hypercyclic operators, subspace-hypercyclic tuples exist on finite-dimensional spaces.

Recall that if $T = (T_1, T_2)$ is a commuting pair of operators and $n = (n_1, n_2)$ is a pair of nonnegative integers (a multi-index), then we define T^n to be the pair (T^{n_1}, T^{n_2}). Ansari in [1] showed that if T is a hypercyclic operator, then T^n is also hypercyclic for any positive integer n. Feldmann in [3] showed that if T is a hypercyclic tuple, then T^n may not be hypercyclic for a multi-index $n = (n_1, n_2)$. So, it is natural to ask that if T is a subspace-hypercyclic tuple for a subspace M and $n = (n_1, n_2)$ is a multi-index, then must T^n be an M-hypercyclic tuple?

The following example shows that the answer to the above question in general is not positive.

Example 3.6. Let $T_1 = -2I_{\mathbb{R}} \oplus I_{\mathbb{R}}$ and $T_2 = \frac{1}{3}I_{\mathbb{R}} \oplus I_{\mathbb{R}}$. Then, $T = (T_1, T_2)$ is a subspace-hypercyclic tuple with respect to $M := \mathbb{R} \oplus \{0\}$. But if we consider $n = (2, 1)$, then $T^n = (T_1^2, T_2)$ is not subspace-hypercyclic with respect to $M := \mathbb{R} \oplus \{0\}$. Since by Theorem 3.3, $cl(orb(T^n, (1 \oplus 0)) \cap M) = [0, +\infty) \oplus \{0\}$ and $cl(orb(T^n, (-1 \oplus 0)) \cap M) = (-\infty, 0] \oplus \{0\}$.

It is proved in [7] that compact operators are not subspace-hypercyclic.

Theorem 3.7. (see [7]) *Let $T \in B(H)$, where H is a Hilbert space. If T is compact, then T is not subspace-hypercyclic for any subspace.*

In the following example we show that for a tuple $T = (T_1, T_2, ..., T_n)$, T may be a subspace-hypercyclic tuple while T_1, T_2, ..., T_n are compact.

Example 3.8. Let $T = (-2I_{\mathbb{R}} \oplus I_{\mathbb{R}}, \frac{1}{3}I_{\mathbb{R}} \oplus I_{\mathbb{R}}, T_3, ..., T_n)$ where T_3,..., T_n are compact operators on $\mathbb{R} \oplus \mathbb{R}$. By Theorem 3.3,

$$cl(orb(T^n, (1 \oplus 0)) \cap M) = \mathbb{R} \oplus \{0\}.$$

So T is a subspace-hypercyclic n-tuple with respect to $M := \mathbb{R} \oplus \{0\}$.

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