SUBSPACE-HYPERCYCLIC TUPLES OF OPERATORS

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Abstract: In this paper we introduce subspace-hypercyclic tuples of operators and construct interesting examples of such operators. We state some sufficient conditions for $n$-tuples of operators to be subspace-hypercyclic. Surprisingly, we prove that subspace-hypercyclic tuples exist on finite-dimensional spaces.

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1. Introduction

Let $X$ be a Banach space. An operator $T$ on $X$ is called hypercyclic, if there exists a vector $x \in X$ whose orbit under $T$, $\text{orb}(T, x) = \{x, Tx, T^2x, \ldots\}$, is dense in $X$. We call $x$ a hypercyclic vector for $T$.

A good reference about hypercyclicity is the book of Grosse-Erdmann and Peris [5]. For more information about this concept one can also see [2] and [4].

By an $n$-tuple of operators we mean a finite sequence of length $n$ of commuting continuous linear operators on a Banach space $X$.

Definition 1.1. Let $T = (T_1, T_2, \ldots, T_n)$ be an $n$-tuple of operators acting on an infinite dimensional Banach space $X$ and let

$$F = F_T = \{T_1^{k_1}T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0, i = 1, \ldots, n\},$$

be the semigroup generated by $T$. For $x \in X$, the orbit of $x$ under the tuple $T$
is
\[ \text{orb}(T, x) = \{ Sx : S \in F \} . \]
A vector \( x \in X \) is called a hypercyclic vector for \( T \), if \( \text{orb}(T, x) \) is dense in \( X \). In this case the tuple \( T \) is called hypercyclic. The set of all hypercyclic vectors of \( T \) is denoted by \( HC(T) \).

Feldmann in [3] proved some interesting properties of these operators. Also in [12] one can find more results about hypercyclic tuples. Recently, Madore and Martinez-Avendano in [7] introduced the concept of subspace-hypercyclicity for an operator as follows.

**Definition 1.2.** Let \( T \) be a bounded linear operator on \( X \) and let \( M \) be a nonzero closed subspace of \( X \). We say that \( T \) is subspace-hypercyclic for \( M \) if there exists \( x \in X \) such that \( \text{orb}(T, x) \cap M \) is dense in \( M \). We call \( x \) a subspace-hypercyclic vector for \( T \).

For example let \( T \) be a bounded linear operator on \( X \) and let \( I \) be the identity operator on \( X \). Then \( T \oplus I : X \oplus X \to X \oplus X \) is subspace-hypercyclic for the subspace \( M := X \oplus \{0\} \) with subspace-hypercyclic vector \( x \oplus 0 \). Clearly, \( T \oplus I \) is not hypercyclic.

Rezai in [10] answered some questions of Madore and Avendano asked in [7]. One can also see [6], [8] and [9] for more information.

This paper is organized as follows. In Section 2 we introduce subspace-hypercyclic tuples of operators and we give some interesting examples of such operators. In Section 3 we show that surprisingly subspace-hypercyclic tuples exist on finite-dimensional spaces.

Throughout this paper, we assume that \( B(X) \) is the space of bounded linear operators acting on a separable and infinite dimensional complex Banach space \( X \). We also denote by \( M \) a nonzero closed subspace of \( X \).

## 2. Definitions and Examples

We first introduce the notion of a subspace-hypercyclic tuple with respect to a non-zero closed subspace \( M \) of \( X \) as follows.

**Definition 2.1.** Let \( T = (T_1, T_2, ..., T_n) \) be an \( n \)-tuple of operators acting on \( X \) and let
\[ F = F_T = \{ T_1^{k_1}T_2^{k_2}...T_n^{k_n} : k_i \geq 0, i = 1, ..., n \}, \]
be the semigroup generated by \( T \). For \( x \in X \), the orbit of \( x \) under \( T \) is the set \( \text{orb}(T, x) = \{ Sx : S \in F \} \).
A vector \( x \in X \) is called an \( M \)-hypercyclic vector for \( T \) if \( \text{orb}(T, x) \cap M \) is dense in \( M \). In this case we say that \( T \) is a subspace-hypercyclic tuple with respect to \( M \) or an \( M \)-hypercyclic tuple.

The set of all \( M \)-hypercyclic vectors for \( T \) is denoted by \( HC(T, M) \).

**Theorem 2.2.** Let \( T = (T_1, T_2, ..., T_n) \) be an \( n \)-tuple of operators acting on \( X \). If the semigroup \( F_T \) contains an \( M \)-hypercyclic operator, then \( T \) is a subspace-hypercyclic tuple with respect to \( M \).

**Proof.** Let \( S \in F_T \) be an \( M \)-hypercyclic operator with a hypercyclic vector \( x \). Then, \( M = \text{cl}(\text{orb}(S, x) \cap M) \subseteq \text{cl}(\text{orb}(T, x) \cap M) \subseteq M \). Hence \( \text{cl}(\text{orb}(T, x) \cap M) = M \) and \( T \) is an \( M \)-hypercyclic tuple. \( \square \)

We first give a simple example of a subspace-hypercyclic tuple.

**Example 2.3.** Let \( A \) and \( B \) be hypercyclic operators on \( X \) and let \( I \) be the identity operator on \( X \). Let \( T_1 = A \oplus I, T_2 = I \oplus B \) and \( T_3 = I \oplus C \) where \( C \) is an arbitrary operator on \( X \). Then \( T = (T_1, T_2, T_3) \) is a subspace-hypercyclic tuple with respect to \( X \oplus \{0\} \) and \( \{0\} \oplus X \), since \( T_1 \) is subspace-hypercyclic with respect to \( X \oplus \{0\} \) and \( T_2 \) is subspace-hypercyclic with respect to \( \{0\} \oplus X \).

Using Example 2.3, one can obtain different subspace-hypercyclic tuples. Recall that on \( l^2 \), the Hilbert space of all square summable complex sequences, the backward shift \( B \) is defined as

\[
B(x_0, x_1, x_3, ...) = (x_1, x_2, ...).
\]

It was shown in [11] that if \( \lambda \) is a scalar with \( |\lambda| > 1 \), then \( \lambda B \) is a hypercyclic operator. By using this fact, we give an example of a subspace-hypercyclic tuple that is not a hypercyclic tuple as follows.

**Example 2.4.** Let \( T = (I \oplus 2B, I \oplus 3B) \), where \( I \) is the identity operator and \( B \) is the backward shift on \( l^2 \). Let \( x \) be a hypercyclic vector for \( 2B \). Then \( 0 \oplus x \) is a subspace-hypercyclic vector for the tuple \( T \) with respect to subspace \( M := \{0\} \oplus l^2 \), since

\[
\text{cl}(\text{orb}(T, (0 \oplus x)) \cap M) = \{0\} \oplus l^2.
\]

Also it is clear that \( T \) is not a hypercyclic tuple.
Example 2.5.  Let $\lambda$ be a scalar with $|\lambda| > 1$ and let $B$ be the backward shift on $l^2$. Let $m \in \mathbb{N}$ and let

$$M := \{ \{a_n\}_{n=0}^\infty : a_n = 0 \text{ for } n < m \}.$$

Define $T = (I, \lambda B)$. Note that $\lambda B$ is an $M$-hypercyclic operator by Example 3.8 in [7]. So, $T$ is an $M$-hypercyclic tuple by Theorem 2.2.

Example 2.6.  Let $B$ be the backward shift on $l^2$ and let $\lambda$ be a scalar with $|\lambda| > 1$. By Example 3.7 in [7], $\lambda B$ is $M$-hypercyclic with respect to

$$M := \{ \{a_n\}_{n=0}^\infty : a_{2k} = 0 \text{ for all } k \}.$$

Define $T = (\lambda B, \frac{1}{\lambda} B)$. Note that $\lambda B \in F_T$ and it is an $M$-hypercyclic operator. So $T$ is an $M$-hypercyclic tuple by Theorem 2.2.

3. Finite Dimensions

Subspace-hypercyclic operators does not exist on finite dimensional spaces. Also it is proved in [7] that an operator can not be subspace-hypercyclic with respect to a finite-dimensional subspace. But surprisingly, we prove that subspace-hypercyclic tuples exist on finite-dimensional spaces.

Theorem 3.1. (see [7]) Let $H$ be a finite-dimensional Hilbert space. If $T \in B(H)$, then $T$ is not subspace-hypercyclic for any closed nonzero subspace $M$ of $H$.

Theorem 3.2. (see [7]) Let $T \in B(H)$ where $H$ is a finite-dimensional Hilbert space. If $T$ is subspace-hypercyclic for a subspace $M$, then $M$ is not finite-dimensional.

Theorem 3.3. (see [3]) If $a, b > 1$ are relatively prime integers, then

$$\left\{ \frac{a^n}{b^k} : n, k \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^+$, the positive real numbers.

In the following example we show that subspace-hypercyclic tuples can exist on finite-dimensional spaces.

Example 3.4. Let $I$ be the identity operator on the real Hilbert space $\mathbb{R}$ and let $T = (-2I \oplus I, \frac{1}{3} I \oplus I)$ be a tuple on finite-dimensional space $\mathbb{R} \oplus \mathbb{R}$. By Theorem 3.3, $\text{cl}(\text{orb}(T, (1 \oplus 0)) \cap M) = M$. Then $T$ is a subspace-hypercyclic tuple with respect to finite-dimensional subspace $M := \mathbb{R} \oplus \{0\}$. Note that $T$ is not a hypercyclic tuple on $\mathbb{R} \oplus \mathbb{R}$.

By generalizing previous example, we get the following theorem:
Theorem 3.5. There are subspace-hypercyclic tuples of operators on $\mathbb{R}^n$ for any integer $n$ greater than 1.

Proof. Suppose that $a, b > 1$ are relatively prime integers. Fix $a, b$ and consider $T = (-|a|I_\mathbb{R} \oplus I_\mathbb{R} \oplus ... \oplus I_\mathbb{R}, \frac{1}{|b|}I_\mathbb{R} \oplus I_\mathbb{R} \oplus ... \oplus I_\mathbb{R})$ as an $n$-tuple on $\mathbb{R}^n = \mathbb{R} \oplus ... \oplus \mathbb{R}$, $n$ times. Then, $T$ is a subspace-hypercyclic $n$-tuple with respect to $M := \mathbb{R} \oplus \{0\} \oplus \{0\}$, with $M$-hypercyclic vector $1 \oplus 0 \oplus ... \oplus 0$.

Note that the above theorem shows that, unlike the subspace-hypercyclic operators, subspace-hypercyclic tuples exist on finite-dimensional spaces.

Example 3.6. Let $T_1 = -2I_\mathbb{R} \oplus I_\mathbb{R}$ and $T_2 = \frac{1}{3}I_\mathbb{R} \oplus I_\mathbb{R}$. Then, $T = (T_1, T_2)$ is a subspace-hypercyclic tuple with respect to $M := \mathbb{R} \oplus \{0\}$. But if we consider $n = (2, 1)$, then $T^n = (T_1^2, T_2)$ is not subspace-hypercyclic with respect to $M := \mathbb{R} \oplus \{0\}$. Since by Theorem 3.3, $cl(orb(T^n, (1 \oplus 0)) \cap M) = [0, +\infty) \oplus \{0\}$ and $cl(orb(T^n, (-1 \oplus 0)) \cap M) = (-\infty, 0] \oplus \{0\}$.

It is proved in [7] that compact operators are not subspace-hypercyclic.

Theorem 3.7. (see [7]) Let $T \in B(H)$, where $H$ is a Hilbert space. If $T$ is compact, then $T$ is not subspace-hypercyclic for any subspace.

In the following example we show that for a tuple $T = (T_1, T_2, ..., T_n)$, $T$ may be a subspace-hypercyclic tuple while $T_1, T_2, ..., T_n$ are compact.

Example 3.8. Let $T = (-2I_\mathbb{R} \oplus I_\mathbb{R}, \frac{1}{3}I_\mathbb{R} \oplus I_\mathbb{R}, T_3, ..., T_n)$ where $T_3, ..., T_n$ are compact operators on $\mathbb{R} \oplus \mathbb{R}$. By Theorem 3.3,

$$cl(orb(T^n, (1 \oplus 0)) \cap M) = \mathbb{R} \oplus \{0\}.$$ 

So $T$ is a subspace-hypercyclic $n$-tuple with respect to $M := \mathbb{R} \oplus \{0\}$. 

References


