SOFT $L$-FUZZY QUASI-UNIFORMITIES
INDUCED BY SOFT $L$-NEIGHBORHOOD SYSTEMS

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Abstract: In this paper, we obtain soft $L$-fuzzy quasi-uniformities induced by soft $L$-neighborhood systems in complete residuated lattices. Moreover, every $N$-continuous surjective soft maps are uniformly continuous soft maps. We give their examples.

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1. Introduction

Molodtsov [13] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,3,4,9,10,16,17,19,20]. Pawlak’s rough set [14,15] can be viewed as a special case of soft rough sets [4].

Kim [9,10] introduced a fuzzy soft $F : A \rightarrow L^U$ as an extension as the soft $F : A \rightarrow P(U)$ where $L$ is a complete residuated lattice [2,5,6]. He introduced soft $L$-fuzzy interior and closure operators, quasi-uniformities and soft $L$-fuzzy topogenous orders in complete residuated lattices.

In this paper, we obtain soft $L$-fuzzy quasi-uniformities induced by soft $L$-neighborhood systems in complete residuated lattices. Moreover, every $N$-continuous surjective soft maps are uniformly continuous soft maps. We give
their examples.

2. Preliminaries

Definition 2.1. [2,5,6] An algebra \((L, \land, \lor, \circ, \to, 0, 1)\) is called a complete residuated lattice if it satisfies the following conditions:

(C1) \(L = (L, \leq, \lor, \land, 1, 0)\) is a complete lattice with the greatest element 1 and the least element 0;
(C2) \((L, \circ, 1)\) is a commutative monoid;
(C3) \(x \circ y \leq z\) iff \(x \leq y \to z\) for \(x, y, z \in L\).

In this paper, we assume that \((L, \leq, \circ, \to)\) is a complete residuated lattice and we denote \(L_0 = L - \{0\}\).

Lemma 2.2. [2,5,6] For each \(x, y, z, x_i, y_i, w \in L\), we have the following properties.

(1) \(1 \to x = x, 0 \circ x = 0,\)
(2) If \(y \leq z\), then \(x \circ y \leq x \circ z, x \to y \leq x \to z\) and \(z \to x \leq y \to x,\)
(3) \(x \circ y \leq x \land y \leq x \lor y \leq x \oplus y,\)
(4) \(x \circ (\bigvee_i y_i) = \bigvee_i (x \circ y_i),\)
(5) \(x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i),\)
(6) \((\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y),\)
(7) \(x \to (\bigvee_i y_i) \geq \bigvee_i (x \to y_i),\)
(8) \((\bigwedge_i x_i) \to y \geq \bigvee_i (x_i \to y),\)
(9) \((x \circ y) \to z = x \to (y \to z) = y \to (x \to z),\)
(10) \(x \circ (x \to y) \leq y\) and \(x \to y \leq (y \to z) \to (x \to z),\)
(11) \((x \to y) \circ (z \to w) \leq (x \circ z) \to (y \circ w),\)
(12) \(x \to y \leq (x \circ z) \to (y \circ z)\) and \((x \to y) \circ (y \to z) \leq x \to z.\)

Definition 2.3. [9,10] Let \(X\) be an initial universe of objects and \(E\) the set of parameters (attributes) in \(X\). A pair \((F, A)\) is called a fuzzy soft set over \(X\), where \(A \subset E\) and \(F : A \to L^X\) is a mapping. We denote \(S(X, A)\) as the family of all fuzzy soft sets under the parameter \(A\).

Definition 2.4. [9,10] Let \((F, A)\) and \((G, A)\) be two fuzzy soft sets over a common universe \(X\).

(1) \((F, A)\) is a fuzzy soft subset of \((G, A)\), denoted by \((F, A) \leq (G, A)\) if \(F(a) \leq G(a)\), for each \(a \in A\).
(2) \((F, A) \land (G, A) = (F \land G, A)\) if \((F \land G)(a) = F(a) \land G(a)\) for each \(a \in A\).

(3) \((F, A) \lor (G, A) = (F \lor G, A)\) if \((F \lor G)(a) = F(a) \lor G(a)\) for each \(a \in A\).

(4) \((F, A) \circ (G, A) = (F \circ G, A)\) if \((F \circ G)(a) = F(a) \circ G(a)\) for each \(a \in A\).

(6) \(\alpha \circ (F, A) = (\alpha \circ F, A)\) for each \(\alpha \in L\).

**Definition 2.5.** [9,10] Let \(S(X, A)\) and \(S(Y, B)\) be the families of all fuzzy soft sets over \(X\) and \(Y\), respectively. The mapping \(f_\phi : S(X, A) \to S(Y, B)\) is a soft mapping where \(f : X \to Y\) and \(\phi : A \to B\) are mappings.

(1) The image of \((F, A) \in S(X, A)\) under the mapping \(f_\phi\) is denoted by \(f_\phi((F, A)) = (f_\phi(F), B)\) where

\[
f_\phi(F)(b)(y) = \begin{cases} \bigvee_{a \in \phi^{-1}(\{b\})} f^{-1}(F(a))(y), & \text{if } \phi^{-1}(\{b\}) \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases}
\]

(2) The inverse image of \((G, B) \in S(Y, B)\) under the mapping \(f_\phi\) is denoted by \(f_\phi^{-1}((G, B)) = (f_\phi^{-1}(G), A)\) where

\[
f_\phi^{-1}(G)(a)(x) = f^{-1}(G(\phi(a)))(x), \ \forall a \in A, x \in X.
\]

(3) The soft mapping \(f_\phi : S(X, A) \to S(Y, B)\) is called injective (resp. surjective, bijective) if \(f\) and \(\phi\) are both injective (resp. surjective, bijective).

**Lemma 2.6.** [9,10] Let \(f_\phi : S(X, A) \to S(Y, B)\) be a soft mapping. Then we have the following properties. For \((F, A), (F_i, A) \in S(X, A)\) and \((G, B), (G_i, B) \in S(Y, B),\)

(1) \((G, B) \geq f_\phi(f_\phi^{-1}((G, B)))\) with equality if \(f\) is surjective,

(2) \((F, A) \leq f_\phi^{-1}(f_\phi((F, A)))\) with equality if \(f\) is injective,

(3) \(f_\phi^{-1}(\bigvee_{i \in I}(G_i, B)) = \bigvee_{i \in I} f_\phi^{-1}((G_i, B))\),

(4) \(f_\phi^{-1}(\bigwedge_{i \in I}(G_i, B)) = \bigwedge_{i \in I} f_\phi^{-1}((G_i, B))\),

(5) \(f_\phi(\bigvee_{i \in I}(F_i, A)) = \bigvee_{i \in I} f_\phi((F_i, A))\),

(6) \(f_\phi(\bigwedge_{i \in I}(F_i, A)) \leq \bigwedge_{i \in I} f_\phi((F_i, A))\) with equality if \(f\) is injective,

(7) \(f_\phi^{-1}((G_1, B) \circ (G_2, B)) = f_\phi^{-1}((G_1, B)) \circ f_\phi^{-1}((G_2, B))\),

(8) \(f_\phi((F_1, A) \circ (F_2, A)) \leq f_\phi((F_1, A)) \circ f_\phi((F_2, A))\) with equality if \(f\) is injective.

**Definition 2.7.** [9] A map \(N : X \to (L^A)^{S(X,A)}\) is called a soft \(L\)-neighborhood system on \(X\) if \(N = \{N_x = N(x) \mid x \in X\}\) satisfies the following conditions

- (SN1) \(N_x((1_X, A)) = (1_X, A)(x) = 1_A\) and \(N_x((1_X, A)) = (0_X, A)(x) = 0_A\),
(SN2) \( N_x((F, A) \circ (G, A)) \geq N_x((F, A)) \circ N_x((G, A)) \) for each \((F, A), (G, A) \in S(X, A)\),
(SN3) If \((F, A) \leq (G, A)\), then \( N_x((F, A)) \leq N_x((G, A)) \),
(SN4) \( N_x((F, A)) \leq (F, A)(x) \) for all \((F, A) \in S(X, A)\) where \((F, A)(x) = F(-)(x)\).

A soft \(L\)-neighborhood system is called stratified if
\[ N_x((\alpha \circ (F, A))) \geq \alpha \circ N_x((F, A)) \] for all \((F, A) \in S(X, A)\) and \(\alpha \in L\).

The triple \((X, A, N)\) is called a soft \(L\)-neighborhood space.

Let \((X, A, N)\) and \((Y, B, M)\) be soft \(L\)-neighborhood spaces. A mapping \(f_\phi : (X, A, N) \rightarrow (Y, B, M)\) is said to be an \(N\)-continuous soft map iff
\[ M_f(x)((G, B))(\phi(a)) \leq N_x(f^{-1}_\phi((G, B)))(a) \]
for each \(x \in X, a \in A, (G, B) \in S(Y, B)\).

**Definition 2.8.** [10] A mapping \(U : S(X \times X, A) \rightarrow L\) is called a soft \(L\)-fuzzy quasi-uniformity on \(X\) iff it satisfies the properties.
(SU1) There exists \((U, A) \in S(X \times X, A)\) such that \(U((U, A)) = 1\),
(SU2) If \((V, A) \leq (U, A)\), then \(U((V, A)) \leq U((U, A))\),
(SU3) For every \((U, A), (V, A) \in S(X \times X, A)\),
\[ U((U, A) \circ (V, A)) \geq U((U, A)) \circ U((V, A)) \]
(SU4) If \(U((U, A)) \neq 0\), then \((1_\Delta, A) \leq (U, A)\), where
\[ 1_\Delta(a)(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \]
(SU5) \(\forall \{U((V, A)) \mid (V, A) \circ (V, A) \leq (U, A)\} \geq U((U, A))\)
\[ ((V, A) \circ (V, A))(a)(x, y) = (V(a) \circ V(a))(x, y) \]
\[ = \bigvee_{z \in X} V(a)(z, x) \circ V(a)(x, y), \forall x, y \in X, a \in A. \]

The triple \((X, A, U)\) is called a soft \(L\)-fuzzy quasi-uniform space.

Let \((X, A, U_X)\) and \((Y, B, U_Y)\) be soft \(L\)-fuzzy quasi-uniform spaces and \(f_\phi : (X, A) \rightarrow (Y, B)\) be a soft map. Then \(f_\phi : (X, A, U_X) \rightarrow (Y, B, U_Y)\) is called an uniformly continuous soft map if for all \((V, B) \in S(Y \times Y, B)\),
\[ U_Y((V, B)) \leq U_X((f \times f)^{-1}_\phi((V, B))), \forall (V, B) \in S(Y \times Y, B). \]
3. Soft $L$-Fuzzy Quasi-Uniformities Induced by Soft $L$-Neighborhood Systems

**Lemma 3.1.** For every $(F, A), (G, A) \in S(X, A)$, we define $(U_F, A), (U_F^{-1}, A) \in S(X \times X, A)$ by

$$U_F(a)(x, y) = F(a)(x) \rightarrow F(a)(y).$$

Then we have the following properties.

1. $(1_{X \times X}, A) = (U_0, A) = (U_1, A),$
2. $(1_{\Delta}, A) \leq (U_F, A),$
3. $(U_F, A) \circ (U_F, A) = (U_F, A),$
4. $(U_F, A) \circ (U_G, A) \leq (U_{F \circ G}, A).$

**Proof.**

1. 
   $$(1_{X \times X}, A) = (U_0, A) = (U_1, A),$$
   
   $$(1_{X \times X}, A)(x, y) = 1 = U_0(a)(x, y)$$
   
   $$(1_{X \times X}, A)(x, y) = U_1(a)(x, y).$$

2. 
   Since $U_F(a)(x, x) = F(a)(x) \rightarrow F(a)(x) = 0,$
   
   $$(1_{\Delta}, A) \leq (U_F, A).$$

3. 
   $$(U_F, A) \circ (U_F, A) = (U_F, A),$$
   
   $$(U_F, A) \circ (U_F, A) \leq (U_{F \circ G}, A).$$

4. 
   $(U_F, A) \circ (U_F, A) \geq (U_F, A)$ from
   
   $$(U_F(a) \circ U_F(a))(x, z)$$
   
   $$(U_F(a) \circ U_F(a))(x, z) = \bigvee_{y \in X} (U_F(a)(x, y) \circ U_F(a)(y, z))$$
   
   $$(U_F(a) \circ U_F(a))(x, z) = \bigvee_{y \in X} ((F(a)(x) \rightarrow F(a)(y)) \circ (F(a)(y) \rightarrow F(a)(z)))$$
   
   $$(U_F(a) \circ U_F(a))(x, z) \leq F(a)(x) \rightarrow F(a)(z) = U_F(a)(x, z).$$

4. 
   By Lemma 2.2 (11),

   $$(U_F(a)(x, y) \circ U_G(a)(x, y)$$
   
   $$(U_F(a)(x, y) \circ U_G(a)(x, y) = (F(a)(x) \rightarrow F(a)(y)) \circ (G(a)(x) \rightarrow G(a)(y))$$
   
   $$(U_F(a)(x, y) \circ U_G(a)(x, y) \leq (F(a)(x) \circ G(a)(x) \rightarrow F(a)(y) \circ G(a)(y))$$
   
   $$(U_F(a)(x, y) \circ U_G(a)(x, y) = U_{F \circ G}(a)(x, y).$$
Theorem 3.2. Let \((X, A, N)\) be a soft \(L\)-neighborhood space. Define a map \(\mathcal{U}_N : S(X \times X) \to L\) by

\[
\mathcal{U}_N((U, A)) = \bigvee \{ \bigwedge_{a \in A} \bigvee_{x \in X} \circ_{i=1}^n N_x((F_i, A))(a) \mid \circ_{i=1}^n (U_{F_i}, A) \leq (U, A) \}.
\]

Then

1. \((X, A, \mathcal{U}_N)\) is a soft \(L\)-fuzzy quasi-uniform space.
2. If \(N\) is stratified, then \(\mathcal{U}_N\) is stratified.

Proof. (SU1) Since \((U_{1_X}, A) = (1_{X \times X}, A)\), we have

\[
\mathcal{U}_N((1_{X \times X}, A)) \geq \bigvee_{x \in X, a \in A} N_x((1_X, A))(a) = 1.
\]

(SU2) If \((U_1, A) \leq (U_2, A)\), \((U_1, A), (U_2, A) \in S(X \times X, A)\), then

\[
\mathcal{U}_N((U_1, A)) = \bigvee \{ \bigwedge_{x \in X, a \in A} N_x((F, A)) \mid (U_F, A) \leq (U_1, A) \}
\leq \bigvee \{ \bigwedge_{x \in X, a \in A} N_x((F, A)) \mid (U_F, A) \leq (U_2, A) \} = \mathcal{U}_N((U_2, A)).
\]

(SU3)

\[
\mathcal{U}_N((U, A)) \circ \mathcal{U}_N((V, A))
\leq \bigvee \{ \bigwedge_{a \in A} \bigvee_{x \in X} \circ_{i=1}^n N_x((F_i, A))(a) \mid \circ_{i=1}^n (U_{F_i}, A) \leq (U, A) \}
\bigwedge \{ \bigwedge_{a \in A} \bigvee_{x \in X} \circ_{i=1}^n N_x((G_j, A))(a) \mid \circ_{j=1}^m (U_{G_j}, A) \leq (V, A) \}
\]

\leq \bigvee \{ \bigwedge_{a \in A} \bigvee_{x \in X} N_x((\circ_{i=1}^n (U_{F_i}, A) \circ \circ_{i=1}^n (U_{F_i}, A))(a) \mid (U, A) \leq (U, A) \}
\bigwedge \{ \bigwedge_{a \in A} \bigvee_{x \in X} \circ_{i=1}^n N_x((F_i, A))(a) \mid (U, A) \leq (V, A) \}
\leq \mathcal{U}_N((U, A) \circ (V, A)).
\]

(SU4) If \(\mathcal{U}_N((U, A)) \neq \perp\), then there exist \((F_i, A) \in S(X, A)\) with \(\circ_{i=1}^n (U_{F_i}, A) \leq (U, A)\) such that

\[
\bigwedge_{a \in A} \bigvee_{x \in X} \circ_{i=1}^n N_x((F_i, A))(a) \neq 0.
\]

For all \(a \in A\) and for some \(x \in X\),

\[
\circ_{i=1}^n N_x((F_i, A))(a) \neq 0.
\]

By Lemma 3.1, \(1_{\bigtriangleup}(a) \leq U_{F_i}(a)\), hence \((1_{\bigtriangleup}, A) \leq \circ_{i=1}^n (U_{F_i}, A) \leq (U, A)\).

(SU5) Suppose there exists \((U, A) \in S(X \times X, A)\) such that

\[
\bigvee \{ \mathcal{U}_N((V, A)) \mid (V, A) \circ (V, A) \leq (U, A) \} \neq \mathcal{U}_N((U, A)).
\]
Then there exist \((F_i, A) \in S(X, A)\) with \(\odot_{i=1}^n(U_{F_i}, A) \leq (U, A)\) such that
\[
\bigvee \{\mathcal{U}_N((V, A)) \mid (V, A) \circ (V, A) \leq (U, A)\} \nsubseteq \bigwedge_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n N_x((F_i, A))(a).
\]
Put \((W, A) = \odot_{i=1}^n(U_{F_i}, A)\). Then
\[
(W, A) \circ (W, A) = (\odot_{i=1}^n(U_{F_i}, A)) \circ (\odot_{i=1}^n(U_{F_i}, A)) = \odot_{i=1}^n(U_{F_i}, A) \circ (U_{F_i}, A) \leq (U, A).
\]
It is a contradiction.

(2) Let \((U, A) \in S(X \times X, A)\), \((F, A) \in S(X, A)\) and \(\alpha \in L\), we have
\[
\mathcal{U}_N(\alpha \odot (U, A)) = \bigvee \{\Lambda_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n N_x((F_i, A))(a) \mid \odot_{i=1}^n(U_{F_i}, A) \leq \alpha \odot (U, A)\}
\]
\[
\supseteq \bigvee \{\Lambda_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n N_x((\alpha \odot F_i, A))(a) \mid \odot_{i=1}^n(U_{\alpha \odot F_i}, A) \leq \alpha \odot (U, A)\}
\]
\[
\supseteq \bigvee \{\alpha \odot \Lambda_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n N_x((F_i, A))(a) \mid \odot_{i=1}^n(U_{\alpha \odot F_i}, A) \leq \alpha \odot (U, A)\}
\]
\[
\supseteq \bigvee \{\alpha \odot \Lambda_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n N_x((F_i, A))(a) \mid \odot_{i=1}^n(U_{F_i}, A) \leq (U, A)\}
\]
\[
\supseteq \alpha \odot \mathcal{U}_N((U, A)).
\]

**Theorem 3.3.** Let \((X, A, N)\) and \((Y, B, M)\) be soft \(L\)-neighborhood spaces. Let \(f_\phi : (X, A, N) \to (Y, B, M)\) be \(N\)-continuous surjective soft map. Then \(f_\phi : (X, A, U_N) \to (Y, B, V_M)\) is an uniformly continuous soft map.

**Proof.**

\[
(f \times f)_\phi^{-1}(U_G)(a)(x_1, x_2) = U_G(\phi(a))(f(x_1), f(x_2))
\]
\[
= G(\phi(a))(f(x_1)) \to G(\phi(a))(f(x_2))
\]
\[
= f_\phi^{-1}(G)(a)(x_1) \to f_\phi^{-1}(G)(a)(x_2)
\]
\[
= U_{f_\phi^{-1}(G)}(a)(x_1, x_2).
\]

Thus,
\[
V_M((V, A))
\]
\[
= \bigvee \{\Lambda_{b \in B} \bigvee_{y \in Y} \odot_{i=1}^n M_y((G_i, B))(b) \mid \odot_{i=1}^n(U_{G_i}, A) \leq (V, B)\}
\]
\((f_\phi \text{ is surjective})\)
\[
= \bigvee \{\Lambda_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n M_{f(x)}((G_i, B))(\phi(a)) \mid \odot_{i=1}^n(f \times f)_\phi^{-1}((U_{G_i}, A)) \leq (f \times f)_\phi^{-1}((V, B))\}
\]
\((f_\phi \text{ is \(N\)-continuous})\)
\[
= \bigvee \{\Lambda_{a \in A} \bigvee_{x \in X} \odot_{i=1}^n N_x(f_\phi^{-1}(G_i, B))(a) \mid \odot_{i=1}^n(U_{f_\phi^{-1}(G_i)}, A) \leq (f \times f)_\phi^{-1}((V, B))\}
\]
\[
\leq U_N((f \times f)_\phi^{-1}(V, B))).
\]
Example 3.4. Let $U = \{h_i \mid i = \{1, ..., 6\}\}$ with $h_i$=house and $E = \{e, b, w, c, i\}$ with $e$=expensive, $b$= beautiful, $w$=wooden, $c$= creative, $i$=in the green surroundings.

Define a binary operation $\odot$ on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}$$

$$x \odot y = \min\{1, x + y\}, \quad x^* = 1 - x$$

Then $([0, 1], \wedge, \rightarrow, 0, 1)$ is a complete residuated lattice (ref.[2,7,21]). Let $A = \{b, c\} \subset E$ and $X = \{h_1, h_4, h_5\}$. Put $(H, A)$ be a fuzzy soft set as follow:

$$(H, A) \begin{array}{ccc}
h_1 & h_4 & h_5 \\
b & 0.5 & 0.6 & 0.2 \\
c & 0.4 & 0.5 & 0.6 \\
\end{array}$$

$$(H, A) \odot (H, A) \begin{array}{ccc}
h_1 & h_4 & h_5 \\
b & 0.0 & 0.2 & 0.0 \\
c & 0.0 & 0.0 & 0.2 \\
\end{array}$$

Define a soft $L$-neighborhood system $N : X \rightarrow (L^A)^{S(X, A)}$ as follows

$$N_{h_1}((F, A)) = \begin{cases} (1, 1), & \text{if } (F, A) = (\bar{1}, A) \\ (0.5, 0.4), & \text{if } (H, A) \leq (F, A), \\ (0, 0), & \text{otherwise}; \end{cases}$$

$$N_{h_4}((F, A)) = \begin{cases} (1, 1), & \text{if } (F, A) = (\bar{1}, A) \\ (0.6, 0.5), & \text{if } (H, A) \leq (F, A), \\ (0.2, 0.0), & \text{if } (H, A) \odot (H, A) \leq (F, A), \\ (0, 0), & \text{otherwise}; \end{cases}$$

$$N_{h_5}((F, A)) = \begin{cases} (1, 1), & \text{if } (F, A) = (\bar{1}, A) \\ (0.2, 0.6), & \text{if } (H, A) \leq (F, A), \\ (0.0, 0.2), & \text{if } (H, A) \odot (H, A) \leq (F, A), \\ (0, 0), & \text{otherwise}; \end{cases}$$

We obtain $(U_H, A), (U_{H \odot H}, A) \in S(X \times X, A)$ such that, for $a \in A$, $U_H(a) \in L^{X \times X}$ with $U_H(a)(x, y) = H(a)(x) \rightarrow H(a)(y)$,

$$U_H(b) = \begin{pmatrix} 1 & 1 & 0.7 \\ 0.9 & 1 & 0.6 \\ 1 & 1 & 1 \end{pmatrix}, \quad U_H(c) = \begin{pmatrix} 1 & 1 & 1 \\ 0.9 & 1 & 1 \\ 0.8 & 0.9 & 1 \end{pmatrix}$$
By a similar method, we obtain
\[
U_{H \odot H}(b) = \begin{pmatrix}
1 & 1 & 1 \\
0.8 & 1 & 0.8 \\
1 & 1 & 1 \\
\end{pmatrix}
\quad U_{H \odot H}(c) = \begin{pmatrix}
1 & 1 & 1 \\
& 0.8 & 0.8 & 1 \\
\end{pmatrix}
\]

By Theorem 3.2, for \((U, A) \leq (U, A) \neq (1_{X \times X}, A)\), we have
\[
\begin{align*}
U_N((U, A)) &= \bigvee \{ \bigwedge_{a \in A} \bigvee_{x \in X} N_x((H, A))(a) \mid (U_H, A) \leq (U, A) \} \\
&= N_{h_1}((H, A))(a) \lor N_{h_4}((H, A))(a) \lor N_{h_5}((H, A))(a) \\
&\lor N_{h_1}((H, A))(b) \lor N_{h_4}((H, A))(b) \lor N_{h_5}((H, A))(b) \\
&= 0.5 \lor 0.6 \lor 0.2 \lor 0.4 \lor 0.5 \lor 0.6 = 0.6.
\end{align*}
\]

By a similar method, we obtain \(U_N : S(X \times X, A) \to L\) as follows
\[
U_N((U, A)) = \begin{cases}
1, & \text{if } (U, A) = (1_{X \times X}, A), \\
0.6, & \text{if } (U_H, A) \leq (U, A) \neq (1_{X \times X}, A), \\
0.2, & \text{if } (U_{H \odot H}, A) \leq (U, A) \neq (U_H, A), \\
0.2, & \text{if } (U, A) \odot (U_H, A) \leq (U, A) \neq (U_H, A), \\
0, & \text{otherwise}.
\end{cases}
\]

Since \(U_H(b) \circ U_H(x) = U_H(x)\) and \(U_{H \odot H}(x) \circ U_{H \odot H}(x) = U_{H \odot H}(x)\) for \(x \in \{a, b\}\), \(U_N\) is an soft \(L\)-fuzzy quasi-uniformity.

References


