SOMEWHAT $e$-$\mathcal{I}$-CONTINUOUS AND SOMEWHAT $e$-$\mathcal{I}$-OPEN FUNCTIONS VIA IDEALS

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Abstract: In this paper, new classes of functions are introduced and studied by making use of $e$-$\mathcal{I}$-open sets and $e$-$\mathcal{I}$-closed sets. Relationship between the new classes and other classes of functions are established besides giving examples, counterexamples, properties and characterizations.

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1. Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski [13] and Vaidyanathaswamy [19]. Jankovic and Hamlett [12] investigated further...
properties of ideal topological spaces. In 1992, Jankovic and Hamlett [11] introduced the notion of $I$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] investigated $I$-open sets and $I$-continuous functions. In this paper, using the notion of $e$-$I$-open sets defined in [2] see also [3], [4], the concepts of somewhat $e$-$I$-continuous functions and somewhat $e$-$I$-open functions are introduced and studied. Some characterizations and properties for somewhat $e$-$I$-continuity and somewhat $e$-$I$-openness are obtained besides giving examples and counterexamples.

An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies the following conditions: $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. A topological space $(X, \tau)$ with an ideal $\mathcal{I}$ is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$. Given an ideal topological space $(X, \tau, \mathcal{I})$ on $X$ and if $\wp(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$, called the local function [18, 12] of $A$ with respect to $\tau$ and $\mathcal{I}$, is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{I}, \tau) = \{ x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \}$$

where $\tau(x) = \{ U \in \tau \mid x \in U \}$. It is known in [12] that $\text{Cl}^*(A) = A \cup A^*(\mathcal{I}, \tau)$ is a Kuratowski closure operator. When there is no chance for confusion, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$. $X^*$ is often a proper subset of $X$. A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $R$-$I$-open (resp. $R$-$I$-closed) [20] if $A = \text{Int}(\text{Cl}^*(A))$ (resp. $A = \text{Cl}^*(\text{Int}(A))$). A point $x \in X$ is called a $\delta - I$-cluster point of $A$ if $\text{Int}(\text{Cl}(U)) \cap A \neq \emptyset$ for each open set $U$ containing $x$. The family of all $\delta - I$-cluster points of $A$ is called the $\delta - \mathcal{I}$-closure of $A$ and is denoted by $\delta \text{Cl}_I(A)$. The $\delta - I$-interior of $A$ is the union of all $R$-$I$-open sets of $X$ contained in $A$ and is denoted by $\delta \text{Int}_I(A)$. $A$ is said to be $\delta - I$-closed if $\delta \text{Cl}_I(A) = A$ [20].

**Definition 1.** A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be:

1. $I$-open [1] if $A \subset \text{Int}(A^*)$.
2. semi-$I$-open [10] if $A \subset \text{Cl}(\delta \text{Int}_I(A))$.
3. $e$-$I$-open if [2] $A \subset \text{Cl}(\delta \text{Int}_I(A)) \cup \text{Int}(\delta \text{Cl}_I(A))$.

**Remark 2.** In the following diagram we denote by arrows the implications between the open sets and the above three relations. It is known that (1) $\mathcal{I}$-openness and openness are independent [1, 6], (2) every $\mathcal{I}$-open set is semi-$\mathcal{I}$-
open [9], and (3) every open set is \( e-I \)-open[2].

\[
\begin{align*}
R-I\text{-open} & \longrightarrow \text{semi}^\ast-I\text{-open} \\
\downarrow & \\
on \longrightarrow & \ e-I\text{-open}
\end{align*}
\]

Definition 3. [8] A function \( f : (X, \tau) \longrightarrow (Y, \sigma) \) is said to be somewhat-continuous provided that if for \( U \in \sigma \) and \( f^{-1}(U) \neq \emptyset \) there exists an open set \( V \) in \( \tau \) such that \( V \neq \emptyset \) and \( V \subset f^{-1}(U) \).

Definition 4. A function \( f : (X, \tau, I) \longrightarrow (Y, \sigma) \) is said to be somewhat-\( I \)-continuous (resp. somewhat \( \text{semi}^\ast-I \)-continuous) if for any \( U \in \sigma \) such that \( f^{-1}(U) \neq \emptyset \) there exists an \( I \)-open (resp. \( \text{semi}^\ast-I \)-open) set \( V \) in \( X \) such that \( V \neq \emptyset \) and \( V \subset f^{-1}(U) \).

Definition 5. [8] A function \( f : (X, \tau) \longrightarrow (Y, \sigma) \) is said to be somewhat-open provided that if \( U \in \tau \) and \( U \neq \emptyset \), then there exists an open set \( V \) in \( \sigma \) such that \( V \neq \emptyset \) and \( V \subset f(U) \).

Definition 6. A function \( f : (X, \tau) \longrightarrow (Y, \sigma, I) \) is said to be somewhat-\( I \)-open (resp. somewhat \( \text{semi}^\ast-I \)-open) provided that if \( U \in \tau \) and \( U \neq \emptyset \), then there exists an \( \text{semi}^\ast-I \)-open set \( V \) in \( Y \) such that \( V \neq \emptyset \) and \( V \subset f(U) \).

2. Somewhat \( e-I \)-Continuous Functions

Definition 7. Let \((X, \tau, I)\) be an ideal topological space and \((Y, \sigma)\) be any topological space. A function \( f : (X, \tau, I) \longrightarrow (Y, \sigma) \) is said to be somewhat \( e-I \)-continuous provided that if \( U \in \sigma \) and \( f^{-1}(U) \neq \emptyset \), then there exists an \( e-I \)-open set \( V \) in \( X \) such that \( V \neq \emptyset \) and \( V \subset f^{-1}(U) \).

Theorem 8. If \( f : (X, \tau, I) \longrightarrow (Y, \sigma) \) is somewhat continuous, then it is somewhat \( e-I \)-continuous.

Proof. Trivial

Theorem 9. Every somewhat \( \text{semi}^\ast-I \)-continuous function is somewhat \( e-I \)-continuous.

Proof. Let \( f : (X, \tau, I) \longrightarrow (Y, \sigma) \) be a somewhat \( \text{semi}^\ast-I \)-continuous function. Let \( U \) be any open set in \( Y \) such that \( f^{-1}(U) \neq \emptyset \). Since \( f \) is somewhat
semi*-I-continuous, there exists a semi*-I-open set V in X such that V \neq \emptyset and V \subset f^{-1}(U)$. Since every semi*-I-open set is e-I-open, there exists an e-I-open set V such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$, which implies that f is somewhat e-I-continuous. \hfill \Box

**Remark 10.** The converses of the above theorem, need not be true in general as shown by the following example.

**Example 11.** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $I = \{\emptyset, \{a\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Let $f : (X, \tau, I) \rightarrow (Y, \sigma)$ be an identity function. Then f is somewhat e-I-continuous but it is neither somewhat continuous nor somewhat semi*-I-continuous. Since the inverse image of \{c\} in $(Y, \sigma)$ is an e-I-open but it is neither open nor semi*-I-open.

**Theorem 12.** If $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is a somewhat e-I-continuous surjection and $g : (Y, \sigma) \rightarrow (Z, \zeta)$ is somewhat continuous, then $g \circ f : (X, \tau, I) \rightarrow (Z, \zeta)$ is e-I-continuous.

**Proof.** Let $W$ be any open set of $(Z, \zeta)$ and $(g \circ f)^{-1}(W) \neq \emptyset$. Then $g^{-1}(W) \neq \emptyset$. Since g is somewhat continuous, there exists $V \in \sigma$ such that $\emptyset \neq V \subset g^{-1}(W)$. Since f is surjective, $\emptyset \neq f^{-1}(V) \subset f^{-1}(g^{-1}(W))$. Since f is somewhat e-I-continuous, there exists an e-I-open set $U$ in $(X, \tau)$ such that $\emptyset \neq U \subset f^{-1}(V)$. Therefore, we have $\emptyset \neq U \subset (g \circ f)^{-1}(W))$. This show that $g \circ f$ is somewhat e-I-continuous. \hfill \Box

**Theorem 13.** Let $f : (X, \tau, I) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \zeta)$ be any two functions. If f is somewhat e-I-continuous and g is continuous, then $g \circ f$ is somewhat e-I-continuous.

**Proof.** Let $U \in \zeta$ and $(g \circ f)^{-1}(U) \neq \emptyset$. Then $g^{-1}(U) \neq \emptyset$. Since $U \in \zeta$ and g is continuous $g^{-1}(U) \in \sigma$. Since $f^{-1}(g^{-1}(U)) \neq \emptyset$ and f is somewhat e-I-continuous, there exists an e-I-open set V in X such that $\emptyset \neq V \subset g^{-1}(U) = (g \circ f)^{-1}(U)$. Then $g \circ f$ is somewhat e-I-continuous. \hfill \Box

**Remark 14.** In Theorem 13, if f is a continuous function and g is a somewhat e-I-continuous function, then it is not necessarily true that $g \circ f$ is somewhat e-I-continuous. Since every continuous function is somewhat e-I-continuous, the composition of somewhat e-I-continuous functions need not be somewhat e-I-continuous. The following example serves this purpose.

**Example 15.** Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, $I = \{\emptyset, \{b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\zeta = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $J = \{\emptyset, \{a\}\}$. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, I)$ and $g : (Y, \sigma, J) \rightarrow (Z, \zeta)$ be
the identity functions. Then clearly $f$ is continuous and $g$ is somewhat $e$-$\mathcal{I}$-continuous but $g \circ f$ is not somewhat $e$-$\mathcal{I}$-continuous. Since $\{c\} \in \zeta$ and $(g \circ f)^{-1}(U) = (g \circ f)^{-1}(\{c\}) = \{c\}$ not somewhat $e$-$\mathcal{I}$-open, $g \circ f$ is not somewhat $e$-$\mathcal{I}$-continuous.

**Definition 16.** A subset $S$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $e$-$\mathcal{I}$-dense if $\text{Cl}^e(S) = X$. In other words if there is no proper $e$-$\mathcal{I}$-closed set $M$ in $X$ such that $S \subset M \subset X$.

**Theorem 17.** Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a surjective function. Then the following are equivalent:

1. $f$ is somewhat $e$-$\mathcal{I}$-continuous;

2. If $M$ is a closed subset of $Y$ such that $f^{-1}(M) \neq X$, then there is a proper $e$-$\mathcal{I}$-closed subset $D$ of $X$ such that $D \supset f^{-1}(M)$;

3. If $S$ is an $e$-$\mathcal{I}$-dense subset of $X$, then $f(S)$ is a dense subset of $Y$.

**Proof.** (1)$\Rightarrow$(2): Let $M$ be a closed subset of $Y$ such that $f^{-1}(M) \neq X$. Then $Y - M$ is an open set in $Y$ such that $f^{-1}(Y - M) = X - f^{-1}(M) \neq \emptyset$. By hypothesis (1) there exists an $e$-$\mathcal{I}$-open set $V$ in $X$ such that $V \neq \emptyset$ and $V \subset f^{-1}(Y - M) = X - f^{-1}(M)$. This means that $X - V \supset f^{-1}(M)$ and $X - V = D$ is a proper $e$-$\mathcal{I}$-closed set in $X$. This proves (2).

(2)$\Rightarrow$(3): Let $S$ be an $e$-$\mathcal{I}$-dense set in $X$. Suppose that $f(S)$ is not dense in $Y$. Then there exists a proper closed set $M$ in $Y$ such that $f(S) \subset M \subset Y$. Clearly $f^{-1}(M) \neq X$. Hence by (2) there exists a proper $e$-$\mathcal{I}$-closed set $D$ such that $S \subset f^{-1}(M) \subset D \subset X$. This contradicts fact that $S$ is $e$-$\mathcal{I}$-dense in $X$.

(3)$\Rightarrow$(2): Suppose that (2) is not true. This means there exists a closed set $M$ in $Y$ such that $f^{-1}(M) \neq X$. But there is no proper $e$-$\mathcal{I}$-closed set $D$ in $X$ such that $f^{-1}(M) \subset D$. This means that $f^{-1}(M)$ is $e$-$\mathcal{I}$-dense in $X$. But by (3) $f(f^{-1}(M)) = M$ must be dense in $Y$, which is contradiction to the choice of $M$.

(2)$\Rightarrow$(1): Let $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$. Then $Y - U$ is closed and $f^{-1}(Y - U) = X - f^{-1}(U) \neq X$. By hypothesis of (2) there exists a proper $e$-$\mathcal{I}$-closed set $D$ of $X$ such that $D \supset f^{-1}(Y - U)$. This implies that $X - D \subset f^{-1}(U)$ and $X - D$ is $e$-$\mathcal{I}$-open and $X - D \neq \emptyset$. □

**Theorem 18.** Let $(X, \tau, \mathcal{I})$ be any ideal topological space and $(Y, \sigma)$ any topological space. If $A$ is an open set in $X$ and $f : (A, \tau/A, \mathcal{I}/A) \rightarrow (Y, \sigma)$ is a somewhat $e$-$\mathcal{I}$-continuous function such that $f(A)$ is dense in $Y$, then any extension $F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ of $f$ is somewhat $e$-$\mathcal{I}$-continuous.
Proof. Let $U$ be any open set in $(Y, \sigma)$ such that $F^{-1}(U) \neq \emptyset$. Since $f(A) \subset Y$ is dense in $Y$ and $U \cap f(A) \neq \emptyset$, it follows that $F^{-1}(U) \cap A \neq \emptyset$. That is $f^{-1}(U) \cap A \neq \emptyset$. Hence by hypothesis on $f$, there exists an $e-I$-open set $V$ in $A$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U) \subset F^{-1}(U)$ which implies $F$ is somewhat $e-I$-continuous.

**Theorem 19.** Let $(X, \tau, I)$ and $(Y, \sigma, J)$ be any two ideal topological spaces, $X = A \cup B$ where $A$ and $B$ are open subsets of $X$ and $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a function such that $f/A$ and $f/B$ are somewhat $e-I$-continuous. Then $f$ is somewhat $e-I$-continuous.

Proof. Let $U$ be any open set in $(Y, \sigma, J)$ such that $f^{-1}(U) \neq \emptyset$. Then $(f/A)^{-1}(U) \neq \emptyset$ or $(f/B)^{-1}(U) \neq \emptyset$ or both $(f/A)^{-1}(U) \neq \emptyset$ and $(f/B)^{-1}(U) \neq \emptyset$.

**Case (1)** Suppose $(f/A)^{-1}(U) \neq \emptyset$.

Since $f/A$ is somewhat $e-I$-continuous, there exists an $e-I$-open set $V$ in $A$ such that $V \neq \emptyset$ and $V \subset (f/A)^{-1}(U) \subset f^{-1}(U)$. Since $V$ is $e-I$-open in $A$ and $A$ is open in $X$, $V$ is $e-I$-open in $X$. Thus $f$ is somewhat $e-I$-continuous.

**Case (2)** the proof is similar with Case (1).

**Case (3)** Suppose $(f/A)^{-1}(U) \neq \emptyset$ and $(f/B)^{-1}(U) \neq \emptyset$.

This follows from both the Cases (1) and (2). Thus $f$ is somewhat $e-I$-continuous.

**Definition 20.** An ideal topological space $(X, \tau, I)$ is said to be $e-I$-separable if there exists a countable subset $B$ of $X$ which is $e-I$-dense in $X$.

**Theorem 21.** If $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is a somewhat $e-I$-continuous surjective function and $X$ is $e-I$-separable, then $Y$ is separable.

Proof. Let $f : X \rightarrow Y$ be a somewhat $e-I$-continuous surjection such that $X$ is $e-I$-separable. Then by definition there exists a countable subset $B$ of $X$ which is $e-I$-dense in $X$. Then by Theorem 17, $f(B)$ is dense in $Y$. Since $B$ is countable, $f(B)$ is also a countable set which is dense in $Y$, which indicates that $Y$ is separable.
3. e-\mathcal{I}-Weakly Equivalent Topologies

**Definition 22.** Let \( X \) be a set and \( \tau \) and \( \sigma \) be topologies for \( X \). Then \( \tau \) is said to be weakly equivalent to \( \sigma \) [8] provided that if \( U \in \tau \) and \( U \neq \emptyset \), then there is an open set \( V \) in \((X, \sigma)\) such that \( V \neq \emptyset \) and \( V \subset U \) and if \( U \in \sigma \) and \( U \neq \emptyset \), then there is an open set \( V \) in \((X, \tau)\) such that \( V \neq \emptyset \) and \( V \subset U \).

**Definition 23.** Let \((X, \tau)\) and \((X, \sigma)\) be topological spaces with same ideal \( \mathcal{I} \). Then \( \tau \) is said to be \( e-\mathcal{I} \)-weakly equivalent to \( \sigma \) provided that if \( U \in \tau \) and \( U \neq \emptyset \), then there is an \( e-\mathcal{I} \)-open set \( V \) in \((X, \sigma, \mathcal{I})\) such that \( V \neq \emptyset \) and \( V \subset f(U) \) and if \( U \in \sigma \) and \( U \neq \emptyset \), then there is an \( e-\mathcal{I} \)-open set \( V \) in \((X, \tau, \mathcal{I})\) such that \( V \neq \emptyset \) and \( V \subset f(U) \).

**Theorem 24.** Let \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_1) \) be a somewhat \( e-\mathcal{I} \)-continuous surjective function and let \( \sigma_2 \) be a topology for \( Y \). If \( \sigma_2 \) is weakly equivalent to \( \sigma_1 \), then the function \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_2) \) is somewhat \( e-\mathcal{I} \)-continuous.

**Proof.** Since \( \sigma_2 \) is weakly equivalent to \( \sigma_1 \), the identity function \( i : (Y, \sigma_1) \rightarrow (Y, \sigma_2) \) is somewhat-continuous. Therefore, by Theorem 12,

\[
f = f \circ i : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_2)
\]

is somewhat \( e-\mathcal{I} \)-continuous.

\[\square\]

**Theorem 25.** Let \( f : (X, \tau_1, \mathcal{I}) \rightarrow (Y, \sigma) \) be a somewhat-continuous function and let \( \tau_2 \) be a topology for \( X \). If \( \tau_2 \) is \( e-\mathcal{I} \)-weakly equivalent to \( \tau_1 \), then the function \( f : (X, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma) \) is somewhat \( e-\mathcal{I} \)-continuous.

**Proof.** Since \( \tau_2 \) is \( e-\mathcal{I} \)-weakly equivalent to \( \tau_1 \), the identity function \( i : (X, \tau_2, \mathcal{I}) \rightarrow (X, \tau_1, \mathcal{I}) \) is somewhat \( e-\mathcal{I} \)-continuous. Therefore, by Theorem 12, \( f = f \circ i : (X, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma) \) is somewhat \( e-\mathcal{I} \)-continuous.

\[\square\]

4. Somewhat \( e-\mathcal{I} \)-Open Functions

**Definition 26.** A function \( f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I}) \) is said to be somewhat \( e-\mathcal{I} \)-open provided that for \( U \in \tau \) and \( U \neq \emptyset \) there exists an \( e-\mathcal{I} \)-open set \( V \) in \( Y \) such that \( V \neq \emptyset \) and \( V \subset f(U) \).

**Theorem 27.** If a function \( f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I}) \) is somewhat open, then it is somewhat \( e-\mathcal{I} \)-open.
Theorem 28. Every somewhat semi*-I-open function is somewhat e-I-open.

Proof. Let \( f : (X, \tau) \rightarrow (Y, \sigma, I) \) be a somewhat semi*-I-open function. Let \( U \in \tau \) and \( U \neq \emptyset \). Since \( f \) is somewhat semi*-I-open, there exists a semi*-I-open set \( V \) in \( X \) such that \( V \neq \emptyset \) and \( V \subset f(U) \). Since every semi*-I-open set is e-I-open, there exist an e-I-open set \( V \) such that \( V \neq \emptyset \) and \( V \subset f(U) \), which implies that \( f \) is somewhat e-I-open. \( \square \)

Remark 29. The converses of the above theorems, need not be true in general as shown by the following example.

Example 30. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\} \), \( I = \{\emptyset, \{a\}\} \), and \( \sigma = \{\emptyset, X, \{a\}, \{b, c\}\} \). Then the identity function \( f : (X, \tau) \rightarrow (Y, \sigma, I) \) is somewhat e-I-open but it is neither somewhat semi*-I-open nor somewhat open. The image of \( \{c\} \in \tau \) is e-I-open but it is neither semi*-I-open nor somewhat open.

Theorem 31. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is an somewhat-open function and \( g : (Y, \sigma) \rightarrow (Z, \zeta, I) \) is a somewhat e-I-open function, then \( g \circ f : (X, \tau) \rightarrow (Z, \zeta, I) \) is somewhat e-I-open.

Proof. Suppose that \( U \in \tau \) and \( U \neq \emptyset \). Since \( f \) is somewhat-open, there exists an open set \( G \) of \( (Y, \sigma) \) such that \( \emptyset \neq G \) and \( G \subset f(U) \). Since \( g \) is somewhat e-I-open, there exists an e-I-open set \( V \in \zeta \) such that \( \emptyset \neq V \subset g(G) \subset (g \circ f)(U) \). This implies that \( g \circ f \) is somewhat e-I-open. \( \square \)

Theorem 32. Let \( f : (X, \tau) \rightarrow (Y, \sigma, I) \) be a bijective function. Then the following are equivalent:

1. \( f \) is somewhat e-I-open;
2. If \( M \) is a closed set of \( X \) such that \( f(M) \neq Y \), then there is an e-I-closed set \( D \) of \( Y \) such that \( D \neq Y \) and \( D \supset f(M) \).

Proof. (1) \( \Rightarrow \) (2): Let \( M \) be a closed set of \( X \) such that \( f(M) \neq Y \). Then \( X - M \) is an open set in \( X \) and \( X - M \neq \emptyset \). Since \( f \) is somewhat e-I-open, there exists an e-I-open set \( V \) in \( X \) such that \( \emptyset \neq V \) and \( V \subset f(X - M) \). Put \( D = Y - V \). Clearly \( D \) is e-I-closed in \( Y \) and we claim that \( D \neq Y \). For if \( D = Y \), then \( V = \emptyset \) which is a contradiction. Since \( V \subset f(X - M) \) and \( f \) is bijective, \( D = Y - V \supset Y - [f(X - M)] = f(M) \).

(2) \( \Rightarrow \) (1): Let \( U \) be any nonempty open set in \( X \). Put \( M = X - U \). Then \( M \) is a proper closed set of \( X \) and \( f(M) \neq Y \). Therefore, by (2) there is an
e-I-closed subset $D$ of $Y$ such that $D \neq Y$ and $f(M) \subset D$. Put $V = Y - D$. Clearly $V$ is an $e$-I-open set and $V \neq \emptyset$. Further, $V = Y - D \subset Y - f(M) = Y - [Y - f(U)] = f(U)$.

**Theorem 33.** If $f : (X, \tau) \to (Y, \sigma, I)$ is a somewhat $e$-I-open function and $A$ is an open set of $X$, then $f/A : (A, \tau/A) \to (Y, \sigma, I)$ is also somewhat $e$-I-open.

**Proof.** Let $U \in \tau/A$ such that $U \neq \emptyset$. Since $U$ is open in $A$ and $A$ is open in $(X, \tau)$, $U$ is open in $(X, \tau)$. By hypothesis, $f : (X, \tau) \to (Y, \sigma, I)$ is somewhat $e$-I-open and there exists an $e$-I-open set $V$ in $Y$ such that $\emptyset \neq V \subset f(U)$. Thus for any open set $U$ in $(A, \tau/A)$ with $U \neq \emptyset$, there exists an $e$-I-open set $V$ in $Y$ such that $\emptyset \neq V \subset f(U)$ which implies that $f/A$ is somewhat $e$-I-open.

**Theorem 34.** Let $(X, \tau)$ be a topological space, $(Y, \sigma, I)$ be an ideal topological space and $X = A \cup B$, where $A$ and $B$ are open sets of $X$. If $f : (X, \tau) \to (Y, \sigma, I)$ is a function such that $f/A$ and $f/B$ are somewhat $e$-I-open, then $f$ is somewhat $e$-I-open.

**Proof.** Let $U$ be any open set of $(X, \tau)$ such that $U \neq \emptyset$. Since $X = A \cup B$, there are three Cases (1) $A \cap U \neq \emptyset$, (2) $B \cap U \neq \emptyset$ or (3) both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$.

Case (1). Since $A \cap U \subset \tau/A$ and $f/A$ is somewhat $e$-I-open, there exists an $e$-I-open set $V$ in $Y$ such that $\emptyset \neq V \subset f(U)$. This shows that $f$ is somewhat $e$-I-open. The other cases are similarly proved.

**References**


