International Journal of Pure and Applied Mathematics
Volume 107 No. 2 2016, 393-406
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i2.9

ijpam.eu

A NEW METHOD FOR THE CONSTRUCTION OF SPLINE BASIS FUNCTIONS FOR SAMPLING APPROXIMATIONS

T. Zhanlav[1], R. Mijiddorj[2 §]
[1,2]Institute of Mathematics
National University of Mongolia
MONGOLIA
[2]Department of Informatics
Mongolian State University of Education
MONGOLIA

Abstract: A method for constructing a new kind of spline basis functions ($\varphi_{2n}(x)$) with compact support on R was described by Ueno et al. (2007). These basis functions for sampling approximations consist of a linear combination of the cardinal B splines, but the construction is complicated and therefore were constructed basis functions $\varphi_{2n}(x)$ only for $n \leq 5$. We discuss a new construction of the basis functions and its approximation properties are considered.

AMS Subject Classification: 65D, 41A15
Key Words: construction of basis splines, sampling and local approximations, error analysis

1. Introduction

In [5] the spline basis functions $\varphi_{2n}(x)$ which are symmetric with narrower compact support are constructed satisfying the following conditions

(C1) $\varphi_{2n}(-x) = \varphi_{2n}(x)$,

(C2) $\text{supp}\varphi_{2n}(x) = [-n-1, n+1]$,

Received: December 14, 2015
Published: April 9, 2016

§Correspondence author

(C3) $\varphi_{2n}(k) = \delta_{k,0}, \ (k \in Z)$,

(C4) $\sum_{k\in Z} k^i \varphi_{2n}(x-k) = x^i, \ i = 0, 1, \cdots, 2n.$

In [5] it is also shown that if $\varphi_{2n}(x)$ is a specific linear combination of $\{N_{2n+2}, \cdots, N_{n+2}\}$ and satisfies (C3), it enjoys the condition (C4) which is equivalent to the moment condition. As mentioned in [5], the construction of such basis functions is interesting from both the sampling and interpolating approximation point of view. But the general rule for choosing the specific linear combination mentioned above is complicated. (C1)-(C4) properties can not guarantee a unique basis depending on how a specific linear combination of the cardinal B-spline is chosen.

The aim of this paper is to give a new construction of functions φ_{2n} that can overcome the above mentioned complication. The main idea of our construction is the direct applications of conditions (C1), (C2) and (C4), but not the condition (C3) in general.

The rest of the paper is organized as follows. In Section 2, we propose a new construction of basis splines without (C3) condition. In Section 3, we consider the approximation properties of a local cubic splines. Finally, in Section 4, we present results of numerical tests which confirm the theoretical order of convergence.

2. Construction of Basis Splines

We are looking for φ_{2n} which satisfies the (C1), (C2) and (C4). If φ satisfies Strang-Fix (SF) and moment (M) conditions, then the sampling approximation obtained by the translation of φ gives us a good approximation [1]. We need the following property:

Proposition 1. *The following are equivalent:*

1. *φ satisfies (SF) and (M) of order r,*

2. $\sum_{k\in Z} (x-k)^i \varphi(x-k) = \delta_{i,0}$,

3. $\sum_{k\in Z} k^i \varphi(x-k) = x^i$,

where $i = 0, \cdots, r$.

Proof (see [1], for example.)

The fulfilment of condition (C4) indicates that $\varphi_{2n}(x)$ is differentiable some-times. So we can differentiate (C4) s times ($s \leq 2n$)

$$\sum k^i \varphi_{2n}^{(s)}(x-k) = i(i-1)\dots(i-s+1)x^{i-s}, \qquad s = 1, 2, \cdots . \tag{1}$$

When $i \neq s$, setting $x = 0$ in (1), we obtain

$$\sum_{k=-n}^{n} k^i \varphi_{2n}^{(s)}(-k) = 0, \qquad i = 0, 1, \dots, 2n. \tag{2}$$

Analogously, when $i = s$, setting $x = 0$ in (1), we obtain

$$\sum_{k=-n}^{n} k^i \varphi_{2n}^{(i)}(-k) = i!, \qquad i \geq 1. \tag{3}$$

From (C1) it is clear that

$$\varphi_{2n}^{(s)}(x) = (-1)^s \varphi_{2n}^{(s)}(-x), \qquad s = 1, 2, \dots \tag{4}$$

Setting $x = 0$ in (4) we conclude that

$$\varphi_{2n}^{(s)}(0) = 0, \qquad \text{when } s \text{ is odd.} \tag{5}$$

If we take into account (C2) and (4), the equalities (2) and (3) can be rewritten as

$$\sum_{k=1}^{n} k^i \varphi_{2n}^{(i)}(k) = \frac{(-1)^i}{2} i!, \qquad i \geq 1, \tag{6a}$$

$$\sum_{k=1}^{n} k^i [(-1)^i + (-1)^s] \varphi_{2n}^{(s)}(k) = 0, \qquad i \neq 0,\ i \neq s, \tag{6b}$$

$$\varphi_{2n}^{(s)}(0) + 2\sum_{k=1}^{n} \varphi_{2n}^{(s)}(k) = 0, \qquad \text{when } i = 0 \text{ and } s \text{ is even.} \tag{6c}$$

We summarise the following algorithm to construct the basis splines:

Algorithm 1

1. Set a value of n.

2. Find the $\varphi_{2n}^{(m)}(k)$ values using (5) (when $1 \le s \le n$), (6a) (when $1 \le i \le n$), (6b) (if $i = 1, 2, \cdots, n$ then $s = i+2, i+4, \cdots$ and $s \le n$; if $s = 1, 2, \cdots, n$ then $i = s+2, s+4, \cdots$ and $i \le 2n$), and (6c) (when $2 \le s \le n$). In this step we can find $n^2 + n$ values of $\varphi_{2n}^{(m)}(k)$, where $m = 1, 2, \cdots, n$.

3. Suppose that the polynomial $\varphi_{2n}(x)$ is a form in $p_i(x) = \sum\limits_{j=0}^{2n+1} a_{i,j} x^j$ on the subintervals $[i-1, i]$, where $i = 1, 2, \cdots, n+1$. Solve the system on the subinterval $[i-1, i]$

$$\begin{aligned} &p_i^{(m)}(i-1) = \varphi_{2n}^{(m)}(i-1), \quad p_i^{(m)}(i) = \varphi_{2n}^{(m)}(i), \\ &p_i(i-1) = c_{i-1}, \qquad p_i(i) = c_i \end{aligned} \tag{7}$$

which consists $2n+2$ equations with $2n+2$ unknowns $a_{i,j}$, where c_k are parameters and $p_i(0) = c_0 = 1 - 2\sum_{k=1}^{n} c_k$ which follows from (6c).

4. In particular,

(a) if $c_k = 0$ $(k = 1, 2, \cdots, n)$ then the spline $\varphi_{2n}(x)$ coincides with one, constructed in [5].

(b) find c_k using $p_i^{(n+1)}(i) = p_{i+1}^{(n+1)}(i)$, where $i, k = 1, 2, \cdots, n$. In this case we can construct the spline $\varphi_{2n}(x)$ belongs to $C^{n+1}[-n-1, n+1]$.

Let us explain some cases of the proposed algorithm. For STEP 2, from (5) and (6), the direct calculations for $n = 1,\ 2$ and 3 give us the following values:

$$1. \quad n = 1, \quad \varphi_2'(0) = 0,\ \varphi_2'(1) = -\tfrac{1}{2}. \tag{8}$$

$$\begin{aligned} 2. \quad n = 2, \quad &\varphi_4'(0) = 0,\ \varphi_4'(1) = -\tfrac{8}{12},\ \varphi_4'(2) = \tfrac{1}{12}, \\ &\varphi_4''(0) = -\tfrac{5}{2},\ \varphi_4''(1) = \tfrac{4}{3},\ \varphi_4''(2) = -\tfrac{1}{12}. \end{aligned} \tag{9}$$

These turn out to correspond to the well known 5-point finite difference scheme for the first and second derivatives [2].

$$\begin{aligned} 3. \quad n = 3, \quad &\varphi_6'(0) = 0,\ \varphi_6'(1) = -\tfrac{3}{4},\ \varphi_6'(2) = \tfrac{3}{20},\ \varphi_6'(3) = -\tfrac{1}{60} \\ &\varphi_6''(0) = -\tfrac{49}{18},\ \varphi_6''(1) = \tfrac{3}{2},\ \varphi_6''(2) = -\tfrac{3}{20},\ \varphi_6''(3) = \tfrac{1}{90}, \\ &\varphi_6'''(0) = 0,\ \varphi_6'''(1) = \tfrac{13}{8},\ \varphi_6'''(2) = -1,\ \varphi_6'''(3) = \tfrac{1}{8}, \end{aligned} \tag{10}$$

which correspond to the well known 7-point central difference formulas of order 4 for derivatives of order 1-4 respectively [2].

For STEP 3, we consider the case $n = 1$. Assume that $\varphi_2(x)$ has a form

$$p_1(x) = a_{1,0} + a_{1,1}x + a_{1,2}x^2 + a_{1,3}x^3, \qquad x \in [0, 1]. \tag{11}$$

In order to determine the coefficients $a_{1,i}$ in (11) we use (7) and (8), i,e.,

$$\varphi_2(0) = 1 - 2c_1, \ \ \varphi_2(1) = c_1, \ \ \varphi_2'(0) = 0, \ \ \varphi_2'(1) = -\frac{1}{2}.$$

As a result we have

$$a_{1,0} = 1 - 2c_1, \ \ a_{1,1} = 0, \ \ a_{1,2} = \frac{18c_1 - 5}{2}, \ \ a_{1,3} = \frac{3(1 - 4c_1)}{2}.$$

Similarly, in order to determine the coefficients of $\varphi_2(x)$ on $x \in [1, 2]$ we will use the conditions

$$\varphi_2(1) = c_1, \ \ \varphi_2(2) = 0, \ \ \varphi_2'(1) = -\frac{1}{2}, \ \ \varphi_2'(2) = 0,$$

which give us

$$a_{2,0} = \frac{4 - 8c_1}{2}, \ \ a_{2,1} = \frac{24c_1 - 8}{2}, \ \ a_{2,2} = \frac{5 - 18c_1}{2}, \ \ a_{2,3} = \frac{4c_1 - 1}{2}.$$

Thus, we find $\varphi_2(x)$ in the form

$$\varphi_2(x) = \frac{1}{2}\begin{cases} 4(1 - 2c_1) + 8(1 - 3c_1)x - (18c_1 - 5)x^2 + (1 - 4c_1)x^3, & -2 \le x \le -1, \\ 2(1 - 2c_1) + (18c_1 - 5)x^2 - 3(1 - 4c_1)x^3, & -1 \le x \le 0, \\ 2(1 - 2c_1) + (18c_1 - 5)x^2 + 3(1 - 4c_1)x^3, & 0 \le x \le 1, \\ 4(1 - 2c_1) - 8(1 - 3c_1)x - (18c_1 - 5)x^2 - (1 - 4c_1)x^3, & 1 \le x \le 2. \end{cases} \tag{12}$$

It is easy to show that

$$\varphi_2''(1 - 0) \ne \varphi_2''(1 + 0), \ \ \varphi_2''(2) = -2 + 12c_1 \ne 0 \quad \text{when } c_1 \ne \frac{1}{6}.$$

It means that $\varphi_2(x) \in C^1[-2; 2]$ when $c_1 \ne \frac{1}{6}$. When $c_1 = 0$ the spline $\varphi_2(x)$ given by (12) coincides with one, constructed in [5]. When $c_1 = \frac{1}{6}$ the function $\varphi_2(x)$ belongs to $C^2[-2, 2]$ and it coincides with well-known cubic B-spline of class C^2 [3],[7]. Similarly we find $\varphi_{2n}(x)$ for $n = 2, 3, 4, 5, 6$. For brevity we present some basis functions in the form:

(STEP 4a) For $c_k = 0$ $(k = 1, 2, \cdots, n)$:

$$\varphi_2(x) = \frac{1}{2}\begin{cases} 2 - 5x^2 + 3x^3, & x \in [0,1[, \\ 4 - 8x + 5x^2 - x^3, & x \in [1,2[, \end{cases}$$

$$\varphi_4(x) = \frac{1}{12}\begin{cases} 12 - 15x^2 - 35x^3 + 63x^4 - 25x^5, & x \in [0,1[, \\ \frac{1}{2}(-96 + 450x - 735x^2 + 545x^3 - 189x^4 + 25x^5), & x \in [1,2[, \\ \frac{1}{2}(432 - 918x + 765x^2 - 313x^3 + 63x^4 - 5x^5), & x \in [2,3[. \end{cases}$$

$$\varphi_6(x) = \frac{1}{8640}\begin{cases} 5(144 - 196x^2 - 959x^4 + 2569x^5 - 2181x^6 + 623x^7), & x \in [0,1[, \\ 3(6624 - 34468x + 77238x^2 - 94920x^3 + 68355x^4 \\ -28749x^5 + 6543x^6 - 623x^7), & x \in [1,2[, \\ -316800 + 934164x - 1171310x^2 + 808920x^3 \\ -332185x^4 + 81109x^5 - 10905x^6 + 623x^7, & x \in [2,3[, \\ 523008 - 1073664x + 940576x^2 - 455760x^3 \\ +131915x^4 - 22807x^5 + 2181x^6 - 89x^7, & x \in [3,4[. \end{cases}$$

(STEP 4b) For $\varphi_{2n} \in C^{n+1}[a,b]$:

$$\varphi_2(x) = \frac{1}{6}\begin{cases} 4 - 6x^2 + 3x^3, & x \in [0,1[, \\ 8 - 12x + 6x^2 - x^3, & x \in [1,2[, \end{cases}$$

$$\varphi_4(x) = \begin{cases} \frac{33}{40} - \frac{5}{4}x^2 + \frac{7}{8}x^4 - \frac{x^5}{3}, & x \in [0,1], \\ \frac{51}{80} + \frac{5}{4}x - \frac{35}{8}x^2 + \frac{15}{4}x^3 - \frac{21}{16}x^4 + \frac{x^5}{6}, & x \in [1,2], \\ \frac{243}{80} - \frac{27}{4}x + \frac{45}{8}x^2 - \frac{9}{4}x^3 + \frac{7}{16}x^4 - \frac{x^5}{30}, & x \in [2,3]. \end{cases}$$

$$\varphi_6(x) = \frac{1}{60480}\begin{cases} 5(892 - 1372x^2 + 392x^4 + 931x^5 - 1057x^6 \\ +301x^7), & x \in [0,1[, \\ 3(4086 - 13916x + 29106x^2 - 39200x^3 + 30870x^4 \\ -13671x^5 + 3171x^6 - 301x^7), & x \in [1,2[, \\ -129310 + 400428x - 525770x^2 + 376320x^3 \\ -158270x^4 + 39151x^5 - 5285x^6 + 301x^7, & x \in [2,3[, \\ 238592 - 499968x + 445312x^2 - 218400x^3 \\ +63700x^4 - 11053x^5 + 1057x^6 - 43x^7, & x \in [3,4[. \end{cases}$$

To compare the profiles of φ_2, φ_4, φ_6, and φ_8, we sketch the graph of these functions in Fig. 1 and 2.

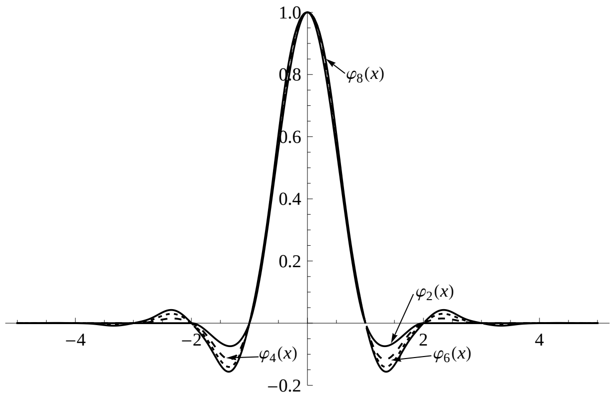

Figure 1: φ_2, φ_4, φ_6 and φ_8 of (4a).

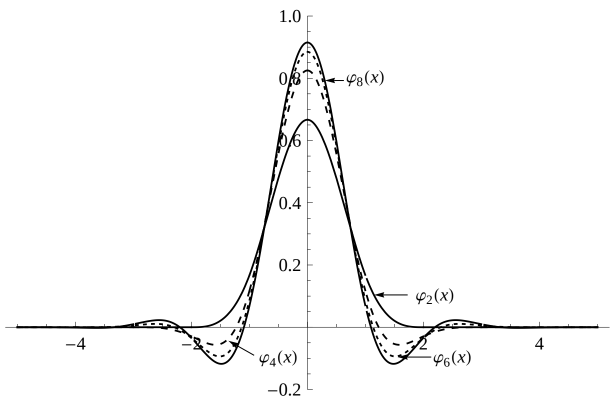

Figure 2: φ_2, φ_4, φ_6 and φ_8 of (4b).

As mentioned above, in [5] was proposed general scheme to construct basis splines, but their construction (method 1) is more complicated and time consuming than our construction. Comparison of these two approaches was made by CPU time (in seconds, processor Intel Core i5 CPU 2.67GHz (4 CPUs)). From Table 1, it is clear that our approach (Algorithm 1) is preferable and very easy to implement. The construction also shows that the basis splines in [5] belong to $\varphi_{2n} \in C^n[a, b]$, while our construction without the requirement of (C3) improved the smoothness of spline.

methods	$\varphi_2(x)$	$\varphi_4(x)$	$\varphi_6(x)$	$\varphi_8(x)$	$\varphi_{10}(x)$
method 1	0.0936	0.3120	6.2556	337.4404	11753.73
Algorithm 1	0.0780	0.1092	0.1404	0.1560	0.1872

Table 1

3. Approximate Splines and their Properties

We now consider the sampling approximation

$$f(x) \approx S_j(x) = \sum_{k\in Z} \alpha_k \varphi_2(2^j x - k). \tag{13}$$

By virtue of (8), (12), and (C1)-(C2) one can obtain at point $x_k = kh,\ h = 2^{-j}$,

$$\begin{aligned} &S_j(x_k) = c_1\alpha_{k-1} + (1-2c_1)\alpha_k + c_1\alpha_{k+1}, \\ &S_j'(x_k) = \frac{\alpha_{k+1}-\alpha_{k-1}}{2h}, \quad S_j''(x_k) = \frac{\alpha_{k+1} - 2\alpha_k + \alpha_{k-1}}{h^2}. \end{aligned} \tag{14}$$

We find the coefficients of sampling approximation (13) such that

$$f^{(r)}(x_k) = S_j^{(r)}(x_k), \quad r = 0, 1, 2. \tag{15}$$

Then it is easy to show that

$$\begin{aligned} \alpha_{k-1} &= f(x_k) - hf'(x_k) + \frac{1-2c_1}{2}h^2 f''(x_k), \\ \alpha_k &= f(x_k) - c_1 h^2 f''(x_k), \\ \alpha_{k+1} &= f(x_k) + hf'(x_k) + \frac{1-2c_1}{2}h^2 f''(x_k). \end{aligned} \tag{16}$$

Let $f(x) \in C^4$. Then from (16) we obtain

$$\begin{aligned} \alpha_{k-1} &= f_{k-1} - c_1 h^2 f''_{k-1} + \frac{1-6c_1}{6}h^3 f^{(3)}_{k-1} + O(h^4), \\ \alpha_{k+1} &= f_{k+1} - c_1 h^2 f''_{k+1} - \frac{1-6c_1}{6}h^3 f^{(3)}_{k+1} + O(h^4). \end{aligned}$$

From this and from (16) it follows that

$$\alpha_i = f_i - c_1 h^2 f_i'' + O(h^l), \qquad i = 1, 2, \cdots, N-1, \tag{17}$$

where

$$l = \begin{cases} 3, & \text{when } c_1 \neq \frac{1}{6}, \\ 4, & \text{when } c_1 = \frac{1}{6}. \end{cases} \tag{18}$$

Using

$$f_i'' = \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + O(h^2)$$

in (17) we obtain

$$\alpha_i = \widehat{\alpha}_i + O(h^l), \tag{19}$$

where

$$\widehat{\alpha}_i = (1 + 2c_1)f_i - c_1(f_{i-1} + f_{i+1}), \qquad i = 1, 2, \cdots, N-1. \tag{20a}$$

It remains to determine the coefficients $\widehat{\alpha}_i$ for $i = -1, 0$ and $i = N, N+1$. If we use the well-known one-sided approximation formulas [2]

$$f_0' = \frac{1}{h}(-\frac{11}{6}f_0 + 3f_1 - \frac{3}{2}f_2 + \frac{1}{3}f_3) + O(h^3),$$

$$f_0'' = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + O(h^2)$$

in (16) for $k = 0$ then we obtain formulas

$$\begin{aligned} \widehat{\alpha}_0 &= (1 - 2c_1)f_0 - c_1(-5f_1 + 4f_2 - f_3), \\ \widehat{\alpha}_{-1} &= \frac{23 - 12c_1}{6}f_0 - \frac{11 - 10c_1}{2}f_1 + \frac{7 - 8c_1}{2}f_2 - \frac{5 - 6c_1}{6}f_3 \end{aligned} \tag{20b}$$

with accuracy $O(h^4)$. Analogously, we obtain

$$\widehat{\alpha}_N = (1 - 2c_1)f_N - c_1(-5f_{N-1} + 4f_{N-2} - f_{N-3}),$$

$$\begin{aligned} \widehat{\alpha}_{N+1} =& \frac{23 - 12c_1}{6}f_N - \frac{11 - 10c_1}{2}f_{N-1} \\ &+ \frac{7 - 8c_1}{2}f_{N-2} - \frac{5 - 6c_1}{6}f_{N-3}. \end{aligned} \tag{20c}$$

From (19), (20) it is clear that instead of (13) one can consider a local cubic spline

$$\widehat{S}_j(f, \varphi) = \sum_k \widehat{\alpha}_k \varphi_2(2^j x - k), \tag{21}$$

where $\widehat{\alpha}_k$ are given by (20). When $c_1 = 0$ the spline $\widehat{S}_j(x)$ coincides with sampling approximation

$$S_j(x) = \sum f(x_k)\varphi_2(2^j x - k) \tag{22}$$

given in [5]. When $c_1 = \frac{1}{6}$, $\widehat{S}_j$ coincides with a local cubic one given in [7],[8]. It is well known that for sampling approximation (22) holds the following error estimation [4],[5].

Theorem 1. *Let φ satisfy (SF) and (M) of order r. If f belongs to $W^{N,p}(R)$, $N \le r$, then for $1 \le p \le \infty$ we have*

$$\|S_j(f,\varphi) - f\|_{L^p(\infty)} \le C_{\varphi,p,N} 2^{-jN} \|f^{(N)}\|_{L^P(R)}, \quad j = 1, 2, \ldots \tag{23}$$

where $C_{\varphi,p,N}$ is a constant depending only on φ, p, and N.

It means that for $\varphi = \varphi_2(x)$ we have

$$S_j(f,\varphi) - f(x) = O(h^2), \quad x \in R, \text{ with } h = 2^{-j}. \tag{24}$$

We shall show that for smooth function $f(x) \in C^4(R)$ the order of accuracy increases. More precisely, we have the following theorem.

Theorem 2. *If f belongs to $C^4(R)$, then for local approximation* (21) *holds the following estimation*

$$\widehat{S}_j^{(r)}(f,\varphi_2) - f^{(r)}(x) = O(h^{l-r}), \qquad x \in R, \ r = 0, 1, \tag{25}$$

where l is given by (18).

Proof. Using Taylor expansion of f_k, $f_{k\pm1}$ at point x in (20) it is easy to show that

$$\begin{aligned} \widehat{\alpha}_k = f(x) - f'(x)h(2^j x - k) + \frac{f''(x)}{2} h^2((2^j x - k)^2 - 2c_1) - \\ - \frac{f'''(x)}{6} h^3((2^j x - k)^3 - 6c_1(2^j x - k)) + O(h^l), \end{aligned} \tag{26}$$

in which we have used $x_k = kh$, $h = 2^{-j}$.

Substituting (26) into (21) we get

$$\begin{aligned}\widehat{S}_j(x) = f(x)\sum \varphi_2(2^j x - k) - hf'(x)\sum_k (2^j x - k)\varphi_2(2^j x - k) + \\ + \frac{f''(x)}{2}h^2[\sum_k (2^j x - k)^2\varphi_2(2^j x - k) - 2c_1\sum_k \varphi_2(2^j x - k)] - \\ - \frac{f'''(x)}{6}h^3[\sum_k (2^j x - k)^3\varphi_2(2^j x - k) \\ - 6c_1\sum_k (2^j x - k)\varphi_2(2^j x - k)] + O(h^l).\end{aligned} \tag{27}$$

If we use (C4) and Proposition 1, then from (27) we obtain

$$\widehat{S}_j(x) - f(x) = -c_1 h^2 f''(x) - \frac{f'''(x)}{6}h^3\sum_k (2^j x - k)^3\varphi_2(2^j x - k) + O(h^l). \tag{28}$$

When $c_1 = 0$ from the last expression we conclude that

$$\widehat{S}_j(x) - f(x) = O(h^3).$$

When $c_1 = \frac{1}{6}$, $\varphi_2(x)$ coincides completely with the cubic B-spline, for which the following relations [7] hold that

$$\sum_p (2^j x - p)^\alpha B_p(x) = l_\alpha^0, \tag{29}$$

with constants:

$$l_0^0 = 1, \quad l_1^0 = l_3^0 = 0, \quad l_2^0 = 1/3.$$

Then using (29) from (27) immediately follows that

$$\widehat{S}_j(x) - f(x) = O(h^4).$$

So (25) is proven for $r = 0$. When $c_1 = 0$, differentiating both sides of (28), we obtain

$$\widehat{S}'_j(x) - f'(x) = O(h^2).$$

Now from (21) we get

$$\widehat{S}'_j(x) = \frac{1}{h}\sum_k \widehat{a}_k\varphi'_2(2^j x - k). \tag{30}$$

Differentiating (29) one time for each α we get

$$\sum_p (2^j x - p)^\alpha B_p'(x) = l_\alpha^1, \tag{31}$$

with constants:

$$l_0^1 = 0, \quad l_1^1 = l_3^1 = -1, \quad l_2^1 = 0.$$

When $c_1 = \frac{1}{6}$, substituting (26) into (30) and using (31) equalities in the obtained equation, we get

$$\widehat{S}_j'(x) - f'(x) = O(h^3).$$

So (25) is proven for $r = 1$. This completes the proof. □

4. Numerical Examples

In this section we present some numerical results to test the accuracy and efficiency of the local cubic spline (21). The tested functions are $y_1(x) = e^x$, $y_2(x) = \cos(\pi x)$ and $y_3(x) = \frac{1}{x+2}$, the interval $[a, b] = [0, 1]$. The respective maximum absolute errors are given in Tables 2-5, numerical convergence order is denoted by

$$NCO = \log_2 |\frac{R_r(j)}{R_r(j+1)}|,$$

where

$$R_r(j) = \max_{x\in[a,b]} |\widehat{S}_j^{(r)}(x) - y^{(r)}(x)|, \quad r = 0, 1.$$

As shown in Table 2-5, the approximation properties of the local cubic spline (Theorem 2) were confirmed by numerical experiments.

5. Conclusion

In this paper, the drawback of the construction of the spline basis functions for sampling approximations in [5] is discussed. By using (C1), C(2), and (C4) properties, this work demonstrates that the construction of the basis spline proposed by T.Ueno, etc., could be simplified. The new method is effective and easy to implement. A local approximation of the basis spline in this work could approximate not only the function values but also derivative values. An open problem and other related problems on this topic remain to be our future work.

j	$\lvert\widehat{S}(x)-y_1(x)\rvert$	NCO	$\lvert\widehat{S}(x)-y_2(x)\rvert$	NCO	$\lvert\widehat{S}(x)-y_3(x)\rvert$	NCO
0	3.00×10^{-1}		1.94×10^{0}			
1	2.98×10^{-2}	3.33	3.86×10^{-1}	2.33	5.56×10^{-3}	
2	5.15×10^{-3}	2.53	6.76×10^{-2}	2.51	5.41×10^{-4}	3.36
3	7.63×10^{-4}	2.76	5.28×10^{-3}	3.68	9.02×10^{-5}	2.59
4	4.67×10^{-5}	4.03	2.04×10^{-4}	4.69	5.81×10^{-6}	3.96
5	1.22×10^{-6}	5.25	1.47×10^{-5}	3.79	1.56×10^{-7}	5.22
6	1.35×10^{-7}	3.18	1.50×10^{-6}	3.29	1.54×10^{-8}	3.34
7	1.87×10^{-8}	2.85	2.24×10^{-7}	2.74	2.33×10^{-9}	2.72

Table 2: $c_1 = 0$

j	$\lvert\widehat{S}'(x)-y_1'(x)\rvert$	NCO	$\lvert\widehat{S}'(x)-y_2'(x)\rvert$	NCO	$\lvert\widehat{S}'(x)-y_3'(x)\rvert$	NCO
0	1.36×10^{0}		7.33×10^{0}			
1	1.32×10^{-1}	3.36	1.27×10^{0}	2.53	2.03×10^{-2}	
2	3.47×10^{-2}	1.93	5.89×10^{-1}	1.11	3.48×10^{-3}	2.54
3	5.43×10^{-3}	2.67	8.01×10^{-2}	2.88	6.26×10^{-4}	2.48
4	1.50×10^{-3}	1.85	2.01×10^{-2}	1.99	1.88×10^{-4}	1.73
5	2.68×10^{-4}	2.49	5.04×10^{-3}	2.00	2.50×10^{-5}	2.91
6	6.71×10^{-5}	2.00	1.26×10^{-3}	2.00	6.25×10^{-6}	2.00
7	1.68×10^{-5}	2.00	3.15×10^{-4}	2.00	1.56×10^{-6}	2.00

Table 3: $c_1 = 0$

j	$\lvert\widehat{S}(x)-y_1(x)\rvert$	NCO	$\lvert\widehat{S}(x)-y_2(x)\rvert$	NCO	$\lvert\widehat{S}(x)-y_3(x)\rvert$	NCO
0	3.24×10^{-1}		2.22×10^{0}			
1	5.25×10^{-3}	5.95	8.10×10^{-2}	4.78	1.02×10^{-3}	
2	3.23×10^{-4}	4.02	8.00×10^{-3}	3.34	6.00×10^{-5}	4.09
3	1.54×10^{-5}	4.39	5.33×10^{-4}	3.91	3.25×10^{-6}	4.21
4	1.06×10^{-6}	3.86	3.99×10^{-5}	3.74	2.55×10^{-7}	3.67
5	6.76×10^{-8}	3.98	2.54×10^{-6}	3.97	1.61×10^{-8}	3.98
6	4.42×10^{-9}	3.93	1.67×10^{-7}	3.93	1.06×10^{-9}	3.93
7	2.64×10^{-10}	4.06	9.96×10^{-9}	4.06	6.32×10^{-11}	4.06

Table 4: $c_1 = \frac{1}{6}$

Acknowledgments

The work was partially supported by Foundation of Science and Technology of Mongolia (No. SST_007/2015).

j	$\lvert\widehat{S}'(x) - y_1'(x)\rvert$	NCO	$\lvert\widehat{S}'(x) - y_2'(x)\rvert$	NCO	$\lvert\widehat{S}'(x) - y_3'(x)\rvert$	NCO
0	1.03×10^{0}		6.67×10^{0}			
1	5.86×10^{-2}	4.14	6.67×10^{-1}	3.32	1.11×10^{-2}	
2	7.92×10^{-3}	2.89	2.09×10^{-1}	1.67	1.52×10^{-3}	2.87
3	1.14×10^{-3}	2.79	4.16×10^{-2}	2.33	2.58×10^{-4}	2.55
4	1.54×10^{-4}	2.89	5.76×10^{-3}	2.86	3.82×10^{-5}	2.76
5	2.00×10^{-5}	2.95	7.37×10^{-4}	2.96	5.22×10^{-6}	2.87
6	2.54×10^{-6}	2.97	9.27×10^{-5}	2.99	6.83×10^{-7}	2.93
7	3.21×10^{-7}	2.99	1.16×10^{-5}	3.00	8.73×10^{-8}	2.97

Table 5: $c_1 = \frac{1}{6}$

References

[1] R. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer, Berlin Heidelberg New York, (1993).

[2] B. Fornberg, Generation of finite difference formulas on arbitrarily spaced grids, *Math. Comp.*, 51 (1998), 699-706.

[3] P. M. Prenter, *Splines and variational methods*, John Wiley, New York, (1975).

[4] H. L. Resnikoff, R. O. Wells, *Wavelet analysis*, Springer-Verlag, New York, (1998).

[5] T. Ueno, S. Truscott, M. Okada, New spline basis functions for sampling approximations, *Numer. Algor.*, 45 (2007), 283-293.

[6] T. Ueno, T. Ide, M. Okada, A wavelet collocation method for evolution equations with energy conservation property, *Bull. Sci. Math.*, 127 (2003), 569-583.

[7] Yu.S. Zav'yalov., B.I. Kvasov and V.L. Miroschnichenko, *Spline Function Methods*, (1980). [in Russian]

[8] T. Zhanlav, B-representation of interpolatory cubic splines, *Vychislitel'nye Systems, Novisibirsk*, 87 (1981), 3-10. [in Russian]