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SKEW N-NORMAL COMPOSITION AND WEIGHTED COMPOSITION OPERATORS ON $L^2(\mu)$

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Abstract: An operator T is called skew n-normal operator if $(T^{\mathsf{n}}T^*)T = T(T^*T^{\mathsf{n}})$, for all natural number n. In this paper, the condition under which composition operators and weighted composition operators become skew n-normal operators have been obtained in terms of radon-nikodym derivative h_{n}. We investigate some basic properties of such operators and study the relation among non normal composition operators and the skew n-normal composition operators.

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1. Introduction

Let H be the infinite dimensional complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. An operator $T \in B(H)$ is said to be self-adjoint (see [3]) if $T^* = T$, normal (see [2]) if $TT^* = T^*T$, quasinormal (see [11]) if $T(T^*T) = (T^*T)T$. An operator T is called n-normal (see [2,6]) if

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$T^nT^{\square} = T^{\square}T^n$, n-quasinormal operator (see [1,9]) if $T^nT^{\square}T = T^{\square}TT^n$, skew n-normal operators (see [10]) if $(T^nT^{\square})T = T(T^{\square}T^n)$ for $n \in N$.

Let $(X\mathcal{A}, \mu)$ be a σ-finite measure space. A transformation T is said to be measurable if $T^{-1}(B) \in \mathcal{A}$ for $B \in \mathcal{A}$. A measurable transformation T is said to be non-singular if

$$\mu(T^{-1}(B)) = 0 \text{ whenever } \mu(B) = 0 \text{ for every } B \in \mathcal{A}.$$

If T is a measurable transformation then T^n is also a measurable transformation, for all natural number n. If T is non-singular, then we say that $\mu(T^{-1})$ is absolutely continuous with respect to μ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to μ. Hence, by Radon-Nikodym theorem there exists a unique non-negative essentially bounded measurable function h_n such that

$$\mu(T^{-1})^n(B) = \int_B h_n d\mu \qquad \text{for } B \in \mathcal{A}$$

and h_n is called the Radon-Nikodym derivative and is denoted by $\frac{d\mu(T^{-1})^{\mathsf{n}}}{d\mu}$.

Let ϕ be an essentially bounded function. The multiplication operator M_ϕ on the space $L^2(\mu)$ induced by ϕ is given by

$$M_\phi f = \phi f \qquad \text{for } f \in L^2(\mu)$$

Let T be a measurable transformation on X. The composition operator C_T on the space $L^2(\mu)$ is given by

$$C_T f = f \circ T \qquad \text{for } f \in L^2(\mu)$$

Let ϕ be a complex-valued measurable function. The weighted composition operator $W_{\phi,T}$ on the space $L^2(\mu)$ induced by ϕ and T is given by

$$W_{\phi,T} f = \phi.f \circ T \qquad \text{for } f \in L^2(\mu).$$

For more on composition operators and weighted composition operators one may refer to [4-8,12].

Proposition 1. *Change of Variables: Let X be a non-empty set and let $\mathcal{A}$ be a σ-algebra on X. Let μ and μT^{-1} be measures on $\mathcal{A}$ and let $h : X \to [0, 1]$ be a measurable function. Then the following are equivalent:*

(i) μT^{-1} is absolutely continuous with respect to μ and h is Radon-Nikodym derivative of μT^{-1} with respect to μ.

(ii) For every measurable function $f : X \to [0; 1]$, the equality

$$\int_X f d\mu T^{-1} = \int_X f h d\mu$$

holds.

The conditional expectation operator $E(.|T^{-1}(\mathcal{A})) = E(f)$ *is defined for each non-negative function* f *in* L^p $(1 \leq p < \infty)$ *and is uniquely determined by the following set of conditions:*

(i) $E(f)$ *is* $T^{-1}(\mathcal{A})$ *measurable.*

(ii) If B *is any* $T^{-1}(\mathcal{A})$ *measurable set for which* $\int_B f d\mu$ *converges then we have*

$$\int_B f d\mu = \int_B E(f) d\mu.$$

The conditional expectation operator E *has the following properties:*

(i) $E(f.g \circ T) = (E(f))(g \circ T)$.

(ii) E *is monotonically increasing, i.e., if* $f \leq g$ *a.e. then* $E(f) \leq E(g)$ *a.e.*

(iii) $E(1) = 1$.

(iv) $E(f)$ *has the form* $E(f) = g \circ T$ *for exactly one* $\mathcal{A}$*-measurable function* g *provided that the support of* g *lies in the support of* h *which is given by*

$$\sigma(h) = \{x : h(x) \neq 0\}.$$

E *is the projection operator onto the closure of the range of the composition operator* C_T *on* $L^2(\mu)$.

Motivated by the approach and direction of research by the mathematicians in [4,5,6,9,10], an effort has been made in the paper to discuss the behaviour of skew n-normal class of operators. In this paper, we study skew n-normal composition operators and weighted composition operators in terms of Radon-Nikodym derivative and expectation operators.

2. Skew n-Normal Composition Operators

Let C_T be the composition operator and C_T^{*} be its adjoint which is given by $C_T^{*} f = h.E(f) \circ T^{-1}$.

Lemma 2. [8, 12] *Let* P *be the projection on* $L^2(X, \mathcal{A}, \mu)$ *onto* $\overline{R(C_T)}$. *Then:*

(i) $C_T^{*} C_T f = hf$ *and* $C_T C_T^{*} f = (h \circ T)Pf$ *for all* $f \in L^2(\mu)$.

(ii) $\overline{R(C_T)} = \{f \in L^2(\mu) : f \quad is \quad T^{-1}(\mathcal{A})\}$ *measurable.*

(iii) If f is $T^{-1}(\mathcal{A})$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).

(iv) $(C_T^{\star}C_T)^k f = h^k f$ for all $k \in \mathcal{N}$.

(v) $(C_T C_T^{\star})^k f = (h \circ T)^k P(f)$.

(vi) E is the identity operator on $L^2(\mu)$ if and if $T^{-1}(\mathcal{A}) = \mathcal{A}$.

Theorem 3. *Let C_T be a composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) C_T is skew n-normal operator.

(ii) $h \circ T^n = h \circ T$.

Proof. For $f \in L^2(\mu)$

$$\begin{aligned}(C_T^n C_T^{\star})C_T f &= (C_T^n C_T^{\star}) f \circ T \\ &= C_T^n (h.E(f \circ T) \circ T^{-1}) \\ &= (h.E(f \circ T) \circ T^{-1}) \circ T^n \\ &= h \circ T^n.E(f \circ T^n).\end{aligned}$$

Also,

$$\begin{aligned}C_T(C_T^{\star} C_T^n) f &= C_T C_T^{\star}(f \circ T^n) \\ &= C_T h.E(f \circ T^n) \circ T^{-1} \\ &= h.E(f \circ T^{n-1}) \circ T \\ &= h \circ T.(E(f \circ T^n)).\end{aligned}$$

If C_T is skew n-normal operator then

$$(C_T^n C_T^{\star})C_T = C_T(C_T^{\star} C_T^n) \Longleftrightarrow h \circ T^n = h \circ T.$$

□

Corollary 4. *Let C_T be a composition operator on $L^2(\mu)$. then the following statements are equivalent:*

(i) C_T is skew n-normal operator.

(ii) $\|\sqrt{h \circ T^n}\| = \|\sqrt{h \circ T}\|$.

Corollary 5. *An operator C_T is skew n-normal operator iff $C_T^{\star}$ is skew n-normal operator.*

Proof. Let C_T be a skew n-normal operator therefore

$$(C_T^n C_T^*)C_T = C_T(C_T^* C_T^n).$$

Taking adjoint on both side, we have

$$C_T^*(C_T C_T^{*n}) = (C_T^{*n} C_T)C_T^*$$

Therefore C_T^* is skew n-normal operator. □

In this theorem we explain the condition under which the adjoint of C_T is skew n-normal operator.

Theorem 6. *An operator $C_T^* \in L^2(\mu)$ is skew n-normal operator iff*

$$h_n.h = h_n.h \circ T^{-n+1}$$

Proof. Suppose C_T^* is skew n-normal operator. Since

$$C_T^*(C_T C_T^{*n}) = (C_T^{*n} C_T)C_T^*.$$

We have

$$\begin{aligned} C_T^*(C_T C_T^{*n})f &= C_T^* C_T(h_n.E(f) \circ T^{-n}) \\ &= C_T^*(h_n.E(f) \circ T^{-n} \circ T) \\ &= hE(h_n \circ TE(f) \circ T^{1-n}) \circ T^{-1} \\ &= hE(h_n.E(f) \circ T^{-n}) \\ &= h.h_n f \circ T^{-n}. \end{aligned}$$

Also,

$$\begin{aligned} (C_T^{*n} C_T)C_T^* f &= C_T^{*n} C_T(h.E(f) \circ T^{-1}) \\ &= C_T^{*n}(h \circ T.E(f) \circ T^{-1} \circ T) \\ &= h_n(h \circ T.E(f) \circ T^{-n}) \\ &= h_n h \circ T^{-n+1} f \circ T^{-n}. \end{aligned}$$

If C_T^* is skew n-normal operator then

$$C_T^*(C_T C_T^{*n}) = (C_T^{*n} C_T)C_T^* \Longleftrightarrow h_n.h = h_n.h \circ T^{-n+1}.$$

□

Theorem 7. *If C_T is skew n-normal operator on $L^2(\mu)$. Then αC_T is skew n-normal operator for every complex number α.*

Proof. Consider

$$
\begin{aligned}
((\alpha C_T)^n (\alpha C_T)^{*})(\alpha C_T) &= \alpha^n \overline{\alpha} \alpha (C_T^n C_T^{*}) C_T. \\
&= \alpha^n \overline{\alpha} \alpha C_T (C_T^{*} C_T^n) \\
&= (\alpha C_T)((\alpha C_T)^{*} (\alpha C_T)^n)
\end{aligned}
$$

so that αC_T is skew n-normal operator. □

Theorem 8. *Let C_T be the skew n-normal composition operator on a Hilbert space $L^2(\mu)$. If $(C_T)^{-1} = C_{T^{-1}}$ then $C_{T^{-1}}$ is skew n-normal composition operator.*

Proof. Since C_T is skew n-normal composition operator, therefore

$$C_T(C_T^{*} C_T^n) = (C_T^n C_T^{*}) C_T.$$

Taking inverse on both sides, we get

$$C_{T^{-1}}^n C_{T^{-1}}^{*} C_{T^{-1}} = C_{T^{-1}} C_{T^{-1}}^{*} C_{T^{-1}}^n.$$

Hence $C_{T^{-1}}$ is skew n-normal composition operator. □

A composition operator C_S is unitarily equivalent to C_T if $C_S = U C_T U^{*}$ where U is some unitary operator.

Theorem 9. *If C_T is skew n-normal operator on a Hilbert space $\mathcal{H}$ and C_S is unitarily equivalent to C_T, then C_S is skew n-normal operator.*

Proof. Since C_S is unitarily equivalent to C_T, then there exists an unitary operator U such that $C_S = U C_T U^{*}$.
Now

$$
\begin{aligned}
(C_S^n C_S^{*}) C_S &= (U C_T^n U^{*} U C_T^{*} U^{*}) U C_T U^{*} \\
&= U (C_T^n C_T^{*}) C_T U^{*} \\
&= U C_T (C_T^{*} C_T^n) U^{*}
\end{aligned}
$$

because C_T is skew n-normal operator. On the other hand

$$C_S (C_S^{*} C_S^n) = U C_T U^{*} (U C_T^{*} U^{*} U C_T^n U^{*}) = U C_T (C_T^{*} C_T^n) U^{*}.$$

This implies that $(C_S^n C_S^{*}) C_S = C_S (C_S^{*} C_S^n)$. Hence C_S is skew n-normal operator. □

Theorem 10. *Let C_S be normal operator and C_T be skew n-normal operator.If C_S and C_T commute, then $C_S C_T$ is skew n-normal operator.*

Proof. Cosider

$$((C_SC_T)^n(C_SC_T)^*)C_SC_T = (C_S^nC_T^nC_T^*C_S^*)C_SC_T.$$

Since C_S is normal operator which commutes with C_T, then by Fuglede-Putnam theorem, C_T commutes with C_S^* therefore

$$\begin{aligned}
(C_S^nC_T^nC_T^*C_S^*)C_SC_T &= C_S^n(C_T^nC_T^*)C_TC_S^*C_S \\
&= C_S^nC_T(C_T^*C_T^n)C_S^*C_S \\
&= C_SC_TC_T^*C_S^{n-1}C_T^nC_S^*C_S \\
&= C_SC_T(C_T^*C_S^*C_S^{n-1}C_T^nC_S) \\
&= C_SC_T(C_T^*C_S^*C_S^nC_T^n) \\
&= C_SC_T((C_SC_T)^*(C_SC_T)^n).
\end{aligned}$$

Thus C_SC_T is skew n-normal operator. □

Theorem 11. *Let C_T be the skew n-normal operator, then C_T is skew $n+k(n-1)$-normal operator, for every natural number k.*

Proof. We prove this result by using the method of induction for every natural number k.
(Base case): when $k=1$

$$\begin{aligned}
(C_T^{n+(n-1)}C_T^*)C_T &= C_T^{n-1}(C_T^nC_T^*)C_T \\
&= C_T^{n-1}C_T(C_T^*C_T^n) \\
&= (C_T^nC_T^*)C_TC_T^{n-1} \\
&= C_T(C_T^*C_T^n)C_T^{n-1} \\
&= C_T(C_T^*C_T^{n+(n-1)}).
\end{aligned}$$

(Inductive step): Suppose the result is true for $n=k$.
To prove the result for $n=k+1$

$$\begin{aligned}
(C_T^{n+(k+1)(n-1)}C_T^*)C_T &= C_T^{n-1}(C_T^{n+k(n-1)}C_T^*)C_T \\
&= C_T^{n-1}C_T(C_T^*C_T^{n+k(n-1)}) \\
&= (C_T^nC_T^*)C_TC_T^{n+k(n-1)-1} \\
&= C_T(C_T^*C_T^n)C_T^{(n-1)(k+1)} \\
&= C_T(C_T^*C_T^{n+(k+1)(n-1)}).
\end{aligned}$$

Therefore C_T is skew $n+k(n-1)$-normal operator. □

Theorem 12. *Every n-normal composition operator is skew n-normal composition operator.*

Proof. Let C_T be n-normal operator therefore $(C_T^n C_T^{\star}) = (C_T^{\star} C_T^n)$ and since every n-normal operator is quasi n-normal operator therefore $C_T(C_T^{\star} C_T^n) = (C_T^{\star} C_T^n)C_T$. Then

$$C_T(C_T^{\star} C_T^n) = (C_T^{\star} C_T^n)C_T = (C_T^n C_T^{\star})C_T.$$

Thus C_T is skew n-normal operator. □

Theorem 13. *Every quasi normal composition operator is skew n-normal composition operator.*

Proof. Suppose that C_T is quasi normal composition operator. Then

$$C_T(C_T^{\star} C_T) = (C_T^{\star} C_T)C_T$$

Now

$$\begin{aligned} C_T^2(C_T^{\star} C_T) &= C_T[(C_T^{\star} C_T)C_T] \\ &= C_T(C_T^{\star} C_T)C_T \\ &= (C_T^{\star} C_T)C_T^2. \end{aligned}$$

Similarly C_T^{n-1} commutes with $C_T^{\star} C_T$ for every n, so that

$$(C_T^n C_T^{\star})C_T = C_T C_T^{n-1}(C_T^{\star} C_T) = C_T(C_T^{\star} C_T)C_T^{n-1} = C_T C_T^{\star} C_T^n.$$

Thus C_T is skew n-normal composition operator. □

3. skew n-Normal Weighted Composition Operator

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $W = W_{\phi,T}$ be the weighted composition operator on $L^2(\mu)$ induced by the complex valued function ϕ and a measurable transformation T. The adjoint $W^{\star}$ of W is given by $W^{\star}f = hE(\phi f) \circ T^{-1}$ for $f \in L^2(\mu)$. We put $\phi_n = \phi.(\phi \circ T).(\phi \circ T^2).......(\phi \circ T^{n-1})$ where n be any natural number. For $f \in L^2(\mu), W^n f = \phi_n.f \circ T^n$ and $W^{\star n} f = h_n.E(\phi_n.f) \circ T^{-n}$.

Theorem 14. *Let W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) W is skew n-normal operator.

(ii) $\phi_n.h \circ T^n E(\phi^2) = \phi.h \circ TE(\phi\phi_n)$.

Proof. For $f \in L^2(\mu)$

$$\begin{aligned}
(W^nW^*)Wf &= (W^nW^*)(\phi.f \circ T) \\
&= W^n(h.E(\phi^2.f \circ T) \circ T^{-1}) \\
&= \phi_n(h.E(\phi^2 f) \circ T^n \\
&= \phi_n h \circ T^n E(\phi^2) f \circ T^n.
\end{aligned}$$

Also,

$$\begin{aligned}
W(W^*W^n)f &= WW^*(\phi_n \circ T^n) \\
&= W(hE(\phi\phi_n.f \circ T^n) \circ T^{-1}) \\
&= \phi(h.E(\phi\phi_n)f \circ T^{n-1}) \circ T \\
&= \phi h \circ TE(\phi\phi_n)f \circ T^n.
\end{aligned}$$

Suppose that W is a skew n-normal operator then

$$(W^nW^*)Wf = W(W^*W^n)f \Longleftrightarrow \phi_n.h \circ T^n E(\phi^2) = \phi.h \circ TE(\phi\phi_n).$$

□

Theorem 15. *Let W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) W^* is skew n-normal operator.

(ii) $hE(\phi^2 h_n E(\phi_n f)) = h_n E(\phi_n \phi h \circ T^{-n+1} E(\phi f))$.

Proof.

$$\begin{aligned}
W^*(WW^{*n})f &= W^*W(h_nE(\phi_n f) \circ T^{-n}) \\
&= W^*\phi(h_nE(\phi_n f) \circ T^{-n}) \circ T \\
&= hE(\phi.\phi h_n \circ TE(\phi_n f) \circ T^{-n+1}) \circ T^{-1} \\
&= hE(\phi^2 h_n E(\phi_n f)) \circ T^{-n}.
\end{aligned}$$

Also,

$$\begin{aligned}
(W^{*n}W)W^*f &= (W^{*n}W)(hE(\phi f) \circ T^{-1}) \\
&= W^{*n}(\phi.h \circ T)E(\phi f) \\
&= h_nE(\phi_n\phi.h \circ TE(\phi f)) \circ T^{-n} \\
&= h_nE(\phi_n\phi h \circ T^{-n+1}E(\phi f) \circ T^{-n}).
\end{aligned}$$

If W^* is a skew n-normal operator. Then

$$W^{*}(WW^{*n}) = (W^{*n}W)W^{*} \Longleftrightarrow hE(\phi^2 h_n E(\phi_n f)) = h_n E(\phi_n \phi h \circ T^{-n+1} E(\phi f)).$$

□

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