A NOTE ON INNER AUTOMORPHISM ON BANACH ALGEBRAS

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Abstract: In this paper we prove that every unital Banach algebra $A$ is commutative if and only if $\text{Inn}(A) = \{I\}$, where $I$ is the identity map on $A$. Some related result are given as well.

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1. Introduction

Let $A$ be a Banach algebra. It is well-known, on the second dual space $A''$ of $A$, there are two multiplications, called the first and second Arens products which make $A''$ into a Banach algebra (see [1], [3]). By definition, the first Arens product □ on $A''$ is induced by the left $A$-module structure on $A$. That is, for each $\Phi, \Psi \in A''$, $f \in A'$ and $a, b \in A$, we have

$$\langle \Phi \Box \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle, \quad \langle \Psi \cdot f, a \rangle = \langle \Psi, f \cdot a \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle.$$ 

Similarly, the second Arens product $\Diamond$ on $A''$ is defined by considering $A$ as a right $A$-module. The Banach algebra $A$ is said to be Arens regular if $\Phi \Box \Psi = \ldots$
Φ♦Ψ on the whole of $A''$. For example, each $C^*$-algebra is Arens regular and for locally compact group $G$, the group algebra $L^1(G)$ is Arens regular if and only if $G$ is finite [7]. Some results about the Arens regularity of Banach algebras obtained in [6]. We denote the canonical embedding of $A$ into $A''$ by $k$, so

$$\langle k(a), \lambda \rangle = \langle \hat{a}, \lambda \rangle = \langle \lambda, a \rangle,$$

$(a \in A, \lambda \in A')$.

Clearly, $a \cdot \Phi = a \square \Phi$ and $\Phi \cdot a = \Phi \square a$, for all $a \in A$ and $\Phi \in A''$.

An automorphism on Banach algebra $A$ is an isomorphism of $A$ onto $A$. The set of automorphisms of $A$ is denoted $Aut(A)$. Note that $Aut(A)$ is a Banach algebra under function composition.

Let $A$ be a unital Banach algebra and let $a \in Inv(A)$, where $Inv(A)$ is the set of all invertible elements of $A$. Let $f_a : A \rightarrow A$ be a mapping defined by $f_a(x) = a^{-1}xa$, for all $x \in A$. Clearly, $f_a$ is an automorphism on $A$. Each such automorphism is called an inner automorphism on $A$. We denote by $Inn(A)$, the set of all inner automorphism on $A$.

A bounded net $(e_\alpha)_{\alpha \in I}$ in $A$ is a bounded approximate identity (BAI for short) if, for each $a \in A$, $ae_\alpha \rightarrow a$ and $e_\alpha a \rightarrow a$.

In this note for every unital Banach algebra, we show that the identity map $I$ is the only inner automorphism on $A$ if and only if $A$ is commutative. Also we investigate some results about the inner automorphism on Arens regular Banach algebras and $C^*$-algebras.

The proof of the following result contained in [7].

**Theorem 1.** Let $G$ be an infinite locally compact group. Then neither $L^1(G)$ nor $M(G)$ is Arens regular.

2. Main Results

**Proposition 2.** Let $A$ be a unital Banach algebra. Then $\varphi : A \rightarrow Aut(A)$ given by $\varphi(a) = f_a$ is an anti-homomorphism onto $Inn(A)$.

**Proof.** Obviously, $\varphi$ maps onto $Inn(A)$. Let $a, b \in A$, then for all $x \in A$,

$$(\varphi(a)\varphi(b))(x) = (f_a \circ f_b)(x) = (ba)^{-1}x(ba) = f_{ba}(x) = \varphi(ba)(x).$$

Thus, $\varphi(ba) = \varphi(a)\varphi(b)$. So, $\varphi$ is a anti-homomorphism.
**Theorem 3.** Let \( A \) be a unital Banach algebra. Then 
\[
A = \text{lin}(\text{Inv}(A)).
\]

**Proof.** Let \( e \) be a unit element of \( A \), and let \( a \in A \) such that \( \|a\| < 1 \). Then by Theorem 1.2.2 of [5], \( e - a \in \text{Inv}(A) \), and so the equality \( a = -(e - a) + e \) implies that \( a \in \text{Inv}(A) \). Therefore for all \( a \in A \) with \( \|a\| < 1 \), \( A = \text{lin}(\text{Inv}(A)) \). Now let \( \|a\| \geq 1 \). Put \( b = \frac{a}{2\|a\|} \), then \( \|b\| < 1 \). Then by the above argument \( b \in \text{Inv}(A) \), which imply that \( a \) is invertible. Therefore \( A = \text{lin}(\text{Inv}(A)) \), for all \( a \in A \). This complete the proof. \( \square \)

Since the second dual of every Arens regular Banach algebra with BAI, is unital [2], so we get the following result.

**Corollary 4.** Let \( A \) be an Arens regular Banach algebra with BAI. Then 
\[
A'' = \text{lin}(\text{Inv}(A'')).
\]

The next result is a characterization of inner automorphism.

**Theorem 5.** Let \( A \) be a unital Banach algebra. Then \( A \) is commutative if and only if \( \text{Inn}(A) = \{I\} \), where \( I \) is the identity map on \( A \).

**Proof.** Suppose that \( A \) is commutative and let \( f_a \) be a arbitrary inner automorphism on \( A \). Then for all \( x \in A \),
\[
f_a(x) = a^{-1}xa = x = I(x).
\]
Thus, \( \text{Inn}(A) = \{I\} \), where \( I \) is the identity map on \( A \).

For the converse, suppose that \( I \) is the only inner automorphism on \( A \), so there exist \( a \in \text{Inv}(A) \) such that \( I = f_a \). Then for all \( x \in A \),
\[
x = I(x) = f_a(x) = a^{-1}xa.
\]
Thus, \( ax = xa \) for all \( x \in A \), and \( a \in \text{Inv}(A) \). Now let \( y \in A \) be arbitrary. Then by Theorem 3,
\[
y = \sum_{i=1}^{n} \lambda_i a_i,
\]
where \( a_i \) belongs in \( \text{Inv}(A) \). Therefore
\[
xy = x(\sum_{i=1}^{n} \lambda_i a_i) = (\sum_{i=1}^{n} \lambda_i a_i)x = yx.
\]
Thus, \( A \) is commutative. \( \square \)
Examples 6. a) Suppose that
\[ A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{C} \right\}. \]
Then \( A \) is unital and commutative Banach algebra, hence by above Theorem the identity map \( I \) is the only inner automorphism on \( A \), that is \( Inn(A) = \{ I \} \).
b) Let \( G \) be a locally compact abelian group and \( A = L^1(G) \) be its group algebra. Suppose that \( G \) is discrete, then \( Inn(A) = \{ I \} \).

As an immediate corollary we have the following.

Corollary 7. Let \( A \) be an Arens regular Banach algebra with a BAI. Then the following statement are equivalent.

1. \( A \) is commutative.
2. \( A'' \) is commutative.
3. \( Inn(A'') = \{ I'' \} \), where \( I'' \) is the identity map on \( A'' \).

Moreover, if \( A \) is unital, then (1), (2) and (3) are equivalent with

4. \( Inn(A) = \{ I \} \), where \( I \) is the identity map on \( A \).

Since every \( C^* \)-algebra is Arens regular and has a BAI, so we have the next result.

Corollary 8. Let \( A \) be a commutative \( C^* \)-algebra. Then \( Inn(A'') = \{ I'' \} \), where \( I'' \) is the identity map on \( A'' \).

Let \( G \) be an infinite locally compact abelian group and let \( A = M(G) \), the Banach algebra of all bounded complex-valued regular Borel measures on \( G \). Then \( A \) is unital and commutative, so \( Inn(A) = \{ I \} \), where \( I \) is the identity map on \( A \). Note that \( A'' \) is unital but it is not commutative, therefore \( Inn(A'') \neq \{ I'' \} \), by Corollary 7. Thus, the Arens regularity of Banach algebra \( A \) in Corollary 7 is essential.

Theorem 9. Let \( A \) be a unital Banach algebra. Let \( p_1 \in Aut(A) \), and suppose that for every homomorphism \( T \) of \( A \) into a unital Banach algebra \( B \), there exists \( p_2 \in Aut(B) \) such that \( p_2 \circ T = T \circ p_1 \). Then

1. If \( T \) is surjective and \( p_1 \in Inn(A) \), then \( p_2 \in Inn(B) \).
2. If \( T \) is injective and \( p_2 \in Inn(B) \), then \( p_1 \in Inn(A) \).
Proof. We prove (1) that the assertion (2) can be proved similarly. Suppose that $T$ is surjective and $p_1 \in \text{Inn}(A)$. Then there exist $a \in \text{Inv}(A)$ such that $p_1 = f_a$. By assumption for every $y \in B$ there exist $x \in A$ such that $T(x) = y$. Hence

$$p_2(y) = p_2(T(x)) = T(p_1(x)) = T(a^{-1}xa) = T^{-1}(a)yT(a),$$

which proves that $p_2 \in \text{Inn}(B)$.

**Theorem 10.** Let $A$ be a unital Banach algebra and let $p \in \text{Aut}(A)$. Then $p \in \text{Inn}(A)$ if and only if $p'' \in \text{Inn}(A'')$, where $p''$ is the second adjoint of $p$.

Proof. Suppose that $p'' \in \text{Inn}(A'')$. Since the canonical embedding $k : A \rightarrow A''$ is monomorphism and $p'' \circ k = k \circ p$, hence by Theorem 9, $p \in \text{Inn}(A)$.

For the converse, let $p \in \text{Inn}(A)$, so there exist $a \in \text{Inv}(A)$ such that $p = f_a$. Let $\Phi \in A''$ be arbitrary. Then by Goldstine’s Theorem there exist bounded net $(x_\alpha) \subseteq A$ that $x_\alpha \rightarrow \Phi$ in the $w^*$-topology of $A''$. Then

$$p''(\Phi) = f''_a(\Phi) = f''_a(w^* - \lim_{\alpha} x_\alpha) = w^* - \lim_{\alpha} f_a(x_\alpha) = a^{-1} \cdot \Phi \cdot a.$$

Thus, $p'' \in \text{Inn}(A'')$.

We denote by $A^{op}$ the opposite algebra of $A$, so that $A^{op}$ is the same linear space as $A$, but the product is $\diamond$, where $a \diamond b = ba$.

**Proposition 11.** Let $A$ be a unital Banach algebra. Then $\text{Inn}(A) = \text{Inn}(A^{op})$.

Proof. Suppose that $p \in \text{Inn}(A)$, so there exist $a \in \text{Inv}(A)$ such that $p = f_a$. Then for all $x \in A$,

$$p(x) = f_a(x) = a^{-1}xa = a \circ x \circ a^{-1}.$$

Take $b = a^{-1}$. Then $p = f_b$, which proves that $p \in \text{Inn}(A^{op})$. The converse is similar.

Let $A = K(c_0)$, the operator algebra of all compact linear operators on the sequence space $c_0$. Then by Example 2.5 of [4] $A$ has a BAI, $(A'', \Box)$ is unital but $(A'', \Diamond)$ is not unital. Note that $A$ false to be Arens regular. However, $\text{Inn}(A'', \Box) \neq \text{Inn}(A'', \Diamond)$.

**Corollary 12.** Suppose that $A$ is a unital Banach algebra which is commutative. Then $\text{Inn}(A'', \Box) = \text{Inn}(A'', \Diamond)$. 
Proof. Since \( \mathcal{A} \) is commutative, we have \( (\mathcal{A}''', \Diamond) = (\mathcal{A}'', \Box)^{\text{op}} \). Thus, the result follows from Proposition 11. \( \square \)

References