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A NOTE ON INNER AUTOMOROPHISM ON BANACH ALGEBRAS

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Abstract: In this paper we prove that every unital Banach algebra $\mathcal{A}$ is commutative if and only if $Inn(\mathcal{A}) = \{\mathcal{I}\}$, where $\mathcal{I}$ is the identity map on $\mathcal{A}$. Some related result are given as well.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra. It is well-known, on the second dual space $\mathcal{A}''$ of $\mathcal{A}$, there are two multiplications, called the first and second Arens products which make $\mathcal{A}''$ into a Banach algebra (see [1], [3]). By definition, the first Arens product $\Box$ on $\mathcal{A}''$ is induced by the left $\mathcal{A}$-module structure on $\mathcal{A}$. That is, for each $\Phi, \Psi \in \mathcal{A}''$, $f \in \mathcal{A}'$ and $a, b \in \mathcal{A}$, we have

$$\langle \Phi \Box \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle, \quad \langle \Psi \cdot f, a \rangle = \langle \Psi, f \cdot a \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle.$$

Similarly, the second Arens product $\Diamond$ on $\mathcal{A}''$ is defined by considering $\mathcal{A}$ as a right $\mathcal{A}$-module. The Banach algebra $\mathcal{A}$ is said to be Arens regular if $\Phi \Box \Psi =$

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$\Phi\Diamond\Psi$ on the whole of $\mathcal{A}''$. For example, each C^*-algebra is Arens regular and for locally compact group G, the group algebra $L^1(G)$ is Arens regular if and only if G is finite [7]. Some results about the Arens regularity of Banach algebras obtained in [6]. We denote the canonical embedding of $\mathcal{A}$ into $\mathcal{A}''$ by k, so

$$\langle k(a), \lambda\rangle = \langle \widehat{a}, \lambda\rangle = \langle \lambda, a\rangle, \qquad (a \in \mathcal{A}, \quad \lambda \in \mathcal{A}').$$

Clearly, $a \cdot \Phi = a\Box\Phi$ and $\Phi \cdot a = \Phi\Box a$, for all $a \in \mathcal{A}$ and $\Phi \in \mathcal{A}''$.

An automorphism on Banach algebra $\mathcal{A}$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{A}$. The set of automorphisms of $\mathcal{A}$ is denoted $Aut(\mathcal{A})$. Note that $Aut(\mathcal{A})$ is a Banach algebra under function composition.

Let $\mathcal{A}$ be a unital Banach algebra and let $a \in Inv(\mathcal{A})$, where $Inv(\mathcal{A})$ is the set of all invertible elements of $\mathcal{A}$. Let $f_a : \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping defined by $f_a(x) = a^{-1}xa$, for all $x \in \mathcal{A}$. Clearly, f_a is an automorphism on $\mathcal{A}$. Each such automorphism is called an inner automorphism on $\mathcal{A}$. We denote by $Inn(\mathcal{A})$, the set of all inner automorphism on $\mathcal{A}$.

A bounded net $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ is a bounded approximate identity (BAI for short) if, for each $a \in \mathcal{A}$, $ae_\alpha \longrightarrow a$ and $e_\alpha a \longrightarrow a$.

In this note for every unital Banach algebra, we show that the identity map $\mathcal{I}$ is the only inner automorphism on $\mathcal{A}$ if and only if $\mathcal{A}$ is commutative. Also we investigate some results about the inner automorphism on Arens regular Banach algebras and C^*-algebras.

The proof of the following result contained in [7].

Theorem 1. *Let G be an infinite locally compact group. Then neither $L^1(G)$ nor $M(G)$ is Arens regular.*

2. Main Results

Proposition 2. *Let $\mathcal{A}$ be a unital Banach algebra. Then $\varphi : \mathcal{A} \longrightarrow Aut(\mathcal{A})$ given by $\varphi(a) = f_a$ is an anti-homomorphism onto $Inn(\mathcal{A})$.*

Proof. Obviously, φ maps onto $Inn(\mathcal{A})$. Let $a, b \in \mathcal{A}$, then for all $x \in \mathcal{A}$,

$$(\varphi(a)\varphi(b))(x) = (f_a \circ f_b)(x) = (ba)^{-1}x(ba) = f_{ba}(x) = \varphi(ba)(x).$$

Thus, $\varphi(ba) = \varphi(a)\varphi(b)$. So, φ is a anti-homomorphism. $\Box$

Theorem 3. *Let $\mathcal{A}$ be a unital Banach algebra. Then*

$$\mathcal{A} = lin(Inv(\mathcal{A})).$$

Proof. Let e be a unit element of $\mathcal{A}$, and let $a \in \mathcal{A}$ such that $\|a\| < 1$. Then by Theorem 1.2.2 of [5], $e - a \in Inv(\mathcal{A})$, and so the equality $a = -(e - a) + e$ implies that $a \in Inv(\mathcal{A})$. Therefore for all $a \in \mathcal{A}$ with $\|a\| < 1$, $\mathcal{A} = lin(Inv(\mathcal{A}))$. Now let $\|a\| \geq 1$. Put $b = \frac{a}{2\|a\|}$, then $\|b\| < 1$. Then by the above argument $b \in Inv(\mathcal{A})$, which imply that a is invertible. Therefore $\mathcal{A} = lin(Inv(\mathcal{A}))$, for all $a \in \mathcal{A}$. This complete the proof. □

Since the second dual of every Arens regular Banach algebra with BAI, is unital [2], so we get the following result.

Corollary 4. *Let $\mathcal{A}$ be an Arens regular Banach algebra with BAI. Then*

$$\mathcal{A}'' = lin(Inv(\mathcal{A}'')).$$

The next result is a characterization of inner automorphism.

Theorem 5. *Let $\mathcal{A}$ be a unital Banach algebra. Then $\mathcal{A}$ is commutative if and only if $Inn(\mathcal{A}) = \{\mathcal{I}\}$, where $\mathcal{I}$ is the identity map on $\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ is commutative and let f_a be a arbitrary inner automorphism on $\mathcal{A}$. Then for all $x \in \mathcal{A}$,

$$f_a(x) = a^{-1}xa = x = \mathcal{I}(x).$$

Thus, $Inn(\mathcal{A}) = \{\mathcal{I}\}$, where $\mathcal{I}$ is the identity map on $\mathcal{A}$.

For the converse, suppose that $\mathcal{I}$ is the only inner automorphism on $\mathcal{A}$, so there exist $a \in Inv(\mathcal{A})$ such that $\mathcal{I} = f_a$. Then for all $x \in \mathcal{A}$,

$$x = \mathcal{I}(x) = f_a(x) = a^{-1}xa.$$

Thus, $ax = xa$ for all $x \in \mathcal{A}$, and $a \in Inv(\mathcal{A})$. Now let $y \in \mathcal{A}$ be arbitrary. Then by Theorem 3,

$$y = \sum_{i=1}^{n} \lambda_i a_i,$$

where a_i belongs in $Inv(\mathcal{A})$. Therefore

$$xy = x(\sum_{i=1}^{n} \lambda_i a_i) = (\sum_{i=1}^{n} \lambda_i a_i)x = yx.$$

Thus, $\mathcal{A}$ is commutative. □

Examples 6. a) Suppose that

$$\mathcal{A} = \Big\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \quad a, b \in \mathbb{C} \Big\}.$$

Then $\mathcal{A}$ is unital and commutative Banach algebra, hence by above Theorem the identity map $\mathcal{I}$ is the only inner automorphism on $\mathcal{A}$, that is $Inn(\mathcal{A}) = \{\mathcal{I}\}$.
b) Let G be a locally compact abelian group and $\mathcal{A} = L^1(G)$ be its group algebra. Suppose that G is discrete, then $Inn(\mathcal{A}) = \{\mathcal{I}\}$.

As an immediate corollary we have the following.

Corollary 7. *Let $\mathcal{A}$ be an Arens regular Banach algebra with a BAI. Then the following statement are equivalent.*

1. *$\mathcal{A}$ is commutative.*
2. *$\mathcal{A}''$ is commutative.*
3. *$Inn(\mathcal{A}'') = \{\mathcal{I}''\}$, where $\mathcal{I}''$ is the identity map on $\mathcal{A}''$.*

Moreover, if $\mathcal{A}$ is unital, then (1), (2) and (3) are equivalent with

4. *$Inn(\mathcal{A}) = \{\mathcal{I}\}$, where $\mathcal{I}$ is the identity map on $\mathcal{A}$.*

Since every C^*-algebra is Arens regular and has a BAI, so we have the next result.

Corollary 8. *Let $\mathcal{A}$ be a commutative C^*-algebra. Then $Inn(\mathcal{A}'') = \{\mathcal{I}''\}$, where $\mathcal{I}''$ is the identity map on $\mathcal{A}''$.*

Let G be an infinite locally compact abelian group and let $\mathcal{A} = M(G)$, the Banach algebra of all bounded complex-valued regular Borel measures on G. Then $\mathcal{A}$ is unital and commutative, so $Inn(\mathcal{A}) = \{\mathcal{I}\}$, where $\mathcal{I}$ is the identity map on $\mathcal{A}$. Note that $\mathcal{A}''$ is unital but it is not commutative, therefore $Inn(\mathcal{A}'') \neq \{\mathcal{I}''\}$, by Corollary 7. Thus, the Arens regularity of Banach algebra $\mathcal{A}$ in Corollary 7 is essential.

Theorem 9. *Let $\mathcal{A}$ be a unital Banach algebra. Let $p_1 \in Aut(\mathcal{A})$, and suppose that for every homomorphism T of $\mathcal{A}$ into a unital Banach algebra $\mathcal{B}$, there exists $p_2 \in Aut(\mathcal{B})$ such that $p_2 \circ T = T \circ p_1$. Then*

1. *If T is surjective and $p_1 \in Inn(\mathcal{A})$, then $p_2 \in Inn(\mathcal{B})$.*
2. *If T is injective and $p_2 \in Inn(\mathcal{B})$, then $p_1 \in Inn(\mathcal{A})$.*

Proof. We prove (1) that the assertion (2) can be proved similarly. Suppose that T is surjective and $p_1 \in Inn(\mathcal{A})$. Then there exist $a \in Inv(\mathcal{A})$ such that $p_1 = f_a$. By assumption for every $y \in \mathcal{B}$ there exist $x \in \mathcal{A}$ such that $T(x) = y$. Hence

$$p_2(y) = p_2(T(x)) = T(p_1(x)) = T(a^{-1}xa) = T^{-1}(a)yT(a),$$

which proves that $p_2 \in Inn(\mathcal{B})$. □

Theorem 10. *Let $\mathcal{A}$ be a unital Banach algebra and let $p \in Aut(\mathcal{A})$. Then $p \in Inn(\mathcal{A})$ if and only if $p'' \in Inn(\mathcal{A}'')$, where p'' is the second adjoint of p.*

Proof. Suppose that $p'' \in Inn(\mathcal{A}'')$. Since the canonical embedding $k : \mathcal{A} \longrightarrow \mathcal{A}''$ is monomorphism and $p'' \circ k = k \circ p$, hence by Theorem 9, $p \in Inn(\mathcal{A})$.

For the converse, let $p \in Inn(\mathcal{A})$, so there exist $a \in Inv(\mathcal{A})$ such that $p = f_a$. Let $\Phi \in \mathcal{A}''$ be arbitrary. Then by Goldstine's Theorem there exist bounded net $(x_\alpha) \subseteq \mathcal{A}$ that $x_\alpha \longrightarrow \Phi$ in the w^*-topology of $\mathcal{A}''$. Then

$$p''(\Phi) = f''_{\widehat{a}}(\Phi) = f''_{\widehat{a}}(w^* - lim_\alpha x_\alpha) = w^* - lim_\alpha f_a(x_\alpha) = a^{-1} \cdot \Phi \cdot a.$$

Thus, $p'' \in Inn(\mathcal{A}'')$. □

We denote by $\mathcal{A}^{op}$ the opposite algebra of $\mathcal{A}$, so that $\mathcal{A}^{op}$ is the same linear space as $\mathcal{A}$, but the product is $\diamond$, where $a \diamond b = ba$.

Proposition 11. *Let $\mathcal{A}$ be a unital Banach algebra. Then $Inn(\mathcal{A}) = Inn(\mathcal{A}^{op})$.*

Proof. Suppose that $p \in Inn(\mathcal{A})$, so there exist $a \in Inv(\mathcal{A})$ such that $p = f_a$. Then for all $x \in \mathcal{A}$,

$$p(x) = f_a(x) = a^{-1}xa = a \diamond x \diamond a^{-1}.$$

Take $b = a^{-1}$. Then $p = f_b$, which proves that $p \in Inn(\mathcal{A}^{op})$. The converse is similar. □

Let $\mathcal{A} = K(c_0)$, the operator algebra of all compact linear operators on the sequence space c_0. Then by Example 2.5 of [4] $\mathcal{A}$ has a BAI, $(\mathcal{A}'', \Box)$ is unital but $(\mathcal{A}'', \Diamond)$ is not unital. Note that $\mathcal{A}$ false to be Arens regular. However, $Inn(\mathcal{A}'', \Box) \neq Inn(\mathcal{A}'', \Diamond)$.

Corollary 12. *Suppose that $\mathcal{A}$ is a unital Banach algebra which is commutative. Then $Inn(\mathcal{A}'', \Box) = Inn(\mathcal{A}'', \Diamond)$.*

Proof. Since $\mathcal{A}$ is commutative, we have $(\mathcal{A}'', \diamondsuit) = (\mathcal{A}'', \Box)^{op}$. Thus, the result follows from Proposition 11. $\Box$

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