GRACEFUL LABELING OF HANGING MODIFIED THETA GRAPHS

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Abstract: Hanging modified theta graphs are obtained from attaching a modified theta graph to each teeth of comb graph $P_{a_i} \odot L_1$, where P_{a_i} denotes a path on a_i vertices. In this paper it is proved that symmetric hanging modified theta graphs and non-decreasing symmetric hanging modified theta graphs are graceful.

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Key Words: graceful labeling, theta graph, modified theta graph

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1. Introduction

Graphs considered in this paper are simple finite and undirected. In general $G(V, E)$ denotes the graph G with vertex set $V(G)$ and edge set $E(G)$, such that $|V(G)| = p$, $|E(G)| = q$. A labeling of the vertices of G with the numbers from 0 to q is an injective map $\phi: V \rightarrow \{0, 1, \dots, q\}$. Let $\phi: E \rightarrow \{1, 2, \dots, q\}$ be a bijective. Then a graph G is graceful if there exists a labeling of its vertices and edges with the condition for all $u, v \in V(G)$ with $uv \in E(G)$ such that $\phi(uv) = |\phi(u) - \phi(v)|$.

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The concept of graceful labeling was introduced Rosa [2] with the name β **valuation**. Gallian [1] has given the extensive survey of contributions to graceful labeling of variety of graphs.

Definition 1. Theta graph: A theta graph is the union of three internally disjoint paths of length 2 between two vertices.

Definition 2. modified theta graph: A modified theta graph consisting of n internal disjoint paths, in which each path contains m vertices and path end vertices are merged with the vertices v_1 and v_2 respectively.

2. Main Result

Hanging modified theta graphs are results from attaching a modified theta graph to each teeth of comb graph $P_{a_i} \odot L_1$, where P_{a_i} denotes a path on a_i vertices. Designate the j^{th} modified theta graph hanging from b_j as $H\theta(n_j, m_j)$ and each $H\theta(n_j, m_j)$ has $(m_j - 1)n_j$ edges. Hanging modified theta graphs satisfy $n \geq 3$, $m = 2j + 1$ and $j = 1, 2, \dots, n$. For $j = 1, 2$ n may be odd or even and for $j = 3, 4, \dots, n$, n must be odd.

If the internally disjoint paths are not of same length then

$$\theta_i = \theta(n_i, m_1^{l_1(i)}, m_2^{l_2(i)}, \dots, m_k^{l_k(i)}),$$

where $l_k(i)$ denotes number of internally disjoint paths and (m_i) denotes paths of length $m_i - 1$ for $i = 1, 2, \dots, k$, joining v_1 and v_2 and $l_1(i) + l_2(i) + \dots + l_k(i) = n_i$. The resultant hanging theta graph denoted by $H(\theta_1, \theta_1, \dots, \theta_n)$ and if all $m_j^{l_j(i)}$ are same then it becomes $H\theta(n_j, m_j)$.

Let $n_1, n_2, \dots, n_n$ be the number of paths contained in each hanging modified theta graph respectively. Let $q_1, q_2, \dots, q_{a_i}$ ($q_i = n_i(m_i - 1) + 1$, $1 \leq i \leq a_i$) be the number of edges of each hanging modified theta graph from $b_1, b_2, \dots, b_a$ respectively. This results $q = \sum_{d=1}^{a_i} q_d + (a_i - 1)$. The path vertices in modified theta graph are labeled in the form of $(c_{(ba_i)(pa_i)}^{le})$ where (ba_i) denotes base vertex, (pa_i) denotes internal path index of theta graph and (le) level index.

Condition for **symmetric hanging modified theta graphs** are all hanging modified theta graphs are isomorphic or at least even number of them must be consecutive.

Condition for **non-decreasing symmetric hanging modified theta graphs** are 1. The number of edges in j^{th} component and $(j + 1)^{th}$ component

of $H\theta(n, m)$ must be same, for $j = 1, 2, \dots, n-1$.

2. The number of paths in each modified theta graph are factors or multiples of factors in relation with the number of edges. Theta's are non-decreasing symmetric denoted as $H\theta((n_1, m_1), (n_2, m_2), \dots, (n_k, m_k))$.

Let $E(A)$ be the collection of base edge($\phi(b_k b_{k+1})$) values, $E(B)$ be the collection supporting edge($\phi(b_d t_d)$) values for $1 \leq k \leq a_i - 1, 1 \leq d \leq a_i$. Let $E(D)$ be the collection of first part of modified theta graph edge($\phi(t_d c_{(d)(i)}^d)$) values, $E(F)$ be the collection of intermediary edge($\phi(c_{(d)(n_{(d-i)})}^j c_{(d)(n_{d-i})}^{j+1})$) values and $E(H)$ be the collection of last part of modified theta graph edge($\phi(v_d c_{(d)(i)}^{m_d})$) values, for $1 \leq d \leq a_i, 1 \leq i \leq n_d, 1 \leq j \leq m_d - 3$. Now, $E(G) = \{E(A) \cup E(B) \cup E(D) \cup E(F) \cup E(H)\}$

The hanging modified theta graphs is shown in the following Figure 1.

Figure 1: The general form of symmetric or non-decreasing symmetric hanging modified theta graphs

Theorem: $H\theta(n_j, m_j), j = 1, 2, \dots, n$ is graceful.

Proof. Labeling of vertices in first hanging modified theta graph is as follows :-

Step 1 : $\phi(b_1) = 0, \phi(t_1) = q, \phi(v_1) = \phi(t_1) - \frac{q_1-1}{2}$.

Step 2 : $\phi(c_{(1)(1)}^1) = 1, \phi(c_{(1)(i)}^1) = i, 2 \leq i \leq n_1$.

Step 3 : $\phi(c_{(1)(n_1)}^2) = q - 1, \phi(c_{(1)(n_1-i)}^2) = \phi(c_{(1)(n_1+1-i)}^2) - 2, 1 \leq i \leq n_1 - 1$.

Step 4 : $\phi (c_{(1)(1)}^{2j+1}) = 1 + n_1j, 1 \leq j \leq \frac{m_1-3}{2}.$

Step 5 : $\phi (c_{(1)(i)}^{2j+1}) = \phi (c_{(1)(i-1)}^{2j+1}) + 1, 2 \leq i \leq n_1, 1 \leq j \leq \frac{m_1-3}{2}.$

Step 6 : $\phi (c_{(1)(n)}^{2j+2}) = q - 1 - n_1j, 1 \leq j \leq \frac{m_1-5}{2}.$

Step 7 : $\phi (c_{(1)(n-i)}^{2j+2}) = \phi (c_{(1)(n+1-i)}^{2j+2}) - 2, 1 \leq i \leq n_1-1, 1 \leq j \leq \frac{m_1-5}{2}.$

Now, labeling of vertices in second hanging modified theta graph has given as under.

Step 1 : $\phi (b_2) = q - (q_1 + 1), \phi (t_2) = q_1 + 1. \phi (v_2) = \phi (t_2) - \frac{q_2-1}{2}.$

Step 2 : $\phi (c_{(2)(1)}^1) = \phi (b_2) + 1, \phi (c_{(2)(i)}^1) = \phi (b_2) + i, 2 \leq i \leq n_2.$

Step 3 : $\phi (c_{(2)(n_2)}^2) = \phi (t_2) - 1. \phi (c_{(2)(n_2-i)}^2) = \phi (c_{(2)(n_2+1-i)}^2) - 2, 1 \leq i \leq n_2 - 1.$

Step 4 : $\phi (c_{(2)(1)}^{2j+1}) = \phi (c_{(2)(1)}^1) + n_2j, 1 \leq j \leq \frac{m_2-3}{2}.$

Step 5 : $\phi (c_{(2)(i)}^{2j+1}) = \phi (c_{(2)(i-1)}^{2j+1}) + 1, 2 \leq i \leq n_2, 1 \leq j \leq \frac{m_2-3}{2}.$

Step 6 : $\phi (c_{(2)(n)}^{2j+2}) = \phi (c_{(2)(n_2)}^2) - n_2j, 1 \leq j \leq \frac{m_2-5}{2}.$

Step 7 : $\phi (c_{(2)(n-i)}^{2j+2}) = \phi (c_{(2)(n+1-i)}^{2j+2}) - 2, 1 \leq i \leq n_2-1, 1 \leq j \leq \frac{m_2-5}{2}.$

(Let $a_i = 2\lambda + 1$ for odd and $a_i = 2\lambda$ for even)

Labeling of vertices in odd segments of hanging modified theta graphs are as follows :-

Step 1 : $\phi (b_{2d+1}) = \phi (b_{2d-1}) + q_{2d} + 1, 1 \leq d \leq \lambda, \text{ for } n \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } n \text{ is even.}$

$\phi (t_{2d+1}) = q - \phi (b_{2d+1}), 1 \leq d \leq \lambda, \text{ for } n \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } n \text{ is even.}$

$\phi (v_{2d+1}) = \phi (t_{2d+1}) - \frac{q_{2d+1}-1}{2}, 1 \leq d \leq \lambda, \text{ for } n \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } n \text{ is even.}$

Step 2 : $\phi (c_{(2d+1)(1)}^1) = \phi (b_{2d+1}) + 1. \phi (c_{(2d+1)(i)}^1) = \phi (b_{2d+1}) + i, 2 \leq i \leq n_{2d+1}, 1 \leq d \leq \lambda, \text{ for } n \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } n \text{ is even.}$

Step 3 : $\phi (c_{(2d+1)(n_{2d+1})}^2) = \phi (t_{2d+1}) - 1. \phi (c_{(2d+1)(n_{2d+1}-i)}^2) = \phi (c_{(2d+1)(n_{2d+1}+1-i)}^2) - 2, 1 \leq i \leq n_{2d+1} - 1, 1 \leq d \leq \lambda, \text{ for } n \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } n \text{ is even.}$

Step 4 : $\phi (c_{(2d+1)(1)}^{2j+1}) = \phi (c_{(2d+1)(1)}^1) + n_{2d+1} j$, $1 \leq j \leq \frac{m_{2d+1}-3}{2}$, $1 \leq d \leq \lambda$,
for n is odd, $1 \leq d \leq \lambda - 1$, for n is even.

Step 5 : $\phi (c_{(2d+1)(i)}^{2j+1}) = \phi (c_{(2d+1)(i-1)}^{2j+1}) + 1$, $2 \leq i \leq n_{2d+1}$, $1 \leq j \leq \frac{m_{2d+1}-3}{2}$,
 $1 \leq d \leq \lambda$, for n is odd, $1 \leq d \leq \lambda - 1$, for n is even.

Step 6 : $\phi (c_{(2d+1)(n)}^{2j+2}) = \phi (t_{2d+1}) - n_{2d+1} j$, $1 \leq j \leq \frac{m_{2d+1}-5}{2}$, $1 \leq d \leq \lambda$, for n
is odd, $1 \leq d \leq \lambda - 1$, for n is even.

Step 7 : $\phi (c_{(2d+1)(n-i)}^{2j+2}) = \phi (c_{(2d+1)(n+1-i)}^{2j+2}) - 2$, $1 \leq i \leq n_{2d+1} - 1$, $1 \leq j \leq$
 $\frac{m_{2d+1}-5}{2}$, $1 \leq d \leq \lambda$, for n is odd, $1 \leq d \leq \lambda - 1$, for n is even.

The labeling of vertices in even segments of hanging modified theta graphs
has given as under.

Step 1 : $\phi (b_{2d+2}) = \phi (b_{2d}) - (q_{2d+1} + 1)$. $\phi (t_{2d+2}) = \phi (t_{2d}) + q_{2d+1} + 1$.
 $\phi (v_{2d+2}) = \phi (t_{2d+2}) - \frac{q_{2d+2}-1}{2}$, $1 \leq d \leq \lambda - 1$.

Step 2 : $\phi (c_{(2d+2)(1)}^1) = \phi (b_{2d+2}) + 1$.

$\phi (c_{2d+2i}^1) = \phi (b_{2d+2}) + i$, $2 \leq i \leq n_{2d+2}$, $1 \leq d \leq \lambda - 1$.

Step 3 : $\phi (c_{(2d+2)(n_{2d+2})}^2) = \phi (t_{2d+2}) - 1$. $\phi (c_{(2d+2)(n_{2d+2}-i)}^2) =$
 $\phi (c_{(2d+2)(n_{2d+2}+1-i)}^2) - 2$, $1 \leq i \leq n_{2d+2} - 1$, $1 \leq d \leq \lambda - 1$.

Step 4 : $\phi (c_{(2d+2)(1)}^{2j+1}) = \phi (c_{(2d+2)(1)}^1) + n_{2d+2} j$, $1 \leq j \leq \frac{m_{2d+2}-3}{2}$, $1 \leq d \leq$
 $\lambda - 1$.

Step 5 : $\phi (c_{(2d+2)(i)}^{2j+1}) = \phi (c_{(2d+2)(i-1)}^{2j+1}) + 1$, $2 \leq i \leq n_{2d+2}$, $1 \leq j \leq \frac{m_{2d+2}-3}{2}$,
 $1 \leq d \leq \lambda - 1$.

Step 6 : $\phi (c_{(2d+2)(n)}^{2j+2}) = \phi (c_{(2d+2)(n_{2d+2})}^2) - n_{2d+2} j$, $1 \leq j \leq \frac{m_{2d+2}-5}{2}$, $1 \leq d \leq$
 $\lambda - 1$.

Step 7 : $\phi (c_{(2d+2)(n-i)}^{2j+2}) = \phi (c_{(2d+2)(n+1-i)}^{2j+2}) - 2$, $1 \leq i \leq n_{2d+2} - 1$, $1 \leq j \leq$
 $\frac{m_{2d+2}-5}{2}$, $1 \leq d \leq \lambda - 1$.

Now, induced labeling of edge values are as follows:-

1 : $\phi (b_1 t_1) = q$. **[2]:** $\phi (t_1 c_{(1)(i)}^1) = q - i$, $1 \leq i \leq n_1$.

3 : $\phi (c_{(1)(n+1-i)}^j c_{(1)(n+1-i)}^{j+1}) = q - j n_1 - i$, $1 \leq i \leq n_1$, $1 \leq j \leq m_1 - 3$.

4 : $\phi (v_1 c_{(1)(i)}^{m_1-2}) = q - (m_1 - 2) n_1 - i$, $1 \leq i \leq n_1$.

$$\mathbf{5} : \phi (b_1 b_2) = q - q_1. [\mathbf{6}]: \phi (v_2 c_{(2)(n_2+1-i)}^{m_2-2}) = q - q_1 - i, 1 \leq i \leq n_2.$$

$$\mathbf{7} : \phi (c_{(2)(i)}^{m_2-j} c_{(2)(i)}^{m_2-(j+1)}) = q - q_1 - (j - 2) n_2 - i, 1 \leq i \leq n_2, 2 \leq j \leq m_2 - 3.$$

$$\mathbf{8} : \phi (t_2 c_{(2)(i)}^1) = q - (q_1 + q_2) + n_2 + 1 - i, 1 \leq i \leq n_2.$$

$$\mathbf{9} : \phi (b_2 t_2) = q - (q_1 + q_2). [\mathbf{10}]: \phi (b_2 b_3) = q - (q_1 + q_2) - 1.$$

Odd indexed base vertices are termed as odd segments of $H\theta (n_j, m_j)$. Similarly, from the even indexed base vertices are termed as even segments of $H\theta (n_j, m_j)$.

Now, induced edge values of odd segments are as follows :-

$$\mathbf{11} : \phi (b_{2d} b_{2d+1}) = q - \sum_{k=1}^{2d} q_k - d, 1 \leq d \leq \lambda, \text{ for } a \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } a \text{ is even.}$$

$$\mathbf{12} : \phi (b_{2d+1} t_{2d+1}) = q - \sum_{k=1}^{2d} q_k - 2d, 1 \leq d \leq \lambda, \text{ for } a \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } a \text{ is even.}$$

$$\mathbf{13} : \phi (t_{2d+1} c_{(2d+1)(i)}^{2d+1}) = q - i - \sum_{k=1}^{2d} q_k - 2d, 1 \leq d \leq \lambda, \text{ for } a \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } a \text{ is even, } 1 \leq i \leq n_1.$$

$$\mathbf{14} : \phi (c_{(2d+1)((n_{2d+1})+1-i)}^j c_{(2d+1)((n_{2d+1})+1-i)}^{j+1}) = q - j n_{2d+1} - i - \sum_{k=1}^{2d} q_k - 2d, 1 \leq d \leq \lambda, \text{ for } a \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } a \text{ is even, } 1 \leq i \leq n_{2d+1}, 1 \leq j \leq m_{2d+1} - 3.$$

$$\mathbf{15} : \phi (v_{2d+1} c_{(2d+1)(i)}^{m_{2d+1}-2}) = q - (m_{2d+1} - 2) n_{2d+1} - \sum_{k=1}^{2d} q_k - 2d, 1 \leq d \leq \lambda, \text{ for } a \text{ is odd, } 1 \leq d \leq \lambda - 1, \text{ for } a \text{ is even, } 1 \leq i \leq n_{2d+1}.$$

$$\phi (b_{2d+1} b_{2d+2}) = q - \sum_{k=1}^{2d+1} q_k - 2d, 1 \leq d \leq \lambda - 1.$$

Induced edge values of even segments are as follows :-

$$\mathbf{16} : \phi (v_{2d+2} c_{(2d+2)(n_{2d+2}+1-i)}^{m_{2d+2}-2}) = q - \sum_{k=1}^{2d+1} q_k - 2d - i, 1 \leq d \leq \lambda - 1, 1 \leq i \leq n_{2d+2}.$$

$$\mathbf{17} : \phi \left(c_{(2d+2)(i)}^{m_{2d+2}-j} c_{(2d+2)(i)}^{m_{2d+2}-(j+1)} \right) = q - \sum_{k=1}^{2d+1} q_k - 2d - 1 - (j-1) n_{2d+2} - i, 1 \leq i \leq n_{2d+2}, 2 \leq j \leq m_{2d+2} - 3, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{18} : \phi \left(t_{2d+2} c_{(2d+2)(n_{2d+2}+1-i)}^1 \right) = q - \sum_{k=1}^{2d+2} q_k - 2d + 1 + n_{2d+2} - i, 1 \leq i \leq n_{2d+2}, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{19} : \phi \left(b_{2d+2} t_{2d+2} \right) = q - \sum_{k=1}^{2d+2} q_k - 2d, 1 \leq d \leq \lambda - 1.$$

$$\mathbf{20} : \phi \left(b_{2d+2} b_{2d+3} \right) = q - \sum_{k=1}^{2d+2} q_k - 2d - 1, 1 \leq d \leq \lambda - 1.$$

The resultant edge values $E(G)$ are distinct.

Result: The hanging modified theta graphs $H\theta(n_j, m_j)$, $m = 2j + 1$ and $j = 1, 2, \dots, n$ which includes symmetric modified theta graphs and non-decreasing symmetric modified theta graphs are graceful. $\square$

Figure 2: Graceful labeling of $H\theta(6m_3^{1,2}, 4m_5^{3,4}, 3m_7^{5,6})$

Example for hanging non-decreasing symmetric modified theta graphs are shown below.

Figure 3: Graceful labeling of $H\theta$ ($6m_3^1, 3m_5^2, 3m_9^3, 6m_5^4$)

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