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THE HECKE ALGEBRA $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ ARISING IN NUMBER THEORY

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Abstract: The algebra $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$ arises in number theory and has been studied in [1] of Bost and Connes. Laca and Raeburn continued the study in [3] and gave an improvement of the theorem of Bost and Connes. This leads us to consider a closely related algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ because of its interesting connections with C^*-algebras and group algebras.

In this paper we give a detailed proof of Laca and Raeburn's theorem. Then we define a new Hecke pair $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ and show that the Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is a universal $*$-algebra generated by the elements $\{\mu_n : n \in \mathbb{N}^*\}$, $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and a new element $u = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$.

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1. Introduction

A Hecke pair (G, S) consists of a discrete group G and a subgroup S such that every double coset consists of finitely many left cosets.

The Hecke algebra $H(P^{+}_{\mathbb{Q}}, P_{\mathbb{Z}})$ first arose in Bost and Connes study [2] and they have proved that it is a universal $*$-algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ subject to six relations.

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M. Laca and I. Raeburn studied also the Hecke algebra of Bost and Connes in [1] and they gave an improvement of Bost and Connes Theorem by showing that there are two redundant relations among the six in Bost and Connes Theorem.

In this paper, we first introduce Hecke algebras, and prove some results about them. We then studied Bost and Connes Hecke algebra $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$, where

$$P_{\mathbb{Q}}^{+} = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r > 0 \right\},$$

$$P_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

and then we gave a proof for the improvement of their Theorem, given by Laca and Raeburn in [1], after we stated and proved some lemmas that lead to the proof.

In the last part of this work we define a new Hecke pair $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, where

$$P_{\mathbb{Q}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r \neq 0 \right\}.$$

and we show that this Hecke algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is a universal $*$-algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$, $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and a new element which we called u.

2. Hecke Algebras

Definition 2.1. Let G be a discrete group and S a subgroup of G. The pair (G, S) ia called a *Hecke pair* if each double coset StS can be written as a finite union of left cosets.

Proposition 2.2. *Let (G, S) be a Hecke pair.Then the set*

$$H(G, S) = \{ f : S\backslash G/S \longrightarrow \mathbb{C} : f \text{ has finite support} \}$$

is a $*$-algebra with

$$(f * g)(StS) = \sum_{rS \in G/S} f(SrS)g(Sr^{-1}tS) \tag{2.1}$$

and

$$f^*(StS) = \overline{f(St^{-1}S)}.$$

This $*$-algebra is called a Hecke algebra.

Lemma 2.3. *The sum in* (2.1) *is finite and depends only on* StS.

Proof. The sum is finite because $f(SrS) \neq 0$ for only finitely many $SrS \in S\backslash G/S$ and these finitely many double cosets contain finitely many left cosets. For the second part of this lemma replace t in the right-hand side by $s_1 t s_2$ for some $s_1, s_2 \in S$. Then

$$\sum_{rS\in G/S} f(SrS)g(Sr^{-1}s_1ts_2S) = \sum_{rS\in G/S} f(SrS)g(Sr^{-1}s_1tS).$$

By writing $r_1 = s_1^{-1}r$,

$$\begin{aligned}\sum_{rS\in G/S} f(SrS)g(Sr^{-1}s_1tS) &= \sum_{r_1S\in G/S} f(Ss_1r_1S)g(Sr_1^{-1}tS)\\ &= \sum_{r_1S\in G/S} f(Sr_1S)g(Sr_1^{-1}tS).\end{aligned}$$

□

Proof of proposition 2.2. Since $H(G,S)$ is obviously a vector space, and the distributive laws are pretty clear, we need to check that $(f*g)*h = f*(g*h)$ and the properties of the involution. On the one hand, we have

$$\begin{aligned}((f*g)*h)(StS) &= \sum_{rS\in G/S} (f*g)(SrS)h(Sr^{-1}tS)\\ &= \sum_{rS\in G/S}\Big(\sum_{pS\in G/S} f(SpS)g(Sp^{-1}rS)\Big)h(Sr^{-1}tS),\end{aligned}$$

and on the other hand,

$$\begin{aligned}(f*(g*h))(StS) &= \sum_{qS\in G/S} f(SqS)(g*h)(Sq^{-1}tS)\\ &= \sum_{qS\in G/S} f(SqS)\Big(\sum_{dS\in G/S} g(SdS)h(Sd^{-1}q^{-1}tS)\Big).\end{aligned}$$

By writing $r = qd$, and noting that for fixed qS, $\sum_{rS} = \sum_{dS}$, we find

$$(f*(g*h))(StS) = \sum_{qS\in G/S}\Big(\sum_{dS\in G/S} f(SqS)g(Sq^{-1}rS)h(Sr^{-1}tS)\Big) = ((f*g)*h).$$

For $\lambda, \mu \in \mathbb{C}$, we have $\big(\lambda f + \mu g\big)^* = \overline{\lambda} f^* + \overline{\mu} g^*$. Next,

$$\begin{aligned}(f^*)^*(StS) &= \overline{f^*(St^{-1}S)} \\ &= \overline{\big(\overline{f(StS)}\big)} \\ &= f(StS).\end{aligned}$$

Finally, we compute

$$\begin{aligned}(f * g)^*(StS) &= \overline{(f * g)(St^{-1}S)} \\ &= \overline{\sum_{rS \in G/S} f(SrS) g(Sr^{-1}t^{-1}S)} \\ &= \sum_{rS \in G/S} f^*(Sr^{-1}S) g^*(StrS)\end{aligned}$$

and by writing $p = tr$ we have that $r = t^{-1}p$, so

$$\begin{aligned}(f * g)^*(StS) &= \sum_{pS \in G/S} f^*(Sp^{-1}tS) g^*(SpS) \\ &= \sum_{pS \in G/S} g^*(SpS) f^*(Sp^{-1}tS) \\ &= (g^* * f^*)(StS).\end{aligned}$$

□

Remark 2.4. Notice that $H(G,S) = \operatorname{span}\Big\{[t] : t \in G\Big\}$, where

$$[t](SpS) = \begin{cases} 1 & \text{if } StS = SpS \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.5. Let S be any subgroup of a group G. Then the map $Sp \mapsto p^{-1}S$ is a one to one correspondence between G/S and $S\backslash G$.

Remark 2.6. If S is a normal subgroup of G, then each double coset StS is equal to tS, so $S\backslash G/S = G/S$, and the Hecke algebra $H(G,S)$ reduces to the group algebra $\mathbb{C}(G/S)$.

Lemma 2.7. *Let (G,S) be a Hecke pair and $r, t \in G$. Then $StS = SrS$ if and only if $t \in SrS$.*

Proof. First suppose that e is the identity element of the group G and that $StS = SrS$. Since $t = ete \in StS$ then $t \in SrS$.
Secondly, suppose that $t \in SrS$. Then $t = s_1 r s_2$ for some $s_1, s_2 \in S$, so

$$StS = S(s_1 r s_2)S = SrS.$$

□

Lemma 2.8. *Suppose that G is a subgroup of the group K and S is a subgroup of G such that (K, S) is a Hecke pair. Then there is an injective $*$-homomorphism ι from $H(G,S)$ into $H(K,S)$ such that*

$$\iota(f)(SkS) := \begin{cases} f(SkS) & \text{if } k \in G \\ 0 & \text{otherwise.} \end{cases}$$

Proof. That ι is linear follows directly from the definition, so we need to check that ι is $*$-preserving, multiplicative and injective. Now, let $f \in H(G,S)$. Then

$$\begin{aligned} \iota(f^*)(SkS) &= \begin{cases} f^*(SkS) & \text{if } k \in G \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \overline{f(Sk^{-1}S)} & \text{if } k \in G \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \overline{\iota(f)(Sk^{-1}S)} & \text{if } k \in G \\ 0 & \text{otherwise} \end{cases} \\ &= \iota(f)^*(SkS). \end{aligned}$$

Thus ι is $*$-preserving.

Next, let $f, h \in H(G,S)$. Then

$$\begin{aligned} \iota(f) * \iota(h)(SkS) &= \sum_{pS \in K/S} \iota(f)(SpS)\iota(h)(Sp^{-1}kS) \\ &= \sum_{pS \in K/S} \left(\begin{Bmatrix} f(SpS) & \text{if } p \in G \\ 0 & \text{otherwise} \end{Bmatrix} \begin{Bmatrix} h(Sp^{-1}kS) & \text{if } p^{-1}k \in G \\ 0 & \text{otherwise} \end{Bmatrix} \right) \\ &= \begin{cases} \sum_{pS \in G/S} f(SpS)h(Sp^{-1}kS) & \text{if } k \in G \\ 0 & \text{otherwise} \end{cases} \\ &= \iota(f * h)(SkS). \end{aligned}$$

Thus ι is multiplicative.

Finally, let $f, h \in H(G,S)$ and suppose that $\iota(f)(SkS) = \iota(h)(SkS)$. Then

$$\iota(f-h)(SkS) = 0, \text{ since } \iota \text{ is linear}$$

and this equivalent to

$$0 = \begin{cases} (f-h)(SkS) & \text{if } k \in G \\ 0 & \text{otherwise} \end{cases}$$

which is equivalent to say that

$$f(SkS) = h(SkS) \text{ for } k \in G.$$

Thus ι is one to one. □

3. Representations of H(G,S)

Let S be any subgroup of a group G, and define

$$\ell^2(G/S) = \Big\{ f : G/S \longrightarrow \mathbb{C} \text{ such that } \sum_{pS \in G/S} |f(pS)|^2 < \infty \Big\}.$$

Then $\ell^2(G/S)$ is a Hilbert space with inner product $(f|h) = \sum_{pS \in G/S} f(pS)\overline{h(pS)}$.

Lemma 3.1. *Let (G,S) be a Hecke pair. Then the formula* $\lambda_t^{G/S}(f)(pS) = f(t^{-1}pS)$ *defines a unitary representation of* G *on* $\ell^2(G/S)$. *It is called a quasi-regular representation.*

Proof. To prove this lemma we need to show that the operator $\lambda_t^{G/S}$ is isometric, onto and that $\lambda_{rt}^{G/S} = \lambda_r^{G/S} \circ \lambda_t^{G/S}$. Let us first show that it is isometric. Let $f \in \ell^2(G/S)$. Then

$$\begin{aligned} \|f\|^2 &= (f|f) \\ &= \sum_{rS \in G/S} |f(rS)|^2 \\ &= \sum_{rS \in G/S} |f(t^{-1}rS)|^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{rS \in G/S} |\lambda_t^{G/S}(f)(rS)|^2 \\
&= \big(\lambda_t^{G/S}(f)|\lambda_t^{G/S}(f)\big) \\
&= \|\lambda_t^{G/S}(f)\|^2.
\end{aligned}
$$

Next, let $f \in \ell^2(G/S)$ and define $f_1 : G/S \longrightarrow \mathbb{C}$ by $f_1(pS) = f(tpS)$. Then

$$
\begin{aligned}
(\lambda_t^{G/S} f_1)(pS) &= f_1(t^{-1}pS) \\
&= f\big(t(t^{-1}pS)\big) \\
&= f(pS).
\end{aligned}
$$

Finally, Let $r, t \in G$ then,

$$
\begin{aligned}
\big(\lambda_r^{G/S} \circ \lambda_t^{G/S}\big)(f)(aS) &= \lambda_r^{G/S}\big(\lambda_t^{G/S} f\big)(aS) \\
&= \big(\lambda_t^{G/S} f\big)(r^{-1}aS) \\
&= f(t^{-1}r^{-1}aS) \\
&= f\big((rt)^{-1}aS\big) \\
&= \big(\lambda_{rt}^{G/S} f\big)(aS).
\end{aligned}
$$

Thus, $\lambda_t^{G/S}$ defines a unitary representation of G on $\ell^2(G/S)$. $\square$

Proposition 3.2. *Let (G, S) be a Hecke pair, and define*

$$R : H(G, S) \longrightarrow B(\ell^2(G/S))$$

by

$$R(h)(f)(pS) = \sum_{rS \in G/S} f(rS)h(Sr^{-1}pS), \text{ where } h \in H(G, S) \text{ and } f \in \ell^2(G/S). \tag{3.1}$$

Then R is a $*$-anti representation of $H(G, S)$, and $R(h)$ commutes with every $\lambda_t^{G/S}$.

Proof. Since R is obviously a linear map, we need to check that $R(h * g) = R(g) \circ R(h)$ and $R(h)^* = R(h^*)$. Let $h, g \in H(G, S)$. On the one hand we have

$$R(h * g)(f)(tS) = \sum_{pS \in G/S} f(pS)(h * g)(Sp^{-1}tS)$$

$$= \sum_{pS\in G/S} f(pS)\Big(\sum_{bS\in G/S} h(SbS)g(Sb^{-1}p^{-1}tS)\Big)$$
$$= \sum_{pS\in G/S}\Big(\sum_{bS\in G/S} f(pS)h(SbS)g(S(pb)^{-1}tS)\Big),$$

and on the other,

$$R(g)\circ R(h)(f)(tS) = R(g)\Big(R(h)(f)\Big)(tS)$$
$$= \sum_{dS\in G/S} R(h)(f)(dS)g(Sd^{-1}tS)$$
$$= \sum_{dS\in G/S}\Big(\sum_{qS\in G/S} f(qS)h(Sq^{-1}dS)g(Sd^{-1}tS)\Big).$$

By writing $pb = v$, and noting that $b = p^{-1}v$, we find

$$R(h*g)(f)(tS) = \sum_{pS\in G/S}\Big(\sum_{vS\in G/S} f(pS)h(Sp^{-1}vS)g(Sv^{-1}tS)\Big)$$
$$= R(g)\circ R(h)(f)(tS).$$

Next, we compute $R(h)^*$. Since $R(h)^*$ is a bounded linear operator on $\ell^2(G/S)$ then for all $f, g \in \ell^2(G/S)$ we have

$$\big(R(h)^*(f)|g\big) = \big(f|R(h)(g)\big)$$
$$= \sum_{tS\in G/S} f(tS)\overline{R(h)(g)(tS)}$$
$$= \sum_{tS\in G/S} f(tS)\Big(\sum_{pS\in G/S} \overline{g(pS)h(Sp^{-1}tS)}\Big)$$
$$= \sum_{tS\in G/S} f(tS)\Big(\sum_{pS\in G/S} \overline{g(pS)}h^*(St^{-1}pS)\Big)$$
$$= \sum_{tS\in G/S}\Big(\sum_{pS\in G/S} f(tS)h^*(St^{-1}pS)\overline{g(pS)}\Big).$$

By noting that $\sum_{tS\in G/S} f(tS)h^*(St^{-1}pS) = R(h^*)(f)(pS)$, we get

$$\big(R(h)^*(f)|g\big) = \sum_{pS\in G/S} R(h^*)(f)(pS)\overline{g(pS)}$$
$$= \big(R(h^*)(f)|g\big).$$

Hence, $R(h)^* = R(h^*)$. $\square$

4. The Hecke Pair of Bost and Connes

Bost and Connes defined

$$P_{\mathbb{Q}}^{+} = \Big\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r > 0 \Big\},$$

and

$$P_{\mathbb{Z}} = \Big\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \Big\},$$

and showed that $(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$ is a Hecke pair. To see this, let $G = P_{\mathbb{Q}}^{+}$ and $S = P_{\mathbb{Z}}$. Then for arbitrary $t = \left(\begin{smallmatrix} 1 & a \\ 0 & r \end{smallmatrix}\right) \in G$, the left coset associated to t is

$$tS = \Big\{ \begin{pmatrix} 1 & n+a \\ 0 & r \end{pmatrix} : n \in \mathbb{Z} \Big\},$$

and the double coset associated to t is

$$StS = \Big\{ \begin{pmatrix} 1 & n+a+rm \\ 0 & r \end{pmatrix} : n, m \in \mathbb{Z} \Big\}.$$

By expressing $r = p/q$, where p and q are relatively prime in $\mathbb{N}$ (we can do this since $r \in \mathbb{Q}$ and $r > 0$), each double coset StS may be written as the disjoint union of q single cosets $t_i S$ by

$$\begin{aligned} StS &= \bigcup_{i=1}^{q} \Big\{ \begin{pmatrix} 1 & n+a+ri \\ 0 & r \end{pmatrix} : n \in \mathbb{Z} \Big\} \\ &= \bigcup_{i=1}^{q} \Big\{ \begin{pmatrix} 1 & a+ri \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \Big\} \\ &= \bigcup_{i=1}^{q} t_i S, \end{aligned}$$

where $t_i = \left(\begin{smallmatrix} 1 & a+ri \\ 0 & r \end{smallmatrix}\right)$.

5. Presentation of the Hecke Algebra $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$

Let us first consider the Hecke pair $(G, S) = (P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$. We shall use the following notations:

(a) For $n \in \mathbb{N}^*, \mu_n = n^{-1/2}\Big[\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Big]$.

(b) For $r \in \mathbb{Q}/\mathbb{Z}, e(r) = \Big[\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\Big]$.

Theorem 5.1 (Bost and Connes, 1995)**.** *$H(P^+_{\mathbb{Q}}, P_{\mathbb{Z}})$ is the universal $*$-algebra generated by the elements $\{\mu_n : n \in \mathbb{N}^*\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ subject to the relations*

(a) *$\mu_n^*\mu_n = 1$ for all $n \in \mathbb{N}^*$.*

(b) *$\mu_{mn} = \mu_m\mu_n$ for all m, n.*

(c) *$e(r)^* = e(-r)$, $e(r_1 + r_2) = e(r_1)e(r_2)$ for all $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$.*

(d) *$\mu_n e(r)\mu_n^* = (1/n)\sum_{j=1}^n e(r/n + j/n)$ for all n and all r.*

Lemma 5.2. *Consider the Hecke pair (G, S) where $G = P^+_{\mathbb{Q}}$ and $S = P_{\mathbb{Z}}$, then for $f \in H(G, S)$ and $n \in \mathbb{N}^*$ we have*

$$\big(\mu_n * f\big)(StS) = n^{-1/2} f\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\big)tS\big) \text{ for all } t \in G.$$

Proof. Let $t \in G$. Then

$$\big(\mu_n * f\big)(StS) = \sum_{pS \in G/S} \mu_n(SpS) f(Sp^{-1}tS).$$

Now $\mu_n(SpS)$ vanishes unless

$$SpS = S\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}S = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}S,$$

so there is exactly one non zero term in the sum, occuring when $pS = \big(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\big)S$, and then $\mu_n(SpS) = n^{-1/2}$. Thus

$$\begin{aligned} \big(\mu_n * f\big)(StS) &= n^{-1/2} f\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\big)^{-1}tS\big) \\ &= n^{-1/2} f\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\big)tS\big). \end{aligned}$$

□

Lemma 5.3. *Consider the Hecke pair (G, S) where $G = P^+_{\mathbb{Q}}$ and $S = P_{\mathbb{Z}}$, then for $f \in H(G, S)$ and $r \in \mathbb{Q}/\mathbb{Z}$ we have*

$$\big(e(r) * f\big)(StS) = f\big(S\big(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\big)tS\big) \text{ for all } t \in G.$$

Proof. Let $t \in G$. Then

$$\big(e(r) * f\big)(StS) = \sum_{pS \in G/S} e(r)(SpS) f(Sp^{-1}tS).$$

Now $e(r)(SpS)$ vanishes unless

$$SpS = S \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} S = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} S,$$

so there is exactly one non zero term in the sum, occuring when $pS = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) S$, and then $e(r)(SpS) = 1$. Thus

$$\begin{aligned} \big(e(r) * f\big)(StS) &= f\big(S\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)^{-1} tS\big) \\ &= f\big(S\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right) tS\big). \end{aligned}$$

$\square$

Lemma 5.4. *Consider the Hecke pair (G, S) where $G = P_{\mathbb{Q}}^{+}$ and $S = P_{\mathbb{Z}}$, then for $f \in H(G, S)$ and $r \in \mathbb{Q}/\mathbb{Z}$ we have*

$$\big(f * e(r)\big)(StS) = f\big(St\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right)S\big) \text{ for all } t \in G.$$

Proof. Let $t \in G$. Then

$$\big(f * e(r)\big)(StS) = \sum_{pS \in G/S} f(SpS) e(r)(Sp^{-1}tS).$$

Now $e(r)(Sp^{-1}tS)$ vanishes unless

$$Sp^{-1}tS = S \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} S,$$

and by taking the inverse of both sides we have that

$$St^{-1}pS = S \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}^{-1} S = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}^{-1} S,$$

so there is exactly one non zero term in the sum, occuring when $t^{-1}pS = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)^{-1} S$, and then $e(r)(Sp^{-1}tS) = 1$. Thus

$$\begin{aligned} \big(f * e(r)\big)(StS) &= f\big(St\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)^{-1} S\big) \\ &= f\big(St\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right) S\big). \end{aligned}$$

$\square$

Lemma 5.5. *Consider the Hecke pair* (G,S) *where* $G = P_{\mathbb{Q}}^{+}$ *and* $S = P_{\mathbb{Z}}$*, then for* $f \in H(G,S)$ *and* $n \in \mathbb{N}^*$ *we have*

$$\big(\mu_n^* * f\big)(StS) = n^{-1/2} \sum_{k=0}^{n-1} f\big(S\big(\begin{smallmatrix} 1 & k \\ 0 & n \end{smallmatrix}\big)tS\big) \text{ for all } t \in G.$$

Proof. Let $t \in G$. Then

$$\big(\mu_n^* * f\big)(StS) = \sum_{pS \in G/S} \mu_n^*(SpS) f(Sp^{-1}tS).$$

Now $\mu_n^*(SpS)$ vanishes unless

$$Sp^{-1}S = S \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S,$$

and by taking the inverse of both sides we get that

$$SpS = S \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} S = \bigcup_{j=1}^{n} \begin{pmatrix} 1 & j/n \\ 0 & 1/n \end{pmatrix} S,$$

so there are n non zero terms in the sum, occuring when $pS = \big(\begin{smallmatrix} 1 & j/n \\ 0 & 1/n \end{smallmatrix}\big)S$ for $1 \le j \le n$, and $\mu_n^*(SpS) = n^{-1/2}$. Thus

$$\begin{aligned}
\big(\mu_n^* * f\big)(StS) &= n^{-1/2} \sum_{pS = \big(\begin{smallmatrix} 1 & j/n \\ 0 & 1/n \end{smallmatrix}\big)S} f(Sp^{-1}tS) \\
&= n^{-1/2} \sum_{j=1}^{n} f\big(S\big(\begin{smallmatrix} 1 & j/n \\ 0 & 1/n \end{smallmatrix}\big)^{-1} tS\big) \\
&= n^{-1/2} \sum_{j=1}^{n} f\big(S\big(\begin{smallmatrix} 1 & -j \\ 0 & n \end{smallmatrix}\big) tS\big) \\
&= n^{-1/2} \sum_{k=0}^{n-1} f\big(S\big(\begin{smallmatrix} 1 & k \\ 0 & n \end{smallmatrix}\big) tS\big).
\end{aligned}$$

□

Proof of theorem 5.1 . We have to show, first, that $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$ is generated by elements $\mu_n, e(r)$ satisfying (a)-(d).

For (a), let $t \in G$. Then Lemma 5.5 says that

$$\big(\mu_n^* * \mu_n\big)(StS) = n^{-1/2} \sum_{k=0}^{n-1} \mu_n\big(S \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} tS\big).$$

Now $\mu_n\big(S\big(\begin{smallmatrix} 1 & k \\ 0 & n \end{smallmatrix}\big)tS\big)$ vanishes unless

$$S \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} tS = S \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S \text{ for } 0 \le k \le n-1,$$

and $\mu_n\big(S\big(\begin{smallmatrix} 1 & k \\ 0 & n \end{smallmatrix}\big)tS\big) = n^{-1/2}$. Thus

$$\big(\mu_n^* * \mu_n\big)(StS) = n^{-1/2} \sum_{k=0}^{n-1} n^{-1/2} = 1.$$

For (b), let $t \in G$. Then Lemma 5.2 says that

$$\big(\mu_m * \mu_n\big)(StS) = m^{-1/2} \mu_n\big(S \begin{pmatrix} 1 & 0 \\ 0 & 1/m \end{pmatrix} tS\big).$$

Now $\mu_n\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/m \end{smallmatrix}\big)tS\big)$ vanishes unless

$$S \begin{pmatrix} 1 & 0 \\ 0 & 1/m \end{pmatrix} tS = S \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S,$$

and this is equivalent to saying that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/m \end{pmatrix} t \in \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S = \Big\{ \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} : k \in \mathbb{Z} \Big\}.$$

So the non zero value of $\mu_n\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/m \end{smallmatrix}\big)tS\big)$ occurs when

$$t = \begin{pmatrix} 1 & 0 \\ 0 & 1/m \end{pmatrix}^{-1} \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & mn \end{pmatrix} \text{ for some } k \in \mathbb{Z},$$

and then $\mu_n\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/m \end{smallmatrix}\big)tS\big) = n^{-1/2}$. Thus

$$\begin{aligned} \big(\mu_m * \mu_n\big)(StS) &= m^{-1/2} \mu_n\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/m \end{smallmatrix}\big)\big(\begin{smallmatrix} 1 & k \\ 0 & mn \end{smallmatrix}\big)S\big) \\ &= m^{-1/2} \mu_n\big(S\big(\begin{smallmatrix} 1 & k \\ 0 & n \end{smallmatrix}\big)S\big) \\ &= m^{-1/2} n^{-1/2} \end{aligned}$$

$$= \mu_{mn}(StS).$$

For (c) let us start with

$$\begin{aligned} e(r)^*(StS) &= \overline{e(r)(St^{-1}S)} \\ &= \begin{cases} 1 & \text{if } St^{-1}S = S\left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } StS = S\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise} \end{cases} \\ &= e(-r)(StS) \end{aligned}$$

Thus $e(r)^* = e(-r)$. For the second part of (c) Lemma 5.3 says that for $t \in G$ we have

$$e(r_1) * e(r_2)(StS) = e(r_2)\Big(S\begin{pmatrix} 1 & -r_1 \\ 0 & 1 \end{pmatrix} tS\Big).$$

Now $e(r_2)\big(S\left(\begin{smallmatrix} 1 & -r_1 \\ 0 & 1 \end{smallmatrix}\right)tS\big)$ vanishes unless

$$S\begin{pmatrix} 1 & -r_1 \\ 0 & 1 \end{pmatrix} tS = S\begin{pmatrix} 1 & r_2 \\ 0 & 1 \end{pmatrix} S = \begin{pmatrix} 1 & r_2 \\ 0 & 1 \end{pmatrix} S,$$

and this equivalent to saying that

$$\begin{pmatrix} 1 & -r_1 \\ 0 & 1 \end{pmatrix} t \in \begin{pmatrix} 1 & r_2 \\ 0 & 1 \end{pmatrix} S = \Big\{\begin{pmatrix} 1 & k+r_2 \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}\Big\},$$

so the non zero value of $e(r_2)\big(S\left(\begin{smallmatrix} 1 & -r_1 \\ 0 & 1 \end{smallmatrix}\right)tS\big)$ occurs when

$$t = \begin{pmatrix} 1 & -r_1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & k+r_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+r_1+r_2 \\ 0 & 1 \end{pmatrix} \text{ for some } k \in \mathbb{Z}.$$

While $e(r_1 + r_2)(StS)$ vanishes unless

$$StS = S\begin{pmatrix} 1 & r_1+r_2 \\ 0 & 1 \end{pmatrix} S = \begin{pmatrix} 1 & r_1+r_2 \\ 0 & 1 \end{pmatrix} S,$$

and this is equivalent to saying that

$$t \in \begin{pmatrix} 1 & r_1+r_2 \\ 0 & 1 \end{pmatrix} S = \Big\{\begin{pmatrix} 1 & j+r_1+r_2 \\ 0 & 1 \end{pmatrix} : j \in \mathbb{Z}\Big\}.$$

Thus by comparing the two results we get that

$$\big(e(r_1) * e(r_2)\big)(StS) = e(r_1 + r_2)(StS).$$

For part (d), let $t \in G$. By applying first Lemma 5.2 and then Lemma 5.4 we get

$$\begin{aligned}
\big(\mu_n * e(r) * \mu_n^*\big)(StS) &= n^{-1/2}\big(e(r) * \mu_n^*\big)\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\right)tS\big)\\
&= n^{-1/2}\mu_n^*\big(S\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\right)\big)tS\big)\\
&= n^{-1/2}\mu_n^*\big(S\left(\begin{smallmatrix} 1 & -r/n \\ 0 & 1/n \end{smallmatrix}\right)tS\big)\\
&= n^{-1/2}\mu_n\big(St^{-1}\left(\begin{smallmatrix} 1 & -r/n \\ 0 & 1/n \end{smallmatrix}\right)^{-1}S\big)\\
&= n^{-1/2}\begin{cases} n^{-1/2} & \text{if } S\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)S = St^{-1}\left(\begin{smallmatrix} 1 & -r/n \\ 0 & 1/n \end{smallmatrix}\right)^{-1}S \\ 0 & \text{otherwise} \end{cases}\\
&= \begin{cases} n^{-1} & \text{if } S\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\right)S = S\left(\begin{smallmatrix} 1 & -r/n \\ 0 & 1/n \end{smallmatrix}\right)tS \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now

$$S\begin{pmatrix} 1 & -r/n \\ 0 & 1/n \end{pmatrix} tS = S\begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} S \Leftrightarrow \begin{pmatrix} 1 & -r/n \\ 0 & 1/n \end{pmatrix} t \in S\begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} S,$$

and since $(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ is a Hecke pair we have

$$S\begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} S = \bigcup_{j=1}^{n} \begin{pmatrix} 1 & j/n \\ 0 & 1/n \end{pmatrix} S.$$

Thus for some $1 \leq j \leq n$

$$\begin{aligned}
t &= \begin{pmatrix} 1 & -r/n \\ 0 & 1/n \end{pmatrix}^{-1} \begin{pmatrix} 1 & j/n \\ 0 & 1/n \end{pmatrix}\\
&= \begin{pmatrix} 1 & r \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & j/n \\ 0 & 1/n \end{pmatrix}\\
&= \begin{pmatrix} 1 & j/n + r/n \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Hence

$$\big(\mu_n * e(r) * \mu_n^*\big)(StS) = \begin{cases} n^{-1} & \text{if } t = \left(\begin{smallmatrix} 1 & j/n+r/n \\ 0 & 1 \end{smallmatrix}\right) \\ 0 & \text{otherwise.} \end{cases}$$

The right-hand side of (d) gives

$$\begin{aligned} n^{-1}\sum_{j=1}^{n} e(r/n+j/n)(StS) &= \begin{cases} n^{-1} & \text{if } StS = S\left(\begin{smallmatrix} 1 & r/n+j/n \\ 0 & 1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} n^{-1} & \text{if } t \in S\left(\begin{smallmatrix} 1 & r/n+j/n \\ 0 & 1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} n^{-1} & \text{if } t \in \left(\begin{smallmatrix} 1 & r/n+j/n \\ 0 & 1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise} \end{cases} \\ &= \big(\mu_n * e(r) * \mu_n^*\big)(StS). \end{aligned}$$

Our next step is to show that the elements $\{\mu_n : n \in \mathbb{N}^*\}$ and $\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ generate the $*$-algebra $H(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$.

Claim. Let $\left(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\right) \in G$. Then for every $t \in G$

$$\left[\begin{pmatrix} 1 & a \\ 0 & n/m \end{pmatrix}\right](StS) = \sqrt{n/m}\big(\mu_m^* e(a/n)\mu_n\big)(StS).$$

Proof. On one hand, we have

$$\begin{aligned} \left[\begin{pmatrix} 1 & a \\ 0 & n/m \end{pmatrix}\right](StS) &= \begin{cases} 1 & \text{if } S\left(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\right)S = StS \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } t \in S\left(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\right)S \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now

$$t \in S\begin{pmatrix} 1 & a \\ 0 & n/m \end{pmatrix}S = \bigcup_{j=1}^{m}\left\{\begin{pmatrix} 1 & a+\alpha+(n/m)j \\ 0 & n/m \end{pmatrix} : \alpha \in \mathbb{Z}\right\},$$

which means that there exist $\alpha \in \mathbb{Z}$ and $1 \le j \le m$ such that

$$t = \begin{pmatrix} 1 & a+\alpha+(n/m)j \\ 0 & n/m \end{pmatrix}.$$

On the other hand, by applying Lemma 5.5 and Lemma 5.3 we have

$$n^{1/2}m^{-1/2}\big(\mu_m^* * e(a/n) * \mu_n\big)(StS)$$

$$\begin{aligned}
&= n^{1/2} m^{-1/2} m^{-1/2} \sum_{k=0}^{m-1} \big(e(a/n) * \mu_n\big)\big(S\left(\begin{smallmatrix} 1 & k \\ 0 & m \end{smallmatrix}\right) tS\big) \\
&= n^{1/2} m^{-1} \sum_{k=0}^{m-1} \mu_n\big(S\left(\begin{smallmatrix} 1 & -a/n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & k \\ 0 & m \end{smallmatrix}\right) tS\big) \\
&= n^{1/2} m^{-1} \sum_{k=0}^{m-1} \mu_n\big(S\left(\begin{smallmatrix} 1 & k-(m/n)a \\ 0 & m \end{smallmatrix}\right) tS\big).
\end{aligned}$$

Now μ_n vanishes unless

$$S \begin{pmatrix} 1 & k-(m/n)a \\ 0 & m \end{pmatrix} tS = S \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S \tag{5.1}$$

which is equivalent to

$$\begin{pmatrix} 1 & k-(m/n)a \\ 0 & m \end{pmatrix} t \in S \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S,$$

and hence to

$$\begin{pmatrix} 1 & k-(m/n)a \\ 0 & m \end{pmatrix} t \in \Big\{ \begin{pmatrix} 1 & i \\ 0 & n \end{pmatrix} : i \in \mathbb{Z} \Big\}.$$

Thus (5.1) holds if and only if there exists $i \in \mathbb{Z}$ such that

$$\begin{pmatrix} 1 & k-(m/n)a \\ 0 & m \end{pmatrix} t = \begin{pmatrix} 1 & i \\ 0 & n \end{pmatrix}$$

and thus if and only if

$$\begin{aligned}
t &= \begin{pmatrix} 1 & k-(m/n)a \\ 0 & m \end{pmatrix}^{-1} \begin{pmatrix} 1 & i \\ 0 & n \end{pmatrix} \\
&= \begin{pmatrix} 1 & -k/m + a/n \\ 0 & 1/m \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & n \end{pmatrix} \\
&= \begin{pmatrix} 1 & a+i-(n/m)k \\ 0 & n/m \end{pmatrix}.
\end{aligned}$$

So $n^{1/2} m^{-1/2} \big(\mu_m^* * e(a/n) * \mu_n\big)(StS) = 0$ unless there exist $i, k \in \mathbb{Z}$ such that $t = \left(\begin{smallmatrix} 1 & a+i-(n/m)k \\ 0 & n/m \end{smallmatrix}\right)$, and if so

$$n^{1/2} m^{-1/2} \big(\mu_m^* * e(a/n) * \mu_n\big)(StS) = n^{1/2} m^{-1} \sum_{k=0}^{m-1} \mu_n\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) S\big)$$

$$= n^{1/2}m^{-1}\sum_{k=0}^{m-1} n^{-1/2}$$
$$= 1.$$

Thus
$$\left[\begin{pmatrix} 1 & a \\ 0 & n/m \end{pmatrix}\right] = n^{1/2}m^{-1/2}\big(\mu_m^* * e(a/n) * \mu_n\big). \tag{5.2}$$

$\square$

This shows that the set $\{\mu_m^* * e(r) * \mu_n\}$ spans $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$, and that any $\mu_m^* e(r)\mu_n$ and $\mu_k^* e(s)\mu_l$ are either multiples of the same characteristic function or have orthogonal supports. So the elements $n^{1/2}m^{-1/2}\big(\mu_m^* * e(a/n) * \mu_n\big)$ are linearly independent. Hence the set $B = \big\{n^{1/2}m^{-1/2}\big(\mu_m^* * e(a/n) * \mu_n\big) : S\big(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\big)S \in S\backslash G/S\big\}$ is a basis for the vector space $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$.

For the last part of the proof suppose that $\{\hat{\mu}_n : n \in \mathbb{N}^*\}$, and $\{\hat{e}(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ are elements in a $*$-algebra A which also satisfy the relations (a)-(d). We have to show that there is a $*$-homomorphism $\phi : H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \longrightarrow A$ such that $\phi(\mu_n) = \hat{\mu}_n$ and $\phi(e(r)) = \hat{e}(r)$. But since B is a basis, there is a linear transformation $\phi : H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \longrightarrow A$ such that $\phi(\mu_n) = \hat{\mu}_n$ and $\phi(e(r)) = \hat{e}(r)$.

Claim. The map $\phi : H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}) \longrightarrow A$ defined by

$$\phi\big(n^{1/2}m^{-1/2}(\mu_m^* e(r)\mu_n)\big) = n^{1/2}m^{-1/2}(\hat{\mu}_m^* \hat{e}(r)\hat{\mu}_n)$$

is a $*$-homomorphism.

Proof. We need to show that ϕ is $*$-preserving and that

$$\phi\big(n^{1/2}m^{-1/2}(\mu_m^* e(r_1)\mu_n) l^{1/2}k^{-1/2}(\mu_k^* e(r_2)\mu_l)\big) = \phi\big(n^{1/2}m^{-1/2}(\mu_m^* e(r_1)\mu_n)\big)$$
$$\phi\big(l^{1/2}k^{-1/2}(\mu_k^* e(r_2)\mu_l)\big)$$

This is enough since it is clear that $\phi(\mu_n) = \hat{\mu}_n$ and $\phi(e(r)) = \hat{e}(r)$.

Next, we compute

$$\begin{aligned}
\phi\big((n^{1/2}m^{-1/2}\mu_m^* e(r_1)\mu_n)^*\big) &= \phi\big((n^{1/2}m^{-1/2}\mu_n^* e(-r_1)\mu_m)\big) \\
&= \big(n^{1/2}m^{-1/2}\hat{\mu}_n^* \hat{e}(-r_1)\hat{\mu}_m\big) \\
&= n^{1/2}m^{-1/2}\big(\hat{\mu}_m^* \hat{e}(-r_1)^* \hat{\mu}_n\big)^* \\
&= n^{1/2}m^{-1/2}\big(\hat{\mu}_m^* \hat{e}(r_1)\hat{\mu}_n\big)^* \\
&= \phi\big(n^{1/2}m^{-1/2}(\mu_m^* e(r_1)\mu_n)\big)^*.
\end{aligned}$$

So ϕ is $*$-preserving.

For the last part we shall use the relations (a)-(d). Now

$$\sqrt{\frac{n}{m}}\big(\mu_m^* e(r_1)\mu_n\big)\sqrt{\frac{l}{k}}\big(\mu_k^* e(r_2)\mu_l\big) = \sqrt{\frac{nl}{mk}}\big(\mu_m^* e(r_1)\mu_n\mu_k^* e(r_2)\mu_l\big).$$

Let $q = [n, k]$ (the least common multiple). Then by using the relations (a),(b) we find

$$\mu_n = \mu_{q/n}^*\mu_q \text{ and } \mu_k^* = \mu_q^*\mu_{q/k}. \tag{5.3}$$

By writing $\nu_{m,r_{\mathbf{1}},n} = \sqrt{\frac{n}{m}}(\mu_m^* e(r_1)\mu_n)$ and $\nu_{k,r_{\mathbf{2}},l} = \sqrt{\frac{l}{k}}(\mu_k^* e(r_2)\mu_l)$ and using relations (a),(c),(d) and equation 5.3 we conclude

$$\begin{aligned}
&(\nu_{m,r_{\mathbf{1}},n})(\nu_{k,r_{\mathbf{2}},l})\\
&\quad= \sqrt{\frac{nl}{mk}}\big(\mu_m^* e(r_1)\mu_{q/n}^*\mu_q\mu_q^*\mu_{q/k} e(r_2)\mu_l\big)\\
&\quad= \sqrt{\frac{nl}{mk}}\big(\mu_m^*\mu_{q/n}^*\mu_{q/n} e(r_1)\mu_{q/n}^*(q^{-1}\sum_{j=1}^{q} e(j/q))\mu_{q/k} e(r_2)\mu_{q/k}^*\mu_{q/k}\mu_l\big)\\
&\quad= \sqrt{\frac{nl}{mk}}\mu_{mq/n}^*\big(nq^{-1}\sum_{i=1}^{q/n} e(nr_1/q + ni/q)\big)\big(q^{-1}\sum_{j=1}^{q} e(j/q)\big)\\
&\qquad\qquad\big(kq^{-1}\sum_{d=1}^{q/k} e(kr_2/q + kd/q)\big)\mu_{lq/k}
\end{aligned}$$

Thus

$$\begin{aligned}
\phi\big((\nu_{m,r_{\mathbf{1}},n})(\nu_{k,r_{\mathbf{2}},l})\big) &= \sqrt{\frac{nl}{mk}}\hat{\mu}_{mq/n}^*\big(nq^{-1}\sum_{i=1}^{q/n}\hat{e}(nr_1/q + ni/q)\big)\\
&\qquad\big(q^{-1}\sum_{j=1}^{q}\hat{e}(j/q)\big)\big(kq^{-1}\sum_{d=1}^{q/k}\hat{e}(kr_2/q + kd/q)\big)\hat{\mu}_{lq/k}
\end{aligned}$$

Since these calculations of $\nu_{m,r_{\mathbf{1}},n}$ and $\nu_{k,r_{\mathbf{2}},l}$ used the relations (a)-(d), we can reverse it for the elements

$$\hat{\nu}_{m,r_{\mathbf{1}},n} = \sqrt{\frac{n}{m}}(\hat{\mu}_m^*\hat{e}(r_1)\hat{\mu}_n) \text{ and } \hat{\nu}_{k,r_{\mathbf{2}},l} = \sqrt{\frac{l}{k}}(\hat{\mu}_k^*\hat{e}(r_2)\hat{\mu}_l)$$

to get that

$$\phi\big((\nu_{m,r_{\mathbf{1}},n})(\nu_{k,r_{\mathbf{2}},l})\big) = \phi\big(\nu_{m,r_{\mathbf{1}},n}\big)\phi\big(\nu_{k,r_{\mathbf{2}},l}\big).$$

Thus ϕ is multiplicative. $\square$

Hence, the proof of the theorem of Bost and Connes is finished.

Remark 5.6. This is actually the improvement of the theorem of Bost and Connes as given in [3] but with a detailed proof. It is proved in [1, proposition 2.8] that the other two relations of Bost and Connes follow.

□

6. The Hecke Pair $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$

Define the group

$$P_{\mathbb{Q}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & r \end{pmatrix} : a, r \in \mathbb{Q}, r \neq 0 \right\},$$

and the subgroup

$$P_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

For every $0 \neq r \in \mathbb{Q}$ express $r = p/q$, where $|p|$ and q are relatively prime in $\mathbb{N}$, then each double coset StS may be written as the disjoint union of q single left cosets as in the Hecke pair $(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}})$.
Thus the pair $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is a Hecke pair.

7. Presentation of the Hecke Algebra $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$

Consider the Hecke pair $(G, S) = (P_{\mathbb{Q}}, P_{\mathbb{Z}})$. We shall use the new notation :

$$u = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].$$

Lemma 7.1. *Consider the Hecke pair (G, S) where $G = P_{\mathbb{Q}}$ and $S = P_{\mathbb{Z}}$, and define $\mu_{-1} = |-1| \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right] = u$. Then for $f \in H(G, S)$ we have*

$$\big(\mu_{-1} * f\big)(StS) = f\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)tS\big) \text{ for all } t \in G.$$

Proof. Let $t \in G$. Then

$$\big(\mu_{-1} * f\big)(StS) = \sum_{pS \in G/S} \mu_{-1}(SpS) f(Sp^{-1}tS).$$

By applying the definition of μ_{-1} and Lemma 5.2 the result follows. □

Corollary 7.2. *Consider the Hecke pair (G, S) where $G = P_{\mathbb{Q}}$ and $S = P_{\mathbb{Z}}$, then for $n \in \mathbb{N}^*$ we have*

$$\big(\mu_n * u\big)(StS) = n^{-1/2} u\big(S\big(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\big)tS\big) \text{ for all } t \in G.$$

Proof. Let $t \in G$. Then

$$\big(\mu_n * u\big)(StS) = \sum_{pS \in G/S} \mu_n(SpS) u(Sp^{-1}tS).$$

Using the proof of Lemma 5.2 the result follows. □

Corollary 7.3. *Consider the Hecke pair (G, S) where $G = P_{\mathbb{Q}}$ and $S = P_{\mathbb{Z}}$, then for $r \in \mathbb{Q}/\mathbb{Z}$ we have*

$$\big(e(r) * u\big)(StS) = u\big(S\big(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\big)tS\big) \text{ for all } t \in G.$$

Proof. This follows directly from the proof of Lemma 5.3. □

Theorem 7.4. *$H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ is the universal $*$-algebra generated by elements $\{\mu_n : n \in \mathbb{N}^*\}$,$\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$, and $u = \big[\big(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\big)\big]$ subject to the relations*
(a) $\mu_n^* \mu_n = 1$ *for all* $n \in \mathbb{N}^*$.
(b) $\mu_{mn} = \mu_m \mu_n$ *for all* m, n.
(c) $e(r)^* = e(-r)$, $e(r_1 + r_2) = e(r_1)e(r_2)$ *for all* $r_1, r_2 \in \mathbb{Q}/\mathbb{Z}$.
(d) $\mu_n e(r) \mu_n^* = (1/n) \sum_{j=1}^{n} e(r/n + j/n)$ *for all* n *and all* r.
(e) $u^* = u$, $u^2 = 1$.
(f) $u\mu_n = \mu_n u$ *for all* $n \in \mathbb{N}^*$.
(g) $e(r)u = ue(-r)$ *for all* $r \in \mathbb{Q}/\mathbb{Z}$.

Proof. Observe that $H(P^+_{\mathbb{Q}}, P_{\mathbb{Z}}) \subset H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, by Lemma 2.8, and this is because we have the same subgroup $P_{\mathbb{Z}}$ and $P^+_{\mathbb{Q}} \subset P_{\mathbb{Q}}$. By Theorem 5.1 we need to show only (e),(f) and (g).

For part (e) let $t \in G$. Then

$$\begin{aligned} u^*(StS) &= \overline{u(St^{-1}S)} \\ &= \begin{cases} 1 & \text{if } St^{-1}S = S\big(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\big)S \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } StS = S\big(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\big)S \\ 0 & \text{otherwise} \end{cases} \\ &= u(StS). \end{aligned}$$

For the second part of (e), we use Lemma 7.1 to get

$$\begin{aligned} u^2(StS) &= (u * u)(StS) \\ &= u\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS\big) \\ &= \begin{cases} 1 & \text{if } S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS = S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)t \in \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)S \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now that t satisfies

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S = \left\{ \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix} : i \in \mathbb{Z} \right\}$$

is equivalent to saying that there exists $i \in \mathbb{Z}$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t = \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix},$$

and thus equivalent to

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} u^2(StS) &= \begin{cases} 1 & \text{if } t = \left(\begin{smallmatrix} 1 & i \\ 0 & 1 \end{smallmatrix}\right) \\ 0 & \text{otherwise} \end{cases} \\ &= 1(StS). \end{aligned}$$

For part (f), let $t \in G$. Then Lemma 7.1 says

$$(u * \mu_n)(StS) = \mu_n\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS\big).$$

Now $\mu_n\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS\big)$ vanishes unless

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t \in \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} S = \left\{ \begin{pmatrix} 1 & j \\ 0 & n \end{pmatrix} : j \in \mathbb{Z} \right\},$$

and this is equivalent to saying that there exists $j \in \mathbb{Z}$ such that

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & j \\ 0 & n \end{pmatrix} = \begin{pmatrix} 1 & j \\ 0 & -n \end{pmatrix}.$$

Hence $(u * \mu_n)(StS) = 0$ unless there exists $j \in \mathbb{Z}$ such that $t = \left(\begin{smallmatrix} 1 & j \\ 0 & -n \end{smallmatrix}\right)$, and if so

$$(u * \mu_n)(StS) = \mu_n\big(S\left(\begin{smallmatrix} 1 & j \\ 0 & n \end{smallmatrix}\right)S\big) = n^{-1/2}.$$

On the other hand, using Corollary 7.2 we get

$$(\mu_n * u)(StS) = n^{-1/2} u\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\right)tS\big).$$

Now $u\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/n \end{smallmatrix}\right)tS\big)$ vanishes unless

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} t \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S = \Big\{ \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} : \alpha \in \mathbb{Z} \Big\}$$

which is equivalent to saying that there exists $\alpha \in \mathbb{Z}$ such that

$$\begin{aligned} t &= \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}^{-1} \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \alpha \\ 0 & -n \end{pmatrix}. \end{aligned}$$

Hence $(\mu_n * u)(StS) = 0$ unless there exists $\alpha \in \mathbb{Z}$ such that $t = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & -n \end{smallmatrix}\right)$, and if so

$$(\mu_n * u)(StS) = n^{-1/2} u\big(S\left(\begin{smallmatrix} 1 & \alpha \\ 0 & -1 \end{smallmatrix}\right)S\big) = n^{-1/2}.$$

Thus $(\mu_n * u) = (u * \mu_n)$.

For part (g), let $t \in G$. On the one hand Corollary 7.3 says

$$(e(r) * u)(StS) = u\big(S\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right)tS\big).$$

Now $u\big(S\left(\begin{smallmatrix} 1 & -r \\ 0 & 1 \end{smallmatrix}\right)tS\big)$ vanishes unless

$$\begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} t \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S = \Big\{ \begin{pmatrix} 1 & \beta \\ 0 & -1 \end{pmatrix} : \beta \in \mathbb{Z} \Big\}$$

and this is equivalent to saying that there exists $\beta \in \mathbb{Z}$ such that

$$\begin{aligned} t &= \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \beta \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & \beta - r \\ 0 & -1 \end{pmatrix}.$$

Hence $(e(r) * u)(StS) = 0$ unless there exists $\beta \in \mathbb{Z}$ such that $t = \left(\begin{smallmatrix} 1 & \beta-r \\ 0 & -1 \end{smallmatrix}\right)$, and if so

$$(e(r) * u)(StS) = u\big(S\left(\begin{smallmatrix} 1 & \beta \\ 0 & -1 \end{smallmatrix}\right)S\big) = 1.$$

On the other hand Lemma 7.1 says

$$\big(u * e(-r)\big)(StS) = e(-r)\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS\big).$$

Now $e(-r)\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS\big)$ vanishes unless

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t \in \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} S = \left\{ \begin{pmatrix} 1 & d-r \\ 0 & 1 \end{pmatrix} : d \in \mathbb{Z} \right\}$$

and this is equivalent to saying that there exists $d \in \mathbb{Z}$ such that

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & d-r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d-r \\ 0 & -1 \end{pmatrix}.$$

Hence $\big(u * e(-r)\big)(StS) = 0$ unless there exists $d \in \mathbb{Z}$ such that $t = \left(\begin{smallmatrix} 1 & d-r \\ 0 & -1 \end{smallmatrix}\right)$, and if so

$$(u * e(-r))(StS) = u\big(S\left(\begin{smallmatrix} 1 & d-r \\ 0 & 1 \end{smallmatrix}\right)tS\big) = 1.$$

Thus

$$e(r) * u = u * e(-r).$$

Next is to show that the elements $\{\mu_n : n \in \mathbb{N}^*\}$,$\{e(r) : r \in \mathbb{Q}/\mathbb{Z}\}$ and $u = \big[\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\big]$ generate the $*$-algebra $(P_{\mathbb{Q}}, P_{\mathbb{Z}})$.

Claim. Let $\left(\begin{smallmatrix} 1 & a \\ 0 & -n/m \end{smallmatrix}\right) \in G$. Then for every $t \in G$

$$\left[\begin{pmatrix} 1 & a \\ 0 & -n/m \end{pmatrix}\right](StS) = \Big(u\sqrt{\frac{n}{m}}(\mu_m^* e(a/n)\mu_n)\Big)(StS).$$

Proof.

$$\begin{aligned} \Big(u\sqrt{\frac{n}{m}}(\mu_m^* e(a/n)\mu_n)\Big)(StS) &= \big(u\big[\left(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\right)\big]\big)(StS) \\ &= \big[\left(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\right)\big]\big(S\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)tS\big) \\ &= \big[\left(\begin{smallmatrix} 1 & a \\ 0 & n/m \end{smallmatrix}\right)\big]\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)StS\big) \\ &= \big[\left(\begin{smallmatrix} 1 & a \\ 0 & -n/m \end{smallmatrix}\right)\big](StS) \end{aligned}$$

$\square$

This shows that the set $u\mu_m^* e(r)\mu_n$ spans $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$, and that the set $B = \left\{\sqrt{\frac{n}{m}}\ (u\mu_m^* e(a/n)\mu_n) : S\left(\begin{smallmatrix} 1 & a \\ 0 & -n/m \end{smallmatrix}\right)S \in S\backslash G/S\right\}$ is a basis for the vector space $H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$ by the proof of Theorem 5.1.

For the last part of the proof suppose that $\left\{\hat{\mu}_n : n \in \mathbb{N}^*\right\}$, $\left\{\hat{e}(r) : r \in \mathbb{Q}/\mathbb{Z}\right\}$ and $\hat{u}$ are elements in a $*$-algebra A which also satisfy the relations (a)-(g). We need to show that there is a $*$-homomorphism $\phi : H(P_{\mathbb{Q}}, P_{\mathbb{Z}}) \longrightarrow A$ such that $\phi(\mu_n) = \hat{\mu}_n$, $\phi(e(r)) = \hat{e}(r)$ and $\phi(u) = \hat{u}$. But since B is a basis, there is a linear transformation ϕ satisfying these things.

Claim. The map $\phi : H(P_{\mathbb{Q}}, P_{\mathbb{Z}}) \longrightarrow A$ defined by $\phi(\sqrt{\frac{n}{m}}(u\mu_m^* e(r)\mu_n)) = \sqrt{\frac{n}{m}}\ (\hat{u}\hat{\mu}_m^* \hat{e}(r)\hat{\mu}_n)$ is a $*$-homomorphism.

Proof. We still need to prove that ϕ is $*$-preserving and that it is multiplicative.

For $*$-preserving we will use relation (g), relation (f) and its adjoint

$$\begin{aligned}
\phi\Big(\sqrt{\frac{n}{m}}(u\mu_m^* e(r)\mu_n)^*\Big) &= \phi\Big(\sqrt{\frac{n}{m}}(\mu_n^* e(-r)\mu_m u)\Big) \\
&= \phi\Big(\sqrt{\frac{n}{m}}(\mu_n^* e(-r)u\mu_m)\Big) \\
&= \phi\Big(\sqrt{\frac{n}{m}}(\mu_n^* u e(r)\mu_m)\Big) \\
&= \phi\Big(\sqrt{\frac{n}{m}}(u\mu_n^* e(r)\mu_m)\Big) \\
&= \sqrt{\frac{n}{m}}(\hat{u}\hat{\mu}_n^* \hat{e}(r)\hat{\mu}_m) \\
&= \sqrt{\frac{n}{m}}(\hat{\mu}_m^* \hat{e}(-r)\hat{\mu}_n \hat{u})^* \\
&= \sqrt{\frac{n}{m}}(\hat{u}\hat{\mu}_m^* \hat{e}(r)\hat{\mu}_n)^* \\
&= \phi\Big(\sqrt{\frac{n}{m}}(u\mu_m^* e(r)\mu_n)\Big)^*.
\end{aligned}$$

So ϕ is $*$-preserving.

For the last part let $q = [n, k]$ (the least common multiple), and let $\nu_1, \nu_2 \in H(P_{\mathbb{Q}}, P_{\mathbb{Z}})$. Then we will have four cases here

1. $\nu_1\nu_2 = \sqrt{\frac{n}{m}}(\mu_m^* e(r_1)\mu_n)\sqrt{\frac{l}{k}}(\mu_k^* e(r_2)\mu_l)$

2. $\nu_1\nu_2 = \sqrt{\frac{n}{m}}(u\mu_m^* e(r_1)\mu_n)\sqrt{\frac{l}{k}}(\mu_k^* e(r_2)\mu_l)$

3. $\nu_1\nu_2 = \sqrt{\frac{n}{m}}(\mu_m^* e(r_1)\mu_n)\sqrt{\frac{l}{k}}(u\mu_k^* e(r_2)\mu_l)$

4. $\nu_1\nu_2 = \sqrt{\frac{n}{m}}(u\mu_m^* e(r_1)\mu_n)\sqrt{\frac{l}{k}}(u\mu_k^* e(r_2)\mu_l)$

We will do case (4) as an example of the calculations. By using the relations (a)-(g) and equation 5.3 we conclude

$$\begin{aligned}
\nu_1\nu_2 &= \sqrt{\frac{n}{m}}(u\mu_m^* e(r_1)\mu_n)\sqrt{\frac{l}{k}}(u\mu_k^* e(r_2)\mu_l) \\
&= \sqrt{\frac{nl}{mk}}\big(u\mu_m^* e(r_1)u\mu_n\mu_k^* e(r_2)\mu_l\big) \\
&= \sqrt{\frac{nl}{mk}}\big(u\mu_m^* u e(-r_1)\mu_n\mu_k^* e(r_2)\mu_l\big) \\
&= \sqrt{\frac{nl}{mk}}\big(\mu_m^* e(-r_1)\mu_n\mu_k^* e(r_2)\mu_l\big).
\end{aligned}$$

Since ϕ is a homomorphism on $H(P_{\mathbb{Q}}^+, P_{\mathbb{Z}})$ we conclude that

$$\phi\big(\nu_1\nu_2\big) = \phi\big(\nu_1\big)\phi\big(\nu_2\big).$$

Thus ϕ is multiplicative. $\square$

Hence, the proof of our theorem is finished. $\square$

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