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OPTIMAL CONTROL ON THE SPREAD OF SLBS COMPUTER VIRUS MODEL

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Abstract: In this paper, we discuss an optimal control on the spread of SLBS computer virus model. Control strategy by installing antivirus on each subpopulation aims to minimize the number of infective computers (latent and breakingout) and the cost associated with the control. Optimal system condition is determined by using Pontryagin principle and is solved numerically by using Forward-Backward Sweep method in combination with the fourth order Runge-Kutta method. Our numerical simulations show that the strategy by installing antivirus on susceptible computers as preventive give a great influence on suppressing the spread of computer viruses.

AMS Subject Classification: 49J15

Key Words: optimal control, spread of computer virus model, Pontryagin principle, forward-backward sweep method

1. Introduction

A computer viruses are malicious program that can replicate themselves and spread among computers. When user on computer network or use removable storage device, the computers are at hight risk of being infective by a computer

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viruses. During the breakingout period, virus may cause damage on whole computer system and give huge financial lose. However, antivirus software as a tool to agains computer virus has limited ability to solve it [1].

Considering the urgency of controlling the spread of computer viruses, many research formulate models of the spread of computer viruses and concerns with qualitative analysis of the model. Various model was adopted form classic epidemic infection disease model, such as SIR or SIRS model [2][3], SIE model [4], SEIR model [5], etc. However, computer viruses have the main characteristics that differentiate from biological virus, i.e., there are no infective computers that do not have infectivity. Therefore, the spread of computer viruses model should not have exposed computers (E). Moreover, infective computers are categorized as latent or breakingout which depends on condition of viruses stay in it [6]. The latent period that begins when the virus enters the computer until just before the virus damages the computer system, whereas the breakingout period is when the virus begins to destroy the computer system until just before cleaned.

Lu-Xing and Xiaofan [1] has proposed a new epidemic model of computer virus where connected computers in this case are categorized into three subpopulations: susceptible computers ($S \equiv S(t)$), latent computers ($L \equiv L(t)$), and breakout computers ($B \equiv B(t)$). By assuming that every new connected computer is in virus free condition, the compartment diagram for the model is shown in Figure 1 and the model is written below,

$$\frac{dS}{dt} = \mu - \beta_1 LS - \beta_2 BS - \theta S + \gamma_1 L + \gamma_2 B - \delta S, \tag{1a}$$

$$\frac{dL}{dt} = \beta_1 LS + \beta_2 BS + \theta S - \gamma_1 L + \alpha L - \delta L, \tag{1b}$$

$$\frac{dB}{dt} = \alpha L - \gamma_2 B - \delta B, \tag{1c}$$

where μ, β_1, β_2, θ, γ_1, γ_2, δ, α are positive constants. In this model, susceptible computers connect to the network at constant rate μ. Each subpopulation disconnects from network with coefficient δ. Due to the interaction, the coefficients of infection from latent computers and breakingout computers to susceptible computers are denoted by β_1 and β_2. The susceptible computers can be infective due to the influence of removable storage device with coefficient θ. Latent computers may break out and enter the B compartment with coefficient α. Finally it is assumed that latent and breakingout computers are cured with coefficient γ_1 and γ_2, respectively.

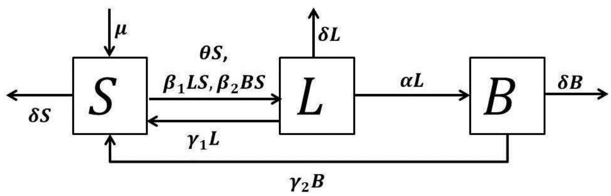

Figure 1: Compartment of the spread of vomputer virus.

Lu-Xing and Xiaofan [1] have shown that system (1) has a unique equilibrium, i.e., the endemic equilibrium which is globally asymptotically stable. This showed that there is no virus-free condition and the virus will always exist in the system. To minimize the number of infective (latent and breakout) computers, other strategies than constant curing have to be implemented to control the spread of computer viruses. Such strategies are usually combined with efforts to minimize the cost related to these strategies. Hence, an important tool in this case is optimal control theory. Lijuan et al. [6] has proposed a control strategy by installing antivirus on breakingout computers to minimize the number of breakingout computers and the costs associated with the control. They found that optimal control solution exists and effective for reducing the number of the breakout computers. However, their numerical simulations showed that the breakout computers still exist and the number of latent computers are increased. In this paper we introduce a different strategy to minimize the number of both latent and breakout computer including the costs associated with the control. Our strategy, the related theoretical optimal control problem and the method to solve this problem will be given in Section 2. In Section 4. numerical simulations are performed to show the influence of control variables. All results are summarized in Section 5.

2. Optimal Control Problem

In this paper we introduce control strategy $u_1 \equiv u_1(t)$, $u_2 \equiv u_2(t)$, and $u_3 \equiv u_3(t)$ that denote control for installing effective antivirus programs on susceptible, latent and breakout computers, respectively. Control u_1 is intended to provide protection for susceptible when interacting with infective computers or removable media. Control u_2 and u_3 are intended to repair infective computers (latent and breakingout). Due to limitation of antivirus abillity, we assume that by applying control u_3 for breakout computers at time t then there will be $\eta u_3 B$ breakout computers which will become susceptibles and $(1-\eta)u_3 B$ breakout computers will become latent, with $\eta \in [0,1]$. By applying such control strategies, model (1) can be written as,

$$\begin{aligned} \frac{dS}{dt} &= \mu - (1-u_1)(\beta_1 LS + \beta_2 BS + \theta S) \\ &\quad + \gamma_1 L + \gamma_2 B - \delta S + u_2 L + \eta u_3 B \end{aligned} \tag{2a}$$

$$\begin{aligned} \frac{dL}{dt} &= (1-u_1)(\beta_1 LS + \beta_2 BS + \theta S) - \gamma_1 L + \\ &\quad \alpha L - \delta L - u_2 L + (1-\eta)u_3 B \end{aligned} \tag{2b}$$

$$\frac{dB}{dt} = \alpha L - \gamma_2 B - \delta B - u_3 B, . \tag{2c}$$

Control variables will be implemented during a time interval $t \in [0,T]$ and they are normalized such that the admissible set is given by,

$$U_{ad} = \{\vec{u} | 0 \le u_1 \le 1, 0 \le u_2 \le 1, 0 \le u_3 \le 1, t \in [0,T]\} \tag{3}$$

where $\vec{u} = (u_1, u_2, u_3)$.

Our objective will be contructed with the following goal:
1. Minimized the number of latent computers and breakingout computers.
2. Minimized the total cost for installing the antivirus program in each subpopulation.

Mathematically, the objective function is to minimize the following functional

$$J(\vec{u}) = \int_0^T [vL + wB + xu_1^2 + yu_2^2 + zu_3^2] dt \tag{4}$$

where v, w, x, y, and z are the weight constants subject to system of equations (2). To solve this optimal control problem we apply the Pontryagin minimum principle. For that, we define adjoint variables $\sigma_1 \equiv \sigma_1(t)$, $\sigma_2 \equiv \sigma_2(t)$, and $\sigma_3 \equiv sigma_3(t)$ for state S, L, and B respectively and consider the following Hamiltonian

$$\begin{aligned} H(S,L,B,\vec{u},\vec{\sigma},t) &= vL + wB + xu_1^2 + yu_2^2 + zu_3^2 + \\ &\quad \sum_{i=1}^{3} \sigma_i f_i(S,L,B,\vec{u}) \end{aligned} \tag{5}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and f_i are the right hand side of system (2), with transversal conditions $\sigma_i(T) = 0$, $i = 1,2,3$.

Theorem 1. *For optimal control problem* (4) *subject to system* (2) *there exists an optimal control* $\vec{u}^{\star}$, *and optimal state solution* $S^{\star}, L^{\star}, B^{\star}$ *so that* $J(\vec{u}^{\star}) = min\ J(\vec{u})$. *Then, there exist adjoint variables* σ_1, σ_2, *and* σ_3 *that satisfy,*

$$\begin{aligned}\frac{\partial \sigma_1}{\partial t} &= \sigma_1((1-u_1)(\beta_1 L + \beta_2 B + \theta) + \delta) - \\ &\quad \sigma_2((1-u_1)(\beta_1 L + \beta_2 B + \theta)) \end{aligned} \tag{6a}$$

$$\begin{aligned}\frac{\partial \sigma_2}{\partial t} &= -v + \sigma_1((1-u_1)\beta_1 S - \gamma_1 - u_2) - \sigma_3 \alpha - \\ &\quad \sigma_2((1-u_1)\beta_1 S - \gamma_1 - \alpha - \delta - u_2) \end{aligned} \tag{6b}$$

$$\begin{aligned}\frac{\partial \sigma_3}{\partial t} &= -w + \sigma_1((1-u_1)\beta_2 S(t) - \gamma_2 - \eta u_3) - \sigma_2 \beta_2 S + \\ &\quad \sigma_2 u_1 \beta_2 S + \sigma_2(1-\eta)u_3 - \sigma_3(\gamma_2 + \delta + u_3) \end{aligned} \tag{6c}$$

with transversal condition $\sigma_i(T) = 0$, $i = 1, 2, 3$.

Proof. The adjoint equations and the transversality conditions are determined from the Hamiltonian (5). Differentiating the Hamiltonian (5) with respect to each state variable, i.e. $\frac{\partial \sigma_1}{\partial t} = -\frac{\partial H}{\partial S}$, $\frac{\partial \sigma_2}{\partial t} = -\frac{\partial H}{\partial L}$ and $\frac{\partial \sigma_3}{\partial t} = -\frac{\partial H}{\partial B}$, we obtain the adjoint system as follows

$$\begin{aligned}\frac{\partial \sigma_1}{\partial t} &= \sigma_1((1-u_1)(\beta_1 L + \beta_2 B + \theta) + \delta) - \\ &\quad \sigma_2((1-u_1)(\beta_1 L + \beta_2 B + \theta)) \end{aligned} \tag{7a}$$

$$\begin{aligned}\frac{\partial \sigma_2}{\partial t} &= -v + \sigma_1((1-u_1)\beta_1 S - \gamma_1 - u_2) - \sigma_3 \alpha - \\ &\quad \sigma_2((1-u_1)\beta_1 S - \gamma_1 - \alpha - \delta - u_2) \end{aligned} \tag{7b}$$

$$\begin{aligned}\frac{\partial \sigma_3}{\partial t} &= -w + \sigma_1((1-u_1)\beta_2 S - \gamma_2 - \eta u_3) - \sigma_3(\gamma_2 + \delta + u_3) - \\ &\quad \sigma_2((1-u_1)\beta_2 S + (1-\eta)u_3) \end{aligned} \tag{7c}$$

In this system, there is no terminal cost and the final state is free. Therefore, the transversality conditions (bounded conditions) for the adjoint variables are

$\sigma_i(T) = 0,\ i = 1,2,3$. Furthermore, the optimal conditions is obtained from the stationary conditions, i.e.,

$$\begin{aligned} 2xu_1^* \quad + \quad & \sigma_1(\beta_1 L^* S^* + \beta_2 B^* S^* + \theta S^*) - \\ & \sigma_2(\beta_1 L^* S^* + \beta_2 B^* S^* + \theta S^*) = 0 \end{aligned} \tag{8a}$$

$$2yu_2^* + \sigma_1 L^* - \sigma_2 L^* = 0 \tag{8b}$$

$$2zu_3^* + \sigma_1 \eta B^* - \sigma_2(1-\eta)B^* - \sigma_3 B^* = 0 \tag{8c}$$

Considering the feature of the admissible set U_{ad}, see equation (3), we obtained

$$u_1^* = min(max(0, \frac{(\sigma_2 - \sigma_1)(\beta_1 LS + \beta_2 BS + \theta S)}{2x}), 1) \tag{9a}$$

$$u_2^* = min(max(0, \frac{(\sigma_2 - \sigma_1)L}{2x}), 1) \tag{9b}$$

$$u_3^* = min(max(0, \frac{(\sigma_3 - \sigma_2(1-\eta) - \sigma_1 \eta)B}{2x}), 1) \tag{9c}$$

Thus the optimal control solution is obtained by solving the following equation system,

$$\frac{dS^*}{dt} = f_1(S^*, L^*, B^*, u_1^*, u_2^*, u_3^*) \tag{10a}$$

$$\frac{dL^*}{dt} = f_2(S^*, L^*, B^*, u_1^*, u_2^*, u_3^*) \tag{10b}$$

$$\frac{dB^*}{dt} = f_3(S^*, L^*, B^*, u_1^*, u_2^*, u_3^*) \tag{10c}$$

$$\begin{aligned} \frac{\partial \sigma_1^*}{\partial t} \quad = \quad & \sigma_1((1-u_1^*)(\beta_1 L^* + \beta_2 B^* + \theta) + \delta) - \\ & \sigma_2^*((1-u_1)(\beta_1 L^* + \beta_2 B^* + \theta)) \end{aligned} \tag{10d}$$

$$\begin{aligned} \frac{\partial \sigma_2^*}{\partial t} \quad = \quad & -v + \sigma_1^*((1-u_1^*)\beta_1 S^* - \gamma_1 - u_2^*) - \\ & \sigma_2((1-u_1^*)\beta_1 S^* - \gamma_1 - \alpha - \delta - u_2^*) - \sigma_3^* \alpha \end{aligned} \tag{10e}$$

$$\begin{aligned} \frac{\partial \sigma_3^*}{\partial t} \quad = \quad & -w + \sigma_1^*((1-u_1^*)\beta_2 S^* - \gamma_2 - \eta u_3^*) - \\ & \sigma_2^*((1-u_1^*)\beta_2 S^* + (1-\eta)u_3^*) - \sigma_3^*(\gamma_2 + \delta + u_3^*. \end{aligned} \tag{10f}$$

□

3. Numerical Simulation

The optimality system (10) in this section is solved numerically using forward-backward Sweep method in combination with the fourth order Runge-Kutta method, see [7] and [8] for the detail. For simulations, we take parameter values from [6] as shown in Tabel 1. The initial population is $S_0 = 15$, $L_0 = 10$, and $B_0 = 10$. The control variables will be implemented during t [0,T] with T = 100.

Table 1: Parameter values for the simulations

μ	δ	β_1	β_2	θ	γ_1	γ_2	α	η
4	0.1	0.02	0.01	0.2	0.2	0.1	0.2	0.8

We first study the influence of each control variable on system (2). For that aim, we carry out simulations by applying one control variable while two other control variables are set to be zero. The results of these simulations are shown in Figure 2. For comparison, we also plot the numerical solution without any control. It is found that without any control, the numerical solution is convergent to endemic equilibrium ($\hat{S}$, $\hat{L}$, $\hat{B}$)=(11.3709, 14.3145, 14.3145). When a control strategy by installing antivirus only on susceptible computers, i.e. only u_1 is implemented, it is optimal to start this strategy at maximum rate. This strategy significantly influences the system which in this case the infective computers (both latent and breakout) are eradicated. At about $t = 90$, u_1 starts to decline. As a result, the number of latent and breakout computers start to increase due to the infection from removable media. If antivirus is installed only on latent computers or only on breakout computers, i.e., applying only u_2 or only u_3, then the optimal control strategies are not at maximum rate. Such optimal control can reduce the number of latent or breakout computers but they are not as effective as u_1. This is because the control strategy by installing antivirus only on latent computers or only on breakout computers can not prevent virus infection from removable storage.

In the second case, we implement all three control variables (u_1, u_2, and u_3) simultaneously, i.e., antivirus is installed on all susceptible, latent and breakout computers. The optimal solution for this case is shown in Figure 3. At the beginning, control u_1 is not at maximum rate due to u_2 and u_3 are also applied. Then, control u_1 is optimal to be implemented at maximum rate, while u_2 and u_3 are set to be zero. This optimal strategy leads to increasing the number of susceptible computers significantly and the number of latent and breakout computers decrease to zero. Since the control variables are stop at $t = 100$, u_1

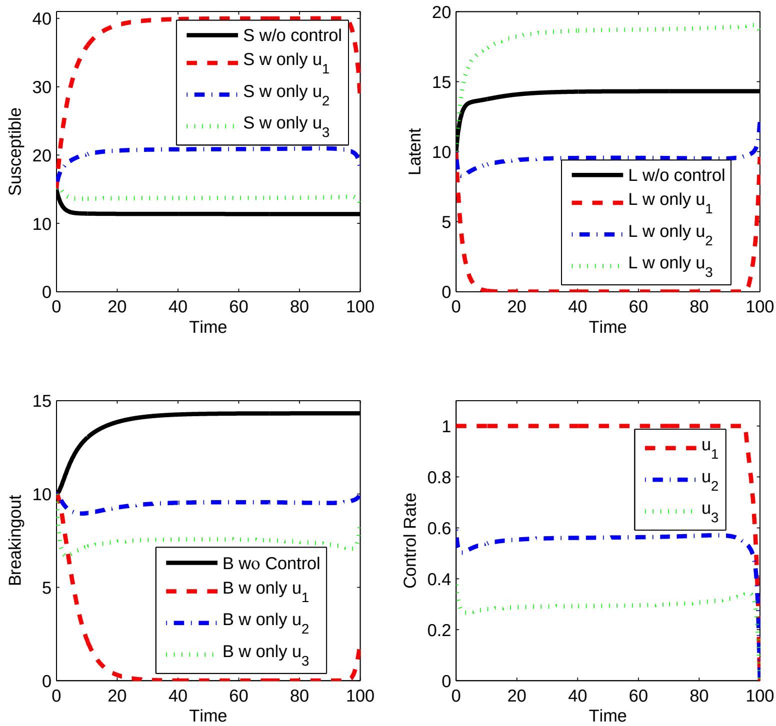

Figure 2: Numerical solutions of system (2) without control (black full-line), with control u_1(red dashed-line), with control u_2 (blue dashed-dotted line), with control u_3 (green dotted-line): (a) susceptible computers, (b) latent computers and (c) breakingout computers. (d) The optimal control rate u_1 (red dahsed-line), u_2 (blue dashed-dotted-line), u_3 (green dotted-line).

starts to decrease at about $t = 90$. With decreasing u_1, latent computers start to appear again due to the infection from removable media so that it is optimal to control the network by targeting the latent and breakout computers. Since all control has to be stop at $t = 100$, while the infection from removable media cannot be stop, the number of latent and breakout computers is getting bigger at the end of control. Figure 2 and Figure 3 show that applying only u_1 and combining all three control variable (u_1, u_2, and u_3) give relatively the same result. This indicates that to suppress the spread of computer viruses is enough

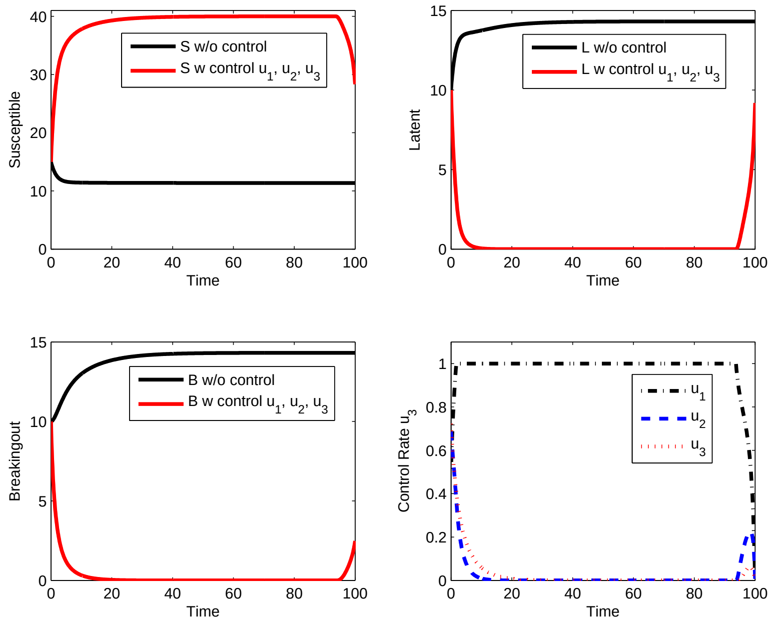

Figure 3: Numerical solutions of system (2) without control (black), with control u_1, u_2 and u_3 (red): (a) susceptible computers, (b) latent computers and (c) breakingout computers. (d) The optimal control rate u_1 (black dashed-dotted line), u_2 (blue dashed-line), u_3 (red dotted-line).

by installing antivirus only on susceptible computers as prevention.

4. Conclusions

In this paper we have discussed an optimal strategy to eradicate virus infection on SLBS computer virus model by installing antivirus program on all sub-population (susceptible, latent and breakout computers). The optimal control strategy is meant to minimize the number of latent and breakout computers as well as the cost associated with installing the antivirus program. The optimal problem is solved using Pontryagin's principle and the obtained optimality system is solved numerically using forward-backward sweep method. Based on the numerical simulations, it is found that installing antivirus program only on susceptible computers as a preventive is the most effective strategy in suppressing the spread of computer viruses.

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