SOFT CLOSURE OPERATORS, SOFT TOPOLOGIES AND SOFT QUASI-UNIFORMITIES

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Abstract: In this paper, we study the notions of soft closure operators in complete residuated lattices. We investigate the relations among soft topologies, soft closure operators and soft $L$-quasi-uniformities in complete residuated lattices. We give their examples.

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1. Introduction

Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structures [5,7-9]. Recently, Molodtsov [11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,3]. Pawlak’s rough set [12,13] can be viewed as a special case of soft rough sets [3]. The topological structures of soft sets have been developed by many researchers [2,7-9,14-17].

Ko [7] introduced a fuzzy soft $F : A \rightarrow L^U$ as an extension as the soft $F : A \rightarrow P(U)$ where $L$ is a complete residuated lattice. Ko [7-9] introduced the soft topological structures, $L$-fuzzy quasi-uniformities and soft $L$-fuzzy topogenous orders in complete residuated lattices.
In this paper, we study the notions of soft closure operators in complete residuated lattices. We investigate the relations among soft topologies, soft closure operators and soft $L$-quasi-uniformities in complete residuated lattices. We give their examples.

2. Preliminaries

**Definition 1.** [4,5] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \rightarrow)$ is a complete residuated lattice.

**Lemma 2.** [4,5] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \rightarrow x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y$,

(4) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(5) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,

(6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,

(7) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$,

(8) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,

(9) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(10) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(11) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,

(12) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$. 
Definition 3. [7-9] Let $X$ be an initial universe of objects and $E$ the set of parameters (attributes) in $X$. A pair $(F, A)$ is called a fuzzy soft set over $X$, where $A \subseteq E$ and $F : A \rightarrow L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter $A$.

Definition 4. [7-9] Let $(F, A)$ and $(G, A)$ be two fuzzy soft sets over a common universe $X$.

1. $(F, A)$ is a fuzzy soft subset of $(G, A)$, denoted by $(F, A) \leq (G, A)$ if $F(a) \leq G(a)$, for each $a \in A$.
2. $(F, A) \wedge (G, A) = (F \wedge G, A)$ if $(F \wedge G)(a) = F(a) \wedge G(a)$ for each $a \in A$.
3. $(F, A) \vee (G, A) = (F \vee G, A)$ if $(F \vee G)(a) = F(a) \vee G(a)$ for each $a \in A$.
4. $(F, A) \odot (G, A) = (F \odot G, A)$ if $(F \odot G)(a) = F(a) \odot G(a)$ for each $a \in A$.
5. $\alpha \odot (F, A) = (\alpha \odot F, A)$ for each $\alpha \in L$.

Definition 5. [7-9] Let $S(X, A)$ and $S(Y, B)$ be the families of all fuzzy soft sets over $X$ and $Y$, respectively. The mapping $f \phi : S(X, A) \rightarrow S(Y, B)$ is a soft mapping where $f : X \rightarrow Y$ and $\phi : A \rightarrow B$ are mappings.

1. The image of $(F, A) \in S(X, A)$ under the mapping $f \phi$ is denoted by $f \phi((F, A)) = (f \phi(F), B)$, where
   \[
   f \phi(F)(b)(y) = \begin{cases} 
   \bigvee_{a \in \phi^{-1}({\{b\}})} f^{-}(F(a))(y), & \text{if } \phi^{-1}({\{b\}}) \neq \emptyset, \\
   0, & \text{otherwise}.
   \end{cases}
   \]
2. The inverse image of $(G, B) \in S(Y, B)$ under the mapping $f \phi$ is denoted by $f^{-1}(G, B) = (f^{-1}(G), A)$ where
   \[
   f^{-1}(G)(a)(x) = f^{\leftarrow}(G(\phi(a)))(x), \forall a \in A, x \in X.
   \]
3. The soft mapping $f \phi : S(X, A) \rightarrow S(Y, B)$ is called injective (resp. surjective, bijective) if $f$ and $\phi$ are both injective (resp. surjective, bijective).

Lemma 6. [7-9] Let $f \phi : S(X, A) \rightarrow S(Y, B)$ be a soft mapping. Then we have the following properties. For $(F, A), (F_i, A) \in S(X, A)$ and $(G, B), (G_i, B) \in S(Y, B)$,

1. $(G, B) \geq f \phi(f^{-1}(G, B))$ with equality if $f$ is surjective,
(2) \((F, A) \leq f_\phi^{-1}(f_\phi((F, A)))\) with equality if \(f\) is injective,
(3) \(f_\phi^{-1}(\bigvee_{i \in I}(G_i, B)) = \bigvee_{i \in I} f_\phi^{-1}((G_i, B)),\)
(4) \(f_\phi^{-1}(\bigwedge_{i \in I}(G_i, B)) = \bigwedge_{i \in I} f_\phi^{-1}((G_i, B)),\)
(5) \(f_\phi(\bigvee_{i \in I}(F, A)) = \bigvee_{i \in I} f_\phi((F, A)),\)
(6) \(f_\phi(\bigwedge_{i \in I}(F, A)) \leq \bigwedge_{i \in I} f_\phi((F, A))\) with equality if \(f\) is injective,
(7) \(f_\phi^{-1}((G_1, B) \odot (G_2, B)) = f_\phi^{-1}((G_1, B)) \odot f_\phi^{-1}((G_2, B)),\)
(8) \(f_\phi((F_1, A) \odot (F_2, A)) \leq f_\phi((F_1, A)) \odot f_\phi((F_2, A))\) with equality if \(f\) is injective.

**Definition 7.** [7-9] A map \(\tau \subset S(X, A)\) is called a soft topology on \(X\) if it satisfies the following conditions.

(ST1) \((0_X, A), (1_X, A) \in \tau, \)where \(0_X(a)(x) = 0, 1_X(a)(x) = 1\) for all \(a \in A, x \in X,\)
(ST2) If \((F, A), (G, A) \in \tau, \)then \((F, A) \odot (G, A) \in \tau,\)
(ST3) If \((F_i, A) \in \tau \)for each \(i \in I, \bigvee_{i \in I}(F_i, A) \in \tau.\)

The triple \((X, A, \tau)\) is called a soft topological space.

Let \((X, A, \tau_X)\) and \((Y, B, \tau_Y)\) be soft topological spaces. A soft map \(f_\phi : (X, A, \tau_X) \rightarrow (Y, B, \tau_Y)\) is called a continuous soft map if \(f_\phi^{-1}(G, B) \in \tau_X,\) for all \((G, B) \in \tau_Y.\)

**Definition 8.** [7-9] A subset \(U \subset S(X \times X, A)\) is called a soft \(L\)-quasi-uniformity on \(X\) if it satisfies the properties.

(SU1) \((1_{X \times X}, A) \in U.\)
(SU2) If \((V, A) \leq (U, A)\) and \((V, A) \in U, \)then \((U, A) \in U.\)
(SU3) For every \((U, A), (V, A) \in U, \)\((U, A) \odot (V, A) \in U.\)
(SU4) If \((U, A) \in U \)then \((1_\Delta, A) \leq (U, A)\) where

\[
1_\Delta(a)(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y, \end{cases}
\]

(SU5) For every \((U, A) \in U, \)there exists \((V, A) \in U\) such that \((V, A) \odot (V, A) \leq (U, A)\) where

\[
((V, A) \odot (V, A))(a)(x, y) = (V(a) \odot V(a))(x, y) = \bigvee_{z \in X} (V(a)(z, x) \odot V(a)(x, y)), \quad \forall x, y \in X, a \in A.
\]
The triple \((X, A, U_X)\) is called a soft \(L\)-quasi-uniform space.

Let \((X, A, U_X)\) and \((Y, B, U_Y)\) be soft quasi-uniform spaces. A soft map \(f_\phi : (X, A, U_X) \to (Y, B, U_Y)\) is called an uniformly continuous soft map if \((f \times f)_{\phi}^{-1}(V, B) \in U_{X}\), for all \((V, B) \in U_Y\).

### 3. Soft Closure Operators, Soft Topologies and Soft Quasi-Uniformities

**Definition 9.** A mapping \(cl : S(X, A) \to S(X, A)\) is called a soft closure operator if it satisfies the following conditions:

1. \(cl(0_X, A) = (0_X, A)\),
2. \(cl(F, A) \geq (F, A)\),
3. If \((F, A) \leq (G, A)\), then \(cl(F, A) \leq cl(G, A)\),
4. \(cl(cl(F, A)) = (F, A)\),
5. \(cl((F, A) \odot (G, A)) \leq cl(F, A) \odot cl(G, A)\).

The triple \((X, A, cl)\) is called a soft closure space.

Let \((X, A, cl_X)\) and \((Y, B, cl_Y)\) be soft closure spaces and \(f_\phi : (X, A) \to (Y, B)\) be a map. Then \(f_\phi\) is called a closure soft map if, for each \((F, A) \in S(X, A)\),

\[ cl_Y(f_\phi(F, A)) \geq f_\phi(cl_X(F, A)) \].

**Theorem 10.** Let \((X, A, U)\) be a soft quasi-uniform space. Define \(cl^r_U, cl^l_U : S(X, A) \to S(X, A)\) as follows

\[
cl^r_U(F, A)(y) = \bigwedge_{(U, A) \in U} \left( \bigvee_{x \in X} (U, A)(y, x) \odot (F, A)(x) \right),
\]

\[
cl^l_U(F, A)(y) = \bigwedge_{(U, A) \in U} \left( \bigvee_{x \in X} (U, A)(x, y) \odot (F, A)(x) \right).
\]

Then, for \(cl \in \{cl^r_U, cl^l_U\}\), we have following properties:

1. \(cl(0_X, A) = (0_X, A)\) and \(cl(F, A) \leq cl(G, A)\) for \((F, A) \leq (G, A)\).
2. \((F, A) \leq cl(F, A)\).
3. \(cl(cl(F, A)) = cl(F, A)\).
4. If \(L\) satisfies \(a \odot \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \odot b_i)\), then \(cl((F, A) \odot (G, A)) \leq cl(F, A) \odot cl(G, A)\).
Proof. (1) Since \((U, A)(y, x) \circ (0_X, A)(x) = (0_X, A)(y)\), \(cl(0_X, A) = (0_X, A)\). Other case it is trivial.

(2) For \((U, A) \in U\), since \((1_\Delta, A) \leq (U, A)\) from (SU4),
\[
\bigvee_{x \in X} (U, A)(y, x) \circ (F, A)(x)) \geq \bigvee_{x \in X} (1_\Delta, A)(y, x) \circ (F, A)(x) = (F, A)(x).
\]
Hence \(cl_U^r(F, A) \geq (F, A)\).

(3)
\[
cl_U(F, A)(y) = \bigwedge_{(U, A) \in U} (\bigvee_{x \in X} (U, A)(y, x) \circ (F, A)(x)) \\
\geq \bigwedge_{(U_1, A) \in U} (\bigvee_{x \in X} \bigvee_{z \in X} (U_1, A)(y, z) \\
\circ (U_1, A)(z, x) \circ (F, A)(x)) \text{ (by (SU5))} \\
\geq \bigwedge_{(U_1, A) \in U} (\bigvee_{x \in X} (U_1, A)(y, z)) \\
\bigwedge_{(U_1, A) \in U} \bigvee_{z \in X} (U_1, A)(z, x) \circ (F, A)(x)) \\
= \bigwedge_{(U_1, A) \in U} (\bigvee_{z \in X} (U_1, A)(y, z) \circ cl_U(F, A)(z)) \\
= cl_U^r(cl_U(F, A))(y).
\]

(4)
\[
cl_U^r((F, A) \circ (G, A))(y) \\
= \bigwedge_{U \in U} (\bigvee_{x \in X} (U, A)(y, x) \circ (F, A)(x) \circ (G, A)(x)) \\
= \bigwedge_{U_1 \circ U_2 \in U} (\bigvee_{x \in X} (U_1, A)(y, x) \circ (U_2, A)(y, x) \\
\circ (F, A)(x) \circ (G, A)(x)) \\
\leq \bigwedge_{U_1 \in U} (\bigvee_{x \in X} (U_1, A)(y, x) \circ (U_2, A)(y, x) \\
\circ (F, A)(x) \circ (G, A)(x)) \\
= \bigwedge_{U_1 \in U} (\bigvee_{x \in X} (U_1, A)(y, x) \circ (F, A)(x)) \\
\bigwedge_{U_2 \in U} (\bigvee_{x \in X} (U_2, A)(y, x) \circ (G, A)(x)) \\
= cl_U(F, A)(y) \circ cl_U^r(G, A)(y).
\]

For \(cl_U^l\), it is similarly proved.

Remark 11. If \((L, \circ)\) is a continuous t-norm, then \(a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i)\).

Theorem 12. Let \((X, A, U)\) be a soft quasi-uniform space and \(a \circ \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \circ b_i)\) for \(a, b_i \in L\). Define \(\tau_U^r, \tau_U^l \subset S(X, A)\) as follows
\[
\tau_U^r = \{(F, A) \in S(X, A) \mid cl_U^r(F, A) = (F, A)\},
\]
\[
\tau_U^l = \{(F, A) \in S(X, A) \mid cl_U^l(F, A) = (F, A)\}.
\]
Then (1) \( \tau^r_X \) is a soft topology on \( X \) such that \( \tau^r_X = \{ cl^r_U(F, A) \mid (F, A) \in S(X, A) \} \).

(2) \( \tau^l_X \) is a soft topology on \( X \) such that \( \tau^l_X = \{ cl^l_U(F, A) \mid (F, A) \in S(X, A) \} \).

Proof. (1) (ST1) Since \( cl^r_U(0_X, A) = (0_X, A) \) and \( cl^r_U(1_X, A) \geq (1_X, A) \), then \((0_X, A), (1_X, A) \in \tau^r_U \).

(1) (ST2) Let \((F, A), (G, A) \in \tau^r_U \). Then \( cl^r_U(F, A) = (F, A) \) and \( cl^r_U(G, A) = (G, A) \). It follows

\[
cl^r_U((F, A) \circ (G, A)) \leq cl^r_U(F, A) \circ cl^r_U(G, A) = (F, A) \circ (G, A).
\]

Thus \((F, A) \circ (G, A) \in \tau^r_U \).

(ST3) Let \((F_i, A) \in \tau^r_U \) for each \( i \in I \). Then

\[
\bigwedge_{U \in U} \left( \bigvee_{x \in X} (U, A)(y, x) \circ (F_i, A)(x) \right) = (F_i, A)(y).
\]

By Theorem 10(2), \( c^r_U(\bigvee_{i \in I}(F_i, A)) = \bigvee_{i \in I}(F_i, A) \). So, \( \bigvee_{i \in I}(F_i, A) \in \tau^r_U \). Thus \( \tau^r_U \) is a soft topology. Put \( \tau = \{ cl^r_U(F, A) \mid (F, A) \in S(X, A) \} \). Let \( cl^r_U(F, A) \in \tau \). Since \( cl^r_U(cl^r_U(F, A)) = cl^r_U(F, A) \), \( cl^r_U(F, A) \in \tau^r_U \). Let \( (F, A) \in \tau^r_U \). Since \( (F, A) = cl^r_U(F, A) \), \( (F, A) \in \tau \). Hence \( \tau = \tau^r_U \).

(2) It is similarly proved in (1).

**Theorem 13.** Let \( f_\phi : (X, A, U_X) \to (Y, B, U_Y) \) be an uniform continuous soft map. Then

(1) \( f_\phi : (X, A, \tau^r_{U_X}) \to (Y, B, \tau^r_{U_Y}) \) is a continuous soft map.

(2) \( f_\phi : (X, A, \tau^l_{U_X}) \to (Y, B, \tau^l_{U_Y}) \) is a continuous soft map.

(3) \( f_\phi(cl^r_U(F, A)) \leq cl^r_{U_Y}(f_\phi(F, A)). \)

(4) \( f_\phi(cl^l_U(F, A)) \leq cl^l_{U_Y}(f_\phi(F, A)). \)
Proof (1) Since \((f \times f)^{-1}(V) \in U_X\) for each \((V, B) \in U_Y\), let \((G, B) \in \tau u_Y\), that is,
\[
\bigwedge_{(V, B) \in U_Y} \left( \bigvee_{w \in Y} (V, B)(y, w) \odot (G, B)(w) \right) = (G, B)(y), \forall y \in Y,
\]
we have
\[
\left( \bigwedge_{(V, B) \in U_X} \left( \bigvee_{x \in X} (U, A)(x, z) \odot f_{\phi}^{-1}(G, B)(z) \right) \right)
\leq \left( \bigwedge_{(V, B) \in U_Y} \left( \bigvee_{x \in X} (V, B)(x, f(x)) \odot (G, B)(f(x)) \right) \right)
\leq (G, B)(f(x)) = f_{\phi}^{-1}(G, B)(x).
\]

By Theorem 10(2), \(f_{\phi}^{-1}(G, B) \in \tau u_Y\).

(3)
\[
cl_{U_Y}(f_{\phi}(F, A))(y) = \left( \bigwedge_{(V, B) \in U_Y} \left( \bigvee_{w \in Y} (V, B)(y, w) \odot f_{\phi}(F, A)(w) \right) \right)
\geq \left( \bigwedge_{(V, B) \in U_Y} \left( \bigvee_{x \in X} (V, B)(x, f(x)) \odot f_{\phi}(F, A)(f(x)) \right) \right)
\geq \left( \bigwedge_{(V, B) \in U_Y} \left( \bigvee_{x \in X} \left( (f \times f)^{-1}(V, B)(f(z), f(x)) \odot f_{\phi}(F, A)(f(x)) \right) \right) \right)
\geq \left( \bigwedge_{(V, B) \in U_Y} \left( \bigvee_{x \in X} \left( (f \times f)^{-1}(V, B)(z, x) \odot (F, A)(x) \right) \right) \right)
\geq \left( \bigwedge_{(V, B) \in U_Y} \left( \bigvee_{x \in X} \left( (U, A)(z, x) \right) \right) \right)
\odot (F, A)(x)) = f_{\phi}(cl_{U}(F, A))(y).
\]

(2) and (4) are similarly proved as (1) and (3), respectively.

Lemma 14. For every \((F, A), (G, A) \in S(X, A)\), we define \((U_F, A) \in S(X \times X, A)\) by
\[
U_F(a)(x, y) = F(a)(x) \to F(a)(y).
\]
then we have the following statements:

1. \((1_{X \times X}, A) = (U_0_X, A) = (U_1_X, A),\)
2. \((1_\triangle, A) \leq (U_F, A),\)
3. \((U_F, A) \circ (U_F, A) = (U_F, A),\)
4. \((U_F, A) \circ (U_G, A) \leq (U_{F \circ G}, A).\)
Proof. (1) 

\(1_{X \times X}(a)(x,y) = 1 = U_{0_X}(a)(x,y) = 0_X(a)(x) \rightarrow 0_X(a)(y) = 1_X(a)(x) \rightarrow 1_X(a)(y) = U_{1_X}(a)(x,y).\)

(2) Since \(U_F(a)(x,x) = F(a)(x) \rightarrow F(a)(x) = 0, (1_{\Delta}, A) \leq (U_F, A).\)

(3) \((U_F, A) \circ (U_F, A) \leq (U_F, A)\) from 

\[ (U_F(a) \circ U_F(a))(x,z) = \bigvee_{y \in X} (U_F(a)(x,y) \circ U_F(a)(y,z)) = \bigvee_{y \in X} (F(a)(x) \rightarrow F(a)(y)) \circ (F(a)(y) \rightarrow F(a)(z)) \leq F(a)(x) \rightarrow F(a)(z) = U_F(a)(x,z). \]

(4) By Lemma 2 (12),

\[ U_F(a)(x,y) \odot U_G(a)(x,y) = (F(a)(x) \rightarrow F(a)(y)) \odot (G(a)(x) \rightarrow G(a)(y)) \leq (F(a)(x) \odot G(a)(x) \rightarrow F(a)(y) \odot G(a)(y)) = U_{F \circ G}(a)(x,y). \]

**Theorem 15.** Let \((X, A, \tau)\) be a soft topological space. Define a function \(U_\tau : S(X \times X, A) \rightarrow L\) by

\[ U_\tau = \{(U, A) \in S(X \times X, A) \mid \bigvee_{i=1}^n (U_{G_i}, A) \leq (U, A), (G_i, A) \in \tau \} \]

where the first \(\bigvee\) is taken over every finite family \(\{U_{(G_i,A)} \mid i = 1, \ldots, n\}\).

Then:

(1) \(U_\tau\) is a soft quasi-uniformity on \(X\).

(2) \(\tau \subset \tau^1_{U_\tau} \).

Proof (1) (SU1) Since \((1_X, A) \in \tau\), there exists \((U_{1_X}, A) \in S(X \times X, A)\) such that \((U_{1_X}, A) \in U_\tau\).

(SU2) It is trivial.
(SU3) For \((U, A), (V, A) \in U_\tau\), there exist two finite families \(\{(F_i, A) \in \tau \mid \circ_i^m(U_{F_i}, A) \leq (U, A)\}\) and \(\{(G_j, A) \in \tau \mid \circ_j^n(U_{G_j}, A) \leq (G, A)\}\). Then \((U, A) \odot (V, A) \geq (\circ_i^m(U_{F_i}, A)) \odot (\circ_j^n(U_{G_j}, A))\). So, \((U, A) \odot (V, A) \in U_\tau\).

(SU4) Let \((U, A) \in U_\tau\). Then there exists a finite family \(\{(F_i, A) \in \tau \mid \circ_i^m(U_{F_i}, A) \leq (U, A)\}\). Since \((U_{F_i}, A) \geq (1_\Delta, A)\) from Lemma 14(2),

\[(1_\Delta, A) \leq \circ_i^m(U_{F_i}, A) \leq (U, A)\).

(SU5) Let \((U, A) \in U_\tau\). Then there exists a finite family \(\{(G_i, A) \in \tau \mid \circ_i^m(U_{G_i}, A) \leq (U, A)\}\). Since \((U_{G_i}, A) \odot (U_{G_i}, A) = (U_{G_i}, A)\) for each \(i \in \{1, \ldots, m\}\) from Lemma 14(3), we have \((\circ_i^m(U_{G_i}, A) \odot (\circ_i^m(U_{G_i}, A)) \leq \circ_i^m(U_{G_i}, A)\) from

\[
\bigvee_{y \in X}((\circ_i^m(U_{G_i}(a)(x), y)) \odot (\circ_i^m(U_{G_i}(a)(y, z))) \\
= \bigvee_{y \in X}((\circ_i^m(U_{G_i}(a)(x) \rightarrow G_i(a)(y))) \odot (\circ_i^m(U_{G_i}(a)(y) \rightarrow G_i(a)(z)))) \\
= \bigvee_{y \in X}((\circ_i^m(U_{G_i}(a)(x) \rightarrow G_i(a)(y))) \odot (G_i(a)(y) \rightarrow G_i(a)(z)))) \\
\leq \circ_i^m(U_{G_i}(a)(x) \rightarrow G_i(a)(z)).
\]

Put \((V, A) = \circ_i^m(U_{G_i}, A)\). Then \((V, A) \in U_\tau\) with \((V, A) \odot (V, A) \leq (U, A)\). Hence \(U_\tau\) is a soft quasi-uniformity on \(X\).

(2) Let \((F, A) \in \tau\). Then \((U_{F}, A) \in U_\tau\). Since

\[
\bigwedge_{U \in U} \left(\bigvee_{y \in X}((U_{F}(A)(y, x)) \odot (F, A)(y)) \leq \bigvee_{y \in X}((U_{F}(A)(y, x) \odot (F, A)(y)) \leq (F, A)(x).
\]

Hence \((F, A) \in \tau_U\).

**Theorem 16.** Let \(f_\phi : (X, A, \tau_X) \rightarrow (Y, B, \tau_Y)\) be a continuous soft map. Then \(f_\phi : (X, A, U_{\tau_X}) \rightarrow (Y, B, U_{\tau_Y})\) is an uniformly continuous soft map.

**Proof.** We have

\[
(f \times f)^{-1}(U_{G}, B)(a)(x, y) = (U_{G}, B)(\phi(a))(f(x), f(y)) \\
= G(\phi(a))(f(x)) \rightarrow G(\phi(a))(f(y)) = f^{-1}(G)(a)(x) \rightarrow f^{-1}(G)(a)(y) \\
= U_{f^{-1}(G)}(a)(x, y).
\]

Let \((U, B) \in U_{\tau_Y}\). Then there exists a finite family \(\{(G_i, B) \in \tau_Y \mid \circ_i^m(U_{G_i}, B) \leq (U, B)\}\).
Since $\circ_{i=1}^{m}(U_{G_i}, B) \leq (U, B)$, we have

$$(f \times f)^{-1}_{\phi}(\circ_{i=1}^{m}(U_{G_i}, B)) = \circ_{i=1}^{m}(f \times f)^{-1}_{\phi}(U_{G_i}, B)$$

$$= \circ_{i=1}^{m}(U_{f^{-1}_{\phi}(G_i)}, B) \leq (f \times f)^{-1}_{\phi}((U, B)).$$

So, $(f \times f)^{-1}_{\phi}((U, B)) \in U_{\tau_X}$.

**Example 17.** Let $X = \{h_i \mid i = \{1, \ldots, 4\}\}$ with $h_i$=house and $E_Y = \{e, b, w, c, i\}$ with $e$=expensive,$b$= beautiful, $w$=wooden, $c$= creative, $i$=in the green surroundings.

Let $(L = [0, 1], \odot, \rightarrow)$ be a complete residuated lattice defined by

$$x \odot y = x \land y, \quad x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Let $X = \{x, y, z\}$ be a set and $W(e), W(b) \in S(X \times X, A)$ such that

$$W(e) = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.7 & 1 & 0.8 \\ 0.4 & 0.4 & 1 \end{pmatrix}, \quad W(b) = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{pmatrix}$$

Define $U = \{(U, A) \in S(X \times X, A) \mid (U, A) \geq (W, A)\}$.

(1) Since $W(e) \circ W(e) = W(e)$ and $W(b) \circ W(b) = W(b)$, $U$ is a soft quasi-uniformity on $X$.

(2) We have $\tau_U = \{cl_U^*(F, A) \mid (F, A) \in S(X, A)\}$ where

$$cl_U^*(F, A)(e) = \left( F(e)(x) \lor (0.5 \land F(e)(y)) \lor (0.5 \land F(e)(z)) \right)$$

$$\left( 0.7 \land F(e)(x) \right) \lor F(e)(y) \lor (0.8 \land F(e)(z))$$

$$\left( 0.4 \land F(e)(x) \right) \lor (0.4 \land F(e)(y)) \lor F(e)(z)$$

$$cl_U^*(F, A)(b) = \left( F(b)(x) \lor (0.6 \land F(b)(y)) \lor (0.7 \land F(b)(z)) \right)$$

$$\left( 0.4 \land F(b)(x) \right) \lor F(b)(y) \lor (0.4 \land F(b)(z))$$

$$\left( 0.5 \land F(b)(x) \right) \lor (0.6 \land F(b)(y)) \lor F(b)(z)$$

We have $\tau_U = \{cl_U^*(F, A) \mid (F, A) \in S(X, A)\}$ where

$$cl_U^*(F, A)(e) = \left( F(e)(x) \lor (0.7 \land F(e)(y)) \lor (0.4 \land F(e)(z)) \right)$$

$$\left( 0.5 \land F(e)(x) \right) \lor F(e)(y) \lor (0.4 \land F(e)(z))$$

$$\left( 0.5 \land F(e)(x) \right) \lor (0.8 \land F(e)(y)) \lor F(e)(z)$$
\[ \text{cl}_U(F, A)(b) = \left\{ \begin{array}{ll}
   F(b)(x) \lor (0.4 \land F(b)(y)) \lor (0.5 \land F(b)(z)) \\
   (0.6 \land F(b)(x)) \lor F(b)(y) \lor (0.6 \land F(b)(z)) \\
   (0.7 \land F(b)(x)) \lor (0.4 \land F(b)(y)) \lor F(b)(z) 
\end{array} \right. \]

(3) Let \( \tau = \{(0, X, A), (1, X, A), (F, A)\} \) a soft topology where

\[ F(e) = (0.4, 0.5, 0.6), \quad F(b) = (0.7, 0.4, 0.9), \]

\[ U_F(e) = \begin{pmatrix}
   1 & 1 & 1 \\
   0.4 & 1 & 1 \\
   0.4 & 0.5 & 1 
\end{pmatrix} \quad U_F(b) = \begin{pmatrix}
   1 & 0.4 & 1 \\
   1 & 1 & 1 \\
   0.7 & 0.4 & 1 
\end{pmatrix} \]

Define \( U_\tau = \{(U, A) \in S(X \times X, A) \mid (U, A) \succeq (U_F, A)\} \). Since \( (U_F, A) \circ (U_F, A) = (U_F, A) \), \( U \) is a soft quasi-uniformity.

We have \( \tau_{U_\tau}^l = \{\text{cl}_U^l(G, A) \mid (G, A) \in S(X, A)\} \) where

\[ \text{cl}^l_{U_\tau}(G, A)(e) = \left\{ \begin{array}{ll}
   G(e)(x) \lor (0.4 \land G(e)(y)) \lor (0.4 \land G(e)(z)) \\
   G(e)(x) \lor G(e)(y) \lor (0.5 \land G(e)(z)) \\
   G(e)(x) \lor G(e)(y) \lor G(e)(z) 
\end{array} \right. \]

\[ \text{cl}^l_{U_\tau}(G, A)(b) = \left\{ \begin{array}{ll}
   G(b)(x) \lor G(b)(y) \lor (0.7 \land G(b)(z)) \\
   (0.4 \land G(b)(x)) \lor G(b)(y) \lor (0.4 \land G(b)(z)) \\
   G(e)(x) \lor G(e)(y) \lor G(e)(z) 
\end{array} \right. \]

Since \( \text{cl}^l_{U_\tau}(F, A) = (F, A) \), \( \tau \subset \tau_{U_\tau}^l \).

References


