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SOFT CLOSURE OPERATORS, SOFT TOPOLOGIES AND SOFT QUASI-UNIFORMITIES

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Abstract: In this paper, we study the notions of soft closure operators in complete residuated lattices. We investigate the relations among soft topologies, soft closure operators and soft L-quasi-uniformities in complete residuated lattices. We give their examples.

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1. Introduction

Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. It is an important mathematical tool for algebraic structures [5,7-9]. Recently, Molodtsov [11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Presently, the soft set theory is making progress rapidly [1,3]. Pawlak's rough set [12,13] can be viewed as a special case of soft rough sets [3]. The topological structures of soft sets have been developed by many researchers [2,7-9,14-17].

Ko [7] introduced a fuzzy soft $F : A \to L^U$ as an extension as the soft $F : A \to P(U)$ where L is a complete residuated lattice. Ko [7-9] introduced the soft topological structures, L-fuzzy quasi-uniformities and soft L-fuzzy topogenous orders in complete residuated lattices.

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2. Preliminaries

Definition 1. [4,5] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, \rightarrow)$ is a complete residuated lattice.

Lemma 2. [4,5] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \rightarrow x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

(3) $x \odot y \leq x \wedge y \leq x \vee y$,

(4) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,

(5) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,

(6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,

(7) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$,

(8) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,

(9) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(10) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(11) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,

(12) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

Definition 3. [7-9] Let X be an initial universe of objects and E the set of parameters (attributes) in X. A pair (F, A) is called a *fuzzy soft set* over X, where $A \subset E$ and $F : A \to L^X$ is a mapping. We denote $S(X, A)$ as the family of all fuzzy soft sets under the parameter A.

Definition 4. [7-9] Let (F, A) and (G, A) be two fuzzy soft sets over a common universe X.

(1) (F, A) is a fuzzy soft subset of (G, A), denoted by $(F, A) \le (G, A)$ if $F(a) \le G(a)$, for each $a \in A$.

(2) $(F, A) \wedge (G, A) = (F \wedge G, A)$ if $(F \wedge G)(a) = F(a) \wedge G(a)$ for each $a \in A$.

(3) $(F, A) \vee (G, A) = (F \vee G, A)$ if $(F \vee G)(a) = F(a) \vee G(a)$ for each $a \in A$.

(4) $(F, A) \odot (G, A) = (F \odot G, A)$ if $(F \odot G)(a) = F(a) \odot G(a)$ for each $a \in A$.
(6) $\alpha \odot (F, A) = (\alpha \odot F, A)$ for each $\alpha \in L$.

Definition 5. [7-9] Let $S(X, A)$ and $S(Y, B)$ be the families of all fuzzy soft sets over X and Y, respectively. The mapping $f_\phi : S(X, A) \to S(Y, B)$ is a soft mapping where $f : X \to Y$ and $\phi : A \to B$ are mappings.

(1) The image of $(F, A) \in S(X, A)$ under the mapping f_ϕ is denoted by $f_\phi((F, A)) = (f_\phi(F), B)$, where

$$f_\phi(F)(b)(y) = \begin{cases} \bigvee_{a \in \phi^{-1}(\{b\})} f^{\to}(F(a))(y), & \text{if } \phi^{-1}(\{b\}) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The inverse image of $(G, B) \in S(Y, B)$ under the mapping f_ϕ is denoted by $f_\phi^{-1}((G, B)) = (f_\phi^{-1}(G), A)$ where

$$f_\phi^{-1}(G)(a)(x) = f^{\leftarrow}(G(\phi(a)))(x), \ \forall a \in A, x \in X.$$

(3) The soft mapping $f_\phi : S(X, A) \to S(Y, B)$ is called injective (resp. surjective, bijective) if f and ϕ are both injective (resp. surjective, bijective).

Lemma 6. [7-9] Let $f_\phi : S(X, A) \to S(Y, B)$ be a soft mapping. Then we have the following properties. For $(F, A), (F_i, A) \in S(X, A)$ and $(G, B), (G_i, B) \in S(Y, B)$,

(1) $(G, B) \ge f_\phi(f_\phi^{-1}((G, B)))$ with equality if f is surjective,

(2) $(F,A) \leq f_\phi^{-1}(f_\phi((F,A)))$ with equality if f is injective,

(3) $f_\phi^{-1}(\bigvee_{i\in I}(G_i,B)) = \bigvee_{i\in I} f_\phi^{-1}((G_i,B))$,

(4) $f_\phi^{-1}(\bigwedge_{i\in I}(G_i,B)) = \bigwedge_{i\in I} f_\phi^{-1}((G_i,B))$,

(5) $f_\phi(\bigvee_{i\in I}(F_i,A)) = \bigvee_{i\in I} f_\phi((F_i,A))$,

(6) $f_\phi(\bigwedge_{i\in I}(F_i,A)) \leq \bigwedge_{i\in I} f_\phi((F_i,A))$ with equality if f is injective,

(7) $f_\phi^{-1}((G_1,B)\odot(G_2,B)) = f_\phi^{-1}((G_1,B))\odot f_\phi^{-1}((G_2,B))$,

(8) $f_\phi((F_1,A)\odot(F_2,A)) \leq f_\phi((F_1,A))\odot f_\phi((F_2,A))$ with equality if f is injective.

Definition 7. [7-9] A map $\tau \subset S(X,A)$ is called a soft topology on X if it satisfies the following conditions.

(ST1) $(0_X,A),(1_X,A)\in\tau$, where $0_X(a)(x)=0, 1_X(a)(x)=1$ for all $a\in A, x\in X$,

(ST2) If $(F,A),(G,A)\in\tau$, then $(F,A)\odot(G,A)\in\tau$,

(ST3) If $(F_i,A)\in\tau$ for each $i\in I$, $\bigvee_{i\in I}(F_i,A)\in\tau$.

The triple (X,A,τ) is called a soft topological space.

Let (X,A,τ_X) and (Y,B,τ_Y) be soft topological spaces. A soft map $f_\phi : (X,A,\tau_X)\to(Y,B,\tau_Y)$ is called a continuous soft map if $f_\phi^{-1}(G,B)\in\tau_X$, for all $(G,B)\in\tau_Y$.

Definition 8. [7-9] A subset $\mathbf{U}\subset S(X\times X,A)$ is called a soft L-quasi-uniformity on X iff it satisfies the properties.

(SU1) $(1_{X\times X},A)\in\mathbf{U}$.

(SU2) If $(V,A)\leq(U,A)$ and $(V,A)\in\mathbf{U}$, then $(U,A)\in\mathbf{U}$.

(SU3) For every $(U,A),(V,A)\in\mathbf{U}$, $(U,A)\odot(V,A)\in\mathbf{U}$.

(SU4) If $(U,A)\in\mathbf{U}$ then $(1_\triangle,A)\leq(U,A)$ where

$$1_\triangle(a)(x,y) = \begin{cases} 1, & \text{if } x=y \\ 0, & \text{if } x\neq y, \end{cases}$$

(SU5) For every $(U,A)\in\mathbf{U}$, there exists $(V,A)\in\mathbf{U}$ such that $(V,A)\circ(V,A)\leq(U,A)$ where

$$\begin{aligned}&((V,A)\circ(V,A))(a)(x,y) = (V(a)\circ V(a))(x,y)\\ &= \bigvee_{z\in X}(V(a)(z,x)\odot V(a)(x,y)),\quad \forall\, x,y\in X, a\in A.\end{aligned}$$

The triple $(X, A, \mathbf{U})$ is called a soft L-quasi-uniform space.

Let $(X, A, \mathbf{U}_X)$ and $(Y, B, \mathbf{U}_Y)$ be soft quasi-uniform spaces. A soft map $f_\phi : (X, A, \mathbf{U}_X) \to (Y, B, \mathbf{U}_Y)$ is called an uniformly continuous soft map if $(f \times f)^{-1}_\phi (V, B) \in \mathbf{U}_X$, for all $(V, B) \in \mathbf{U}_Y$.

3. Soft Closure Operators, Soft Topologies and Soft Quasi-Uniformities

Definition 9. A mapping $cl : S(X, A) \to S(X, A)$ is called a soft closure operator if it satisfies the following conditions:

(C1) $cl(0_X, A) = (0_X, A)$,

(C2) $cl(F, A) \geq (F, A)$,

(C3) If $(F, A) \leq (G, A)$, then $cl(F, A) \leq cl(G, A)$,

(C4) $cl(cl(F, A)) = (F, A)$,

(C5) $cl((F, A) \odot (G, A)) \leq cl(F, A) \odot cl(G, A)$.

The triple (X, A, cl) is called a soft closure space.

Let (X, A, cl_X) and (Y, B, cl_Y) be soft closure spaces and $f_\phi : (X, A) \to (Y, B)$ be a map. Then f_ϕ is called a closure soft map if, for each $(F, A) \in S(X, A)$,

$$cl_Y(f_\phi(F, A)) \geq f_\phi(cl_X(F, A)).$$

Theorem 10. *Let $(X, A, \mathbf{U})$ be a soft quasi-uniform space. Define $cl^r_{\mathbf{U}}, cl^l_{\mathbf{U}} : S(X, A) \to S(X, A)$ as follows*

$$cl^r_{\mathbf{U}}(F, A)(y) = \bigwedge_{(U,A)\in\mathbf{U}} \Big(\bigvee_{x\in X} (U, A)(y, x) \odot (F, A)(x)\Big),$$

$$cl^l_{\mathbf{U}}(F, A)(y) = \bigwedge_{(U,A)\in\mathbf{U}} \Big(\bigvee_{x\in X} (U, A)(x, y) \odot (F, A)(x)\Big).$$

Then, for $cl \in \{cl^r_{\mathbf{U}}, cl^l_{\mathbf{U}}\}$, we have following properties:

(1) $cl(0_X, A) = (0_X, A)$ and $cl(F, A) \leq cl(G, A)$ for $(F, A) \leq (G, A)$.

(2) $(F, A) \leq cl(F, A)$.

(3) $cl(cl(F, A)) = cl(F, A)$.

(4) If L satisfies $a \odot \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \odot b_i)$, then $cl((F, A) \odot (G, A)) \leq cl(F, A) \odot cl(G, A)$.

Proof. (1) Since $(U,A)(y,x)\odot(0_X,A)(x)=(0_X,A)(y)$, $cl(0_X,A)=(0_X,A)$. Other case it is trivial.

(2) For $(U,A)\in\mathbf{U}$, since $(1_\triangle,A)\le(U,A)$ from (SU4),

$$\begin{aligned}&\bigvee_{x\in X}(U,A)(y,x)\odot(F,A)(x))\\&\ge\bigvee_{x\in X}(1_\triangle,A)(y,x)\odot(F,A)(x)=(F,A)(x).\end{aligned}$$

Hence $cl^r_{\mathbf{U}}(F,A)\ge(F,A)$.

(3)

$$\begin{aligned}&cl^r_{\mathbf{U}}(F,A)(y)=\bigwedge_{(U,A)\in\mathbf{U}}(\bigvee_{x\in X}(U,A)(y,x)\odot(F,A)(x))\\&\ge\bigwedge_{(U_1,A)\in\mathbf{U}}(\bigvee_{x\in X}\bigvee_{z\in X}(U_1,A)(y,z)\\&\odot(U_1,A)(z,x)\odot(F,A)(x))\ \text{(by (SU5))}\\&\ge\bigwedge_{(U_1,A)\in\mathbf{U}}(\bigvee_{z\in X}(U_1,A)(y,z)\odot\\&\bigwedge_{(U_1,A)\in\mathbf{U}}\bigvee_{x\in X}(U_1,A)(z,x)\odot(F,A)(x))\\&=\bigwedge_{(U_1,A)\in\mathbf{U}}(\bigvee_{z\in X}(U_1,A)(y,z)\odot cl^r_{\mathbf{U}}(F,A)(z))\\&=cl^r_{\mathbf{U}}(cl^r_{\mathbf{U}}(F,A))(y).\end{aligned}$$

(4)

$$\begin{aligned}&cl^r_{\mathbf{U}}((F,A)\odot(G,A))(y)\\&=\bigwedge_{U\in\mathbf{U}}(\bigvee_{x\in X}(U,A)(y,x)\odot(F,A)(x)\odot(G,A)(x))\\&=\bigwedge_{U_1\odot U_2\in\mathbf{U}}(\bigvee_{x\in X}(U_1,A)(y,x)\odot(U_2,A)(y,x)\\&\odot(F,A)(x)\odot(G,A)(x))\\&\le\bigwedge_{U_1,U_2\in\mathbf{U}}(\bigvee_{x\in X}(U_1,A)(y,x)\odot(U_2,A)(y,x)\\&\odot(F,A)(x)\odot(G,A)(x))\\&=\bigwedge_{U_1\in\mathbf{U}}(\bigvee_{x\in X}(U_1,A)(y,x)\odot(F,A)(x))\\&\odot\bigwedge_{U_2\in\mathbf{U}}(\bigvee_{x\in X}(U_2,A)(y,x)\odot(G,A)(x))\\&=cl^r_{\mathbf{U}}(F,A)(y)\odot cl^r_{\mathbf{U}}(G,A)(y).\end{aligned}$$

For $cl^l_{\mathbf{U}}$, it is similarly proved.

Remark 11. If $(L,\odot)$ is a continuous t-norm, then $a\odot\bigwedge_{i\in I}b_i=\bigwedge_{i\in I}(a\odot b_i)$.

Theorem 12. *Let $(X,A,\mathbf{U})$ be a soft quasi-uniform space and $a\odot\bigwedge_{i\in I}b_i=\bigwedge_{i\in I}(a\odot b_i)$ for $a,b_i\in L$. Define $\tau^r_{\mathbf{U}},\tau^l_{\mathbf{U}}\subset S(X,A)$ as follows*

$$\tau^r_{\mathbf{U}}=\{(F,A)\in S(X,A)\mid cl^r_{\mathbf{U}}(F,A)=(F,A)\},$$

$$\tau^l_{\mathbf{U}}=\{(F,A)\in S(X,A)\mid cl^l_{\mathbf{U}}(F,A)=(F,A)\}.$$

Then (1) $\tau^r_{\mathbf{U}}$ *is a soft topology on* X *such that* $\tau^r_{\mathbf{U}} = \{cl^r_{\mathbf{U}}(F,A) \mid (F,A) \in S(X,A)\}$.

(2) $\tau^l_{\mathbf{U}}$ *is a soft topology on* X *such that* $\tau^l_{\mathbf{U}} = \{cl^l_{\mathbf{U}}(F,A) \mid (F,A) \in S(X,A)\}$.

Proof. (1) (ST1) Since $cl^r_{\mathbf{U}}(0_X,A) = (0_X,A)$ and $cl^r_{\mathbf{U}}(1_X,A) \geq (1_X,A)$, then $(0_X,A),(1_X,A) \in \tau^r_{\mathbf{U}}$.

(ST2) Let $(F,A),(G,A) \in \tau^r_{\mathbf{U}}$. Then $cl^r_{\mathbf{U}}(F,A) = (F,A)$ and $cl^r_{\mathbf{U}}(G,A) = (G,A)$. It follows

$$\begin{aligned} cl^r_{\mathbf{U}}((F,A)\odot(G,A)) &\leq cl^r_{\mathbf{U}}(F,A)\odot cl^r_{\mathbf{U}}(G,A) \\ &= (F,A)\odot(G,A). \end{aligned}$$

Thus $(F,A)\odot(G,A) \in \tau^r_{\mathbf{U}}$.

(ST3) Let $(F_i,A) \in \tau^r_{\mathbf{U}}$ for each $i \in I$. Then

$$\bigwedge_{U\in\mathbf{U}}(\bigvee_{x\in X}(U,A)(y,x)\odot(F_i,A)(x)) = (F_i,A)(y).$$

$$\begin{aligned} &\bigwedge_{U\in\mathbf{U}}(\bigvee_{x\in X}(U,A)(y,x)\odot\bigvee_{i\in I}(F_i,A)(x)) \\ &= \bigwedge_{U\in\mathbf{U}}\bigvee_{i\in I}(\bigvee_{x\in X}(U,A)(y,x)\odot(F_i,A)(x)) \\ &= \bigvee_{i\in I}\bigwedge_{U\in\mathbf{U}}(\bigvee_{x\in X}(U,A)(y,x)\odot(F_i,A)(x)) \\ &\leq \bigvee_{i\in I}(F_i,A)(y). \end{aligned}$$

By Theorem 10(2), $c^r_{\mathbf{U}}(\bigvee_{i\in I}(F_i,A)) = \bigvee_{i\in I}(F_i,A)$. So, $\bigvee_{i\in I}(F_i,A) \in \tau^r_{\mathbf{U}}$. Thus $\tau^r_{\mathbf{U}}$ is a soft topology. Put $\tau = \{cl^r_{\mathbf{U}}(F,A) \mid (F,A) \in S(X,A)\}$. Let $cl^r_{\mathbf{U}}(F,A) \in \tau$. Since $cl^r_{\mathbf{U}}(cl^r_{\mathbf{U}}(F,A)) = cl^r_{\mathbf{U}}(F,A)$, $cl^r_{\mathbf{U}}(F,A) \in \tau^r_{\mathbf{U}}$. Let $(F,A) \in \tau^r_{\mathbf{U}}$. Since $(F,A) = cl^r_{\mathbf{U}}(F,A)$, $(F,A) \in \tau$. Hence $\tau = \tau^r_{\mathbf{U}}$.

(2) It is similarly proved in (1).

Theorem 13. *Let* $f_\phi : (X,A,\mathbf{U}_X) \to (Y,B,\mathbf{U}_Y)$ *be an uniform continuous soft map. Then*

(1) $f_\phi : (X,A,\tau^r_{\mathbf{U}_X}) \to (Y,B,\tau^r_{\mathbf{U}_Y})$ *is a continuous soft map.*

(2) $f_\phi : (X,A,\tau^l_{\mathbf{U}_X}) \to (Y,B,\tau^l_{\mathbf{U}_Y})$ *is a continuous soft map.*

(3) $f_\phi(cl^r_{\mathbf{U}}(F,A)) \leq cl^r_{\mathbf{U}_Y}(f_\phi(F,A))$.

(4) $f_\phi(cl^l_{\mathbf{U}}(F,A)) \leq cl^l_{\mathbf{U}_Y}(f_\phi(F,A))$.

Proof (1) Since $(f\times f)_\phi^{-1}(V)\in \mathbf{U}_X$ for each $(V,B)\in \mathbf{U}_Y$, let $(G,B)\in \tau_{\mathbf{U}_Y}$, that is,

$$\bigwedge_{(V,B)\in \mathbf{U}_Y}(\bigvee_{w\in Y}(V,B)(y,w)\odot (G,B)(w))=(G,B)(y),\forall y\in Y,$$

we have

$$\begin{array}{l}\bigwedge_{(U,A)\in \mathbf{U}_X}(\bigvee_{x\in X}(U,A)(x,z)\odot f_\phi^{-1}(G,B)(z))\\ \leq \bigwedge_{(f\times f)_\phi^{-1}(V,B)\in \mathbf{U}_X}(\bigvee_{x\in X}(f\times f)_\phi^{-1}(V,B)(x,z)\odot f_\phi^{-1}(G,B)(z))\\ \leq \bigwedge_{(f\times f)_\phi^{-1}(V,B)\in \mathbf{U}_X}(\bigvee_{x\in X}(V,B)(f(x),f(z))\odot (G,B)(f(z)))\\ \leq \bigwedge_{(V,B)\in \mathbf{U}_Y}(\bigvee_{x\in X}(V,B)(f(x),f(z))\odot (G,B)(f(z)))\\ \leq (G,B)(f(x))=f_\phi^{-1}(G,B)(x).\end{array}$$

By Theorem 10(2), $f_\phi^{-1}(G,B)\in \tau_{\mathbf{U}_Y}$.

(3)

$$\begin{array}{l}cl^r_{\mathbf{U}_Y}(f_\phi(F,A))(y)\\ =\bigwedge_{(V,B)\in \mathbf{U}_Y}\bigvee_{w\in Y}((V,B)(y,w)\odot f_\phi(F,A)(w))\\ \geq \bigwedge_{(V,B)\in \mathbf{U}_Y}\bigvee_{x\in X}((V,B)(y,f(x))\odot f_\phi(F,A)(f(x)))\\ \geq \bigwedge_{(V,B)\in \mathbf{U}_Y}\bigvee_{x\in X}\bigvee_{z\in f^{-1}(y)}((V,B)(f(z),f(x))\odot f_\phi(F,A)(f(x)))\\ \geq \bigwedge_{(V,B)\in \mathbf{U}_Y}\bigvee_{x\in X}\bigvee_{z\in f^{-1}(y)}((f\times f)_\phi^{-1}(V,B)(z,x)\odot (F,A)(x))\\ \geq \bigvee_{z\in f^{-1}(y)}\bigwedge_{(U,A)\in \mathbf{U}_X}\bigvee_{x\in X}((U,A)(z,x)\\ \odot(F,A)(x))=f_\phi(cl^r_{\mathbf{U}}(F,A))(y).\end{array}$$

(2) and (4) are similarly proved as (1) and (3), respectively.

Lemma 14. For every $(F,A),(G,A)\in S(X,A)$, we define $(U_F,A)\in S(X\times X,A)$ by

$$U_F(a)(x,y)=F(a)(x)\to F(a)(y).$$

then we have the following statements:

(1) $(1_{X\times X},A)=(U_{0_X},A)=(U_{1_X},A)$,

(2) $(1_\triangle,A)\leq (U_F,A)$,

(3) $(U_F,A)\circ(U_F,A)=(U_F,A)$,

(4) $(U_F,A)\odot(U_G,A)\leq (U_{F\odot G},A)$.

Proof. (1)

$$\begin{aligned}&1_{X\times X}(a)(x,y)=1=U_{0_X}(a)(x,y)=0_X(a)(x)\to 0_X(a)(y)\\&=1_X(a)(x)\to 1_X(a)(y)=U_{1_X}(a)(x,y).\end{aligned}$$

(2) Since $U_F(a)(x,x)=F(a)(x)\to F(a)(x)=0$, $(1_\triangle,A)\le(U_F,A)$.

(3) $(U_F,A)\circ(U_F,A)\le(U_F,A)$ from

$$\begin{aligned}&(U_F(a)\circ U_F(a))(x,z)=\bigvee_{y\in X}(U_F(a)(x,y)\circ U_F(a)(y,z))\\&=\bigvee_{y\in X}((F(a)(x)\to F(a)(y))\odot(F(a)(y)\to F(a)(z)))\\&\le F(a)(x)\to F(a)(z)=U_F(a)(x,z).\end{aligned}$$

$(U_F,A)\circ(U_F,A)\ge(U_F,A)$ from

$$\begin{aligned}&(U_F(a)\circ U_F(a))(x,z)=\bigvee_{y\in X}(U_F(a)(x,y)\circ U_F(a)(y,z))\\&\ge((F(a)(x)\to F(a)(x))\odot(F(a)(x)\to F(a)(z)))=U_F(a)(x,z).\end{aligned}$$

(4) By Lemma 2 (12),

$$\begin{aligned}&U_F(a)(x,y)\odot U_G(a)(x,y)\\&=(F(a)(x)\to F(a)(y))\odot(G(a)(x)\to G(a)(y))\\&\le(F(a)(x)\odot G(a)(x)\to F(a)(y)\odot G(a)(y)\\&=U_{F\odot G}(a)(x,y).\end{aligned}$$

Theorem 15. *Let (X,A,τ) be a soft topological space. Define a function $\mathbf{U}_\tau:S(X\times X,A)\to L$ by*

$$\mathbf{U}_\tau=\{(U,A)\in S(X\times X,A)\mid \odot_{i=1}^n(U_{G_i},A)\le(U,A),(G_i,A)\in\tau\}$$

where the first $\bigvee$ is taken over every finite family $\{U_{(G_i,A)}\mid i=1,...,n\}$.
Then:

(1) $\mathbf{U}_\tau$ is a soft quasi-uniformity on X.

(2) $\tau\subset\tau^l_{\mathbf{U}_\tau}$.

Proof (1) (SU1) Since $(1_X,A)\in\tau$, there exists $(U_{1_X},A)\in S(X\times X,A)$ such that $(U_{1_X},A)\in\mathbf{U}_\tau$.

(SU2) It is trivial.

(SU3) For $(U,A),(V,A)\in \mathbf{U}_\tau$, there exist two finite families $\{(F_i,A)\in\tau \mid \odot_{i=1}^m (U_{F_i},A)\le (U,A)\}$ and $\{(G_j,A)\in\tau \mid \odot_{j=1}^n (U_{G_j},A)\le (G,A)\}$. Then $(U,A)\odot(V,A)\ge (\odot_{i=1}^m (U_{F_i},A))\odot(\odot_{j=1}^n (U_{G_j},A))$. So, $(U,A)\odot(V,A)\in\mathbf{U}_\tau$.

(SU4) Let $(U,A)\in\mathbf{U}_\tau$. Then there exists a finite family $\{(F_i,A)\in\tau \mid \odot_{i=1}^m (U_{F_i},A)\le (U,A)\}$. Since $(U_{F_i},A)\ge (1_\triangle,A)$ from Lemma 14(2),

$$(1_\triangle,A)\le \odot_{i=1}^m (U_{F_i},A)\le (U,A).$$

(SU5) Let $(U,A)\in\mathbf{U}_\tau$. Then there exists a finite family $\{(G_i,A)\in \tau \mid \odot_{i=1}^m (U_{G_i},A)\le (U,A)\}$. Since $(U_{G_i},A)\circ(U_{G_i},A)=(U_{G_i},A)$ for each $i\in\{1,...,m\}$ from Lemma 14(3), we have $(\odot_{i=1}^m (U_{G_i},A)\circ(\odot_{i=1}^m (U_{G_i},A))\le \odot_{i=1}^m (U_{G_i},A)$ from

$$\begin{aligned}
&\bigvee_{y\in X}((\odot_{i=1}^m U_{G_i}(a)(x,y))\odot(\odot_{i=1}^m U_{G_i}(a)(y,z)))\\
&=\bigvee_{y\in X}((\odot_{i=1}^m (G_i(a)(x)\to G_i(a)(y))\odot(\odot_{i=1}^m (G_i(a)(y)\to G_i(a)(z))))\\
&=\bigvee_{y\in X}((\odot_{i=1}^m (G_i(a)(x)\to G_i(a)(y))\odot(G_i(a)(y)\to G_i(a)(z))))\\
&\le \odot_{i=1}^m (G_i(a)(x)\to G_i(a)(z)).
\end{aligned}$$

Put $(V,A)=\odot_{i=1}^m (U_{G_i},A)$. Then $(V,A)\in\mathbf{U}_\tau$ with $(V,A)\circ(V,A)\le(U,A)$. Hence $\mathbf{U}_\tau$ is a soft quasi-uniformity on X.

(2) Let $(F,A)\in\tau$. Then $(U_F,A)\in\mathbf{U}_\tau$. Since

$$\begin{aligned}
&\bigwedge_{U\in\mathbf{U}}(\bigvee_{y\in X}(U,A)(y,x)\odot(F,A)(y))\\
&\le \bigvee_{y\in X}((U_F,A)(y,x)\odot(F,A)(y))\\
&=\bigvee_{y\in X}(((F,A)(y)\to(F,A)(x))\odot(F,A)(y))\le (F,A)(x).
\end{aligned}$$

Hence $(F,A)\in\tau^l_{\mathbf{U}_\tau}$.

Theorem 16. *Let $f_\phi:(X,A,\tau_X)\to(Y,B,\tau_Y)$ be a continuous soft map. Then $f_\phi:(X,A,\mathbf{U}_{\tau_X})\to(Y,B,\mathbf{U}_{\tau_Y})$ is an uniformly continuous soft map.*

Proof. We have

$$\begin{aligned}
&(f\times f)_\phi^{-1}(U_G,B)(a)(x,y)=(U_G,B)(\phi(a))(f(x),f(y))\\
&=G(\phi(a))(f(x))\to G(\phi(a))(f(y))=f_\phi^{-1}(G)(a)(x)\to f_\phi^{-1}(G)(a)(y)\\
&=U_{f_\phi^{-1}(G)}(a)(x,y).
\end{aligned}$$

Let $(U,B)\in\mathbf{U}_{\tau_Y}$. Then there exists a finite family $\{(G_i,B)\in\tau_Y \mid \odot_{i=1}^m (U_{G_i},B)\le (U,B)\}$.

Since $\odot_{i=1}^{m}(U_{G_i}, B) \leq (U, B)$, we have

$$(f \times f)_{\phi}^{-1}(\odot_{i=1}^{m}(U_{G_i}, B)) = \odot_{i=1}^{m}(f \times f)_{\phi}^{-1}((U_{G_i}, B))$$
$$= \odot_{i=1}^{m}(U_{f_{\phi}^{-1}(G_i)}, B) \leq (f \times f)_{\phi}^{-1}((U, B)).$$

So, $(f \times f)_{\phi}^{-1}((U, B)) \in \mathbf{U}_{\tau_X}$.

Example 17. Let $X = \{h_i \mid i = \{1, ..., 4\}\}$ with h_i=house and $E_Y = \{e, b, w, c, i\}$ with e=expensive,b= beautiful, w=wooden, c= creative, i=in the green surroundings.

Let $(L = [0, 1], \odot, \rightarrow)$ be a complete residuated lattice defined by

$$x \odot y = x \wedge y, \quad x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Let $X = \{x, y, z\}$ be a set and $W(e), W(b) \in S(X \times X, A)$ such that

$$W(e) = \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.7 & 1 & 0.8 \\ 0.4 & 0.4 & 1 \end{pmatrix}, \; W(b) = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.6 & 1 \end{pmatrix}$$

Define $\mathbf{U} = \{(U, A) \in S(X \times X, A) \mid (U, A) \geq (W, A)\}$.

(1) Since $W(e) \circ W(e) = W(e)$ and $W(b) \circ W(b) = W(b)$, $\mathbf{U}$ is a soft quasi-uniformity on X.

(2) We have $\tau_{\mathbf{U}}^{r} = \{cl_{\mathbf{U}}^{r}(F, A) \mid (F, A) \in S(X, A)\}$ where

$$cl_{\mathbf{U}}^{r}(F, A)(e) = \begin{pmatrix} F(e)(x) \vee (0.5 \wedge F(e)(y)) \vee (0.5 \wedge F(e)(z)) \\ (0.7 \wedge F(e)(x)) \vee F(e)(y) \vee (0.8 \wedge F(e)(z)) \\ (0.4 \wedge F(e)(x)) \vee (0.4 \wedge F(e)(y)) \vee F(e)(z) \end{pmatrix}$$

$$cl_{\mathbf{U}}^{r}(F, A)(b) = \begin{pmatrix} F(b)(x) \vee (0.6 \wedge F(b)(y)) \vee (0.7 \wedge F(b)(z)) \\ (0.4 \wedge F(b)(x)) \vee F(b)(y) \vee (0.4 \wedge F(b)(z)) \\ (0.5 \wedge F(b)(x)) \vee (0.6 \wedge F(b)(y)) \vee F(b)(z) \end{pmatrix}$$

We have $\tau_{\mathbf{U}}^{l} = \{cl_{\mathbf{U}}^{l}(F, A) \mid (F, A) \in S(X, A)\}$ where

$$cl_{\mathbf{U}}^{l}(F, A)(e) = \begin{pmatrix} F(e)(x) \vee (0.7 \wedge F(e)(y)) \vee (0.4 \wedge F(e)(z)) \\ (0.5 \wedge F(e)(x)) \vee F(e)(y) \vee (0.4 \wedge F(e)(z)) \\ (0.5 \wedge F(e)(x)) \vee (0.8 \wedge F(e)(y)) \vee F(e)(z) \end{pmatrix}$$

$$cl^l_{\mathbf{U}}(F,A)(b) = \begin{pmatrix} F(b)(x) \vee (0.4 \wedge F(b)(y)) \vee (0.5 \wedge F(b)(z)) \\ (0.6 \wedge F(b)(x)) \vee F(b)(y) \vee (0.6 \wedge F(b)(z)) \\ (0.7 \wedge F(b)(x)) \vee (0.4 \wedge F(b)(y)) \vee F(b)(z) \end{pmatrix}$$

(3) Let $\tau = \{(0_X, A), (1_X, A), (F, A)\}$ a soft topology where

$$F(e) = (0.4, 0.5.0.6), \quad F(b) = (0.7, 0.4.0.9),$$

$$U_F(e) = \begin{pmatrix} 1 & 1 & 1 \\ 0.4 & 1 & 1 \\ 0.4 & 0.5 & 1 \end{pmatrix}, U_F(b) = \begin{pmatrix} 1 & 0.4 & 1 \\ 1 & 1 & 1 \\ 0.7 & 0.4 & 1 \end{pmatrix}$$

Define $\mathbf{U}_\tau = \{(U,A) \in S(X \times X, A) \mid (U,A) \geq (U_F, A)\}$. Since $(U_F, A) \circ (U_F, A) = (U_F, A)$, $\mathbf{U}$ is a soft quasi-uniformity.

We have $\tau^l_{\mathbf{U}_\tau} = \{cl^l_{\mathbf{U}_\tau}(G,A) \mid (G,A) \in S(X,A)\}$ where

$$cl^l_{\mathbf{U}_\tau}(G,A)(e) = \begin{pmatrix} G(e)(x) \vee (0.4 \wedge G(e)(y)) \vee (0.4 \wedge G(e)(z)) \\ G(e)(x) \vee G(e)(y) \vee (0.5 \wedge G(e)(z)) \\ G(e)(x) \vee G(e)(y) \vee G(e)(z) \end{pmatrix}$$

$$cl^l_{\mathbf{U}_\tau}(G,A)(b) = \begin{pmatrix} G(b)(x) \vee G(b)(y) \vee (0.7 \wedge G(b)(z)) \\ (0.4 \wedge G(b)(x)) \vee G(b)(y) \vee (0.4 \wedge G(b)(z)) \\ G(e)(x) \vee G(e)(y) \vee G(e)(z) \end{pmatrix}$$

Since $cl^l_{\mathbf{U}_\tau}(F,A) = (F,A)$, $\tau \subset \tau^l_{\mathbf{U}_\tau}$.

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