

THEORY OF DISCRETE FOURIER SERIES GENERATED BY GENERALIZED DIFFERENCE OPERATOR

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Abstract: Constant amplitude transforms like Discrete Fourier Transform (DFT), Walsh transform, nonlinear phase Walsh-like transforms and Gold codes have been successfully used in many wire-line and wireless communication technologies including code division multiple access (CDMA), discrete multi-tone (DMT) and orthogonal frequency division multiplexing (OFDM) types. In this paper, we derive the discrete Fourier Series using orthonormal functions and generalized difference operator with its inverse. Suitable examples are provided to illustrate the main results.

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1. Introduction

In 1807, Fourier astounded some of his contemporaries by asserting that an "arbitrary" function could be expressed as a linear combination of sine and cosine

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functions. For a brief but excellent account of the history of this subject and its impact on the development of mathematics, one can refer [1, 2, 3, 4, 5]. These linear combinations, now called Fourier series, have become an indispensable tool in the analysis of certain periodic phenomena (such as vibrations, planetary and wave motion) which are applied in physics and engineering [6, 7, 8, 9].

In 1989, Miller and Rose [10] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [11, 12]. As application of $\Delta_h^{-\nu}$, by taking $\nu = m$ (positive integer) and $h = \ell$, the sum of the m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of an arithmetic progression have been derived using $\Delta_\ell^{-m} u(k)$, where $\Delta_\ell u(k) = u(k + \ell) - u(k)$ [13].

The basic problems in the theory of discrete Fourier series are described in the setting of discrete orthogonal functions. Therefore, first we present some terminology concerning discrete orthogonal function and then we develop the theory of discrete Fourier series. Throughout this paper, we assume that the interval $I = [a, b]$, $a < b$, $\ell = \frac{b-a}{N}$, $i = \sqrt{-1}$ and N is a positive integer.

2. Preliminaries

An n^{th} root of unity is a complex number satisfying the equation

$$z^N = 1, \quad N = 0, 1, 2, \dots \quad (1)$$

If z holds equation (1) but $z^m \neq 1$; $0 < m < N - 1$, then z is defined as a primitive N^{th} root of unity. The complex number $z_0 = e^{j(2\pi/N)}$, where $j^2 = -1$, is the primitive N^{th} root of unity with the smallest positive argument. The other primitive N^{th} roots of unity are expressed as

$$z_n = e^{j(2\pi/N)n}, \quad n = 1, 2, 3, \dots, N - 1, \quad (2)$$

where n and N are co-prime. All N^{th} roots of unity satisfy the unique summation property of a geometric series expressed as

$$\sum_{k=0}^{N-1} z_n^k = \Delta^{-1} z_n^k \Big|_{k=0}^N = \frac{z_n^N - 1}{z_n - 1} = \begin{cases} 1 & N = 1, \\ 0 & N > 1. \end{cases} \quad (3)$$

A periodic with period of N , constant modulus, complex discrete-time sequence $e_r(k)$ is defined as

$$e_r(k) = (z_r)^k = e^{j(2\pi/N)rk}, \quad r, k = 0, 1, 2, \dots, N - 1. \quad (4)$$

This complex sequence over a finite discrete time interval in a geometric series is expressed according to equation (3) as follows.

$$\sum_{k=0}^{N-1} e_r(k) = \Delta^{-1} e_r(k) \Big|_{k=0}^N = \Delta^{-1} e^{j(2\pi r/N)k} \Big|_{k=0}^N = \begin{cases} N & \text{if } r = mN, \\ 0 & \text{if } r \neq mN. \end{cases} \quad (5)$$

This mathematical property is utilized with the factorization into two orthogonal exponential functions, where one defines the Discrete Fourier Transform(DFT) $\{e_n(k)\}$ satisfying

$$\Delta^{-1} e_n(k) e_m^*(k) \Big|_{k=0}^N = \Delta^{-1} e^{j(2\pi/N)(n-m)(k)} \Big|_{k=0}^N = \begin{cases} N & \text{if } n - m = r = mN, \\ 0 & \text{if } n - m = r \neq mN, \end{cases} \quad (6)$$

where m, n are integers and the notation $(*)$ represents the complex conjugate of a function. The equation (6) motivates us to define the generalized discrete orthonormal system and Fourier series by replacing Δ by Δ_ℓ and $e_n(k)$ by $u_n(k)$.

Definition 2.1. [14] Let $u(k)$, $k \in [0, \infty)$, be a real or complex valued function and $\ell \in (0, \infty)$. Then, the generalized difference operator Δ_ℓ on $u(k)$ is defined as

$$\Delta_\ell u(k) = u(k + \ell) - u(k), \quad (7)$$

and the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as

$$\text{if } \Delta_\ell v(k) = u(k), \text{ then } v(k) = \Delta_\ell^{-1} u(k) + c, \quad (8)$$

where c is a constant.

Example 2.2. [14] If we denote $k_\ell^{(n)} = k(k - \ell) \cdots (k - (n - 1)\ell)$ and $k = k_\ell^{(0)}$, then we have

$$\Delta_\ell^{-1} k_\ell^{(n)} = \frac{k_\ell^{(n+1)}}{(n + 1)\ell} + c. \quad (9)$$

Theorem 2.3. [14] For $k \in [a, b]$ if $\ell = \frac{b - a}{N}$, then we have

$$\Delta_\ell^{-1} u(k) \Big|_a^b = \sum_{r=1}^N u(b - r\ell) = \sum_{r=0}^{N-1} u(a + r\ell). \quad (10)$$

Lemma 2.4. Let p be real and $\ell > 0$. If $1 \neq \cos p\ell$, then

$$\Delta_\ell^{-1} \sin pk = \frac{\sin p(k - \ell) - \sin pk}{2(1 - \cos p\ell)} \quad (11)$$

and

$$\Delta_\ell^{-1} \cos pk = \frac{\cos p(k - \ell) - \cos pk}{2(1 - \cos p\ell)}. \quad (12)$$

Proof. Replacing $u(k)$ by $\sin pk$ and $\cos pk$ in (7), we get

$$\Delta_\ell \sin pk = (\cos p\ell - 1) \sin pk + \sin p\ell \cos pk, \quad (13)$$

$$\Delta_\ell \cos pk = (\cos p\ell - 1) \cos pk - \sin p\ell \sin pk. \quad (14)$$

Since Δ_ℓ is linear, i.e., $c\Delta_\ell u(k) = \Delta_\ell cu(k)$ and $(\cos p\ell - 1)$ and $\sin p\ell$ are constants, multiplying (13) by $(\cos p\ell - 1)$, (14) by $\sin p\ell$ and then subtracting the second from the first one, we obtain

$$\Delta_\ell[(\cos p\ell - 1) \sin pk - \sin p\ell \cos pk] = (2 - 2 \cos p\ell) \sin pk. \quad (15)$$

Now (11) follows from (8) and dividing (15) by $(2 - 2 \cos p\ell)$.

Similarly, multiplying (13) by $\sin p\ell$, (14) by $(\cos p\ell - 1)$ and then adding them, we arrive

$$\Delta_\ell[\sin p\ell \sin pk - (\cos p\ell - 1) \cos pk] = (2 - 2 \cos p\ell) \cos pk. \quad (16)$$

Now, (12) follows from (8) and dividing (16) by $(2 - 2 \cos p\ell)$. \square

Definition 2.5. Let $u(k)$ and $v(k)$ be complex valued functions defined on $[a, b]$ and $\ell = \frac{b-a}{N}$. The discrete inner product of u and v with respect to ℓ is defined as

$$(u, v)_\ell = \ell \Delta_\ell^{-1} u(k) v^*(k) \Big|_a^b = \ell \sum_{r=0}^{N-1} u(a + r\ell) v^*(a + r\ell). \quad (17)$$

The number $\|u\|_{(\ell)} = (u, u)_\ell^{1/2} = \left\{ \ell \sum_{r=0}^{N-1} |u(a + r\ell)|^2 \right\}^{1/2}$ is the L_ℓ^2 - norm of u . We denote $L_\ell^2(I)$ as the set of all complex valued functions $u(k)$ which are bounded on I and $\|u\|_\ell < \infty$.

3. Discrete Orthogonal Systems of Functions

The function $u(k) = \frac{1}{k_\ell^{(2)}}$, where $k_\ell^{(2)} = k(k - \ell)$, is not bounded on $[0, 2\ell]$, since $u(\ell) = \frac{1}{\ell_\ell^{(2)}} = \frac{1}{\ell(\ell - \ell)} = \infty$. Hence, we consider only bounded functions on $I = [a, b]$.

Definition 3.1. Let $S_\ell = \{\phi_0, \phi_1, \phi_2, \dots, \phi_m\}$ be a collection of bounded complex valued functions defined on I . If $(\phi_n, \phi_m)_\ell = 0$ whenever $m \neq n$, the collection S_ℓ is said to be a discrete orthogonal system on I with respect to ℓ . If in addition, each u_n has norm 1, then S_ℓ is said to be an orthonormal system.

Example 3.2. Let $I = [0, 2\pi]$, $\ell = \frac{\pi}{N}$ and N is a positive integer. Consider the system of functions $S_\ell = \{\phi_0, \phi_1, \phi_2, \dots, \phi_N\}$, where

$$\phi_0(k) = \frac{1}{\sqrt{2\pi}}, \quad \phi_{2n-1}(k) = \frac{\cos nk}{\sqrt{\pi}}, \quad \phi_{2n}(k) = \frac{\sin nk}{\sqrt{\pi}}, \quad n = 1, 2, 3, \dots, N \quad (18)$$

From (11) and (12) for $n = 1, 2, 3, \dots, N$, we have

$$\Delta_\ell^{-1} \sin nk \Big|_0^{2\pi} = \frac{\sin n(k - \ell) - \sin nk}{2(1 - \cos n\ell)} \Big|_0^{2\pi} = 0$$

and

$$\Delta_\ell^{-1} \cos nk \Big|_0^{2\pi} = \frac{\cos n(k - \ell) - \cos nk}{2(1 - \cos n\ell)} \Big|_0^{2\pi} = 0.$$

When n is multiple of 2π we find that,

$$\Delta_\ell^{-1} \sin nk \Big|_0^{2\pi} = \sin n(2\pi - \ell) + \sin n(2\pi - 2\ell) + \dots + \sin n(2\pi - 2\pi) = 0.$$

From (8), we have $\Delta_\ell^{-1} 1 = \Delta_\ell^{-1} k_\ell^{(0)} = \frac{k_\ell^{(1)}}{\ell}$.

Definition 2.5 yields the following relations:

$$\|\phi_0\|_\ell^2 = (\phi_0, \phi_0)_\ell = \ell \Delta_\ell^{-1} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \Big|_0^{2\pi} = \frac{\ell}{2\pi} \Delta_\ell^{-1} (1) \Big|_0^{2\pi} = \frac{\ell}{2\pi} \frac{k_\ell^{(1)}}{\ell} \Big|_0^{2\pi} = 1. \quad (19)$$

If $\phi_n = \frac{\cos nk}{\sqrt{\pi}}$, then we have

$$\begin{aligned}\|\phi_n\|_\ell^2 &= (\phi_n, \phi_n)_\ell = \ell \Delta_\ell^{-1} \frac{\cos^2 nk}{\sqrt{\pi}\sqrt{\pi}} \Big|_0^{2\pi} = \frac{\ell}{\pi} \Delta_\ell^{-1} \left(\frac{1 + \cos 2nk}{2} \right) \Big|_0^{2\pi} \\ &= \frac{\ell}{\pi} \Delta_\ell^{-1} \frac{1}{2} \Big|_0^{2\pi} + \frac{\ell}{2\pi} \Delta_\ell^{-1} \cos 2nk \Big|_0^{2\pi} = 1 + 0 = 1.\end{aligned}\quad (20)$$

Similarly, if $\phi_n = \frac{\sin nk}{\sqrt{\pi}}$, then we have

$$\|\phi_n\|_\ell^2 = (\phi_n, \phi_n)_\ell = \ell \Delta_\ell^{-1} \frac{\sin^2 nk}{\sqrt{\pi}\sqrt{\pi}} \Big|_0^{2\pi} = 1. \quad (21)$$

$$(\phi_n, \phi_m)_\ell = \ell \Delta_\ell^{-1} \frac{\cos nk}{\sqrt{\pi}} \cdot \frac{\cos mk}{\sqrt{\pi}} \Big|_0^{2\pi} = \frac{\ell}{2\pi} \Delta_\ell^{-1} (\cos(m+n)k + \cos(m-n)k) \Big|_0^{2\pi} = 0. \quad (22)$$

Also, we obtain

$$(\phi_n, \phi_m)_\ell = \ell \Delta_\ell^{-1} \frac{\sin nk}{\sqrt{\pi}} \cdot \frac{\sin mk}{\sqrt{\pi}} \Big|_0^{2\pi} = 0 \quad (23)$$

and

$$(\phi_n, \phi_m)_\ell = \ell \Delta_\ell^{-1} \frac{\sin nk}{\sqrt{\pi}} \cdot \frac{\cos mk}{\sqrt{\pi}} \Big|_0^{2\pi} = 0. \quad (24)$$

From (19)-(24), the system S_ℓ is a discrete orthonormal system on I .

Note. Since

$$\Delta_\ell^{-1} \cos nk \Big|_0^{2\pi} = \frac{\cos n(k-\ell) - \cos nk}{2(1 - \cos n\ell)} \Big|_{k=0}^{2\pi} = 0,$$

if $n\ell \neq m2\pi$,

$$\sin n(k-\ell) - \sin nk \Big|_{k=0}^{2\pi} = 0$$

and

$$\cos n(k-\ell) - \cos nk \Big|_{k=0}^{2\pi} = 0,$$

for all $n\ell$, we take $\Delta_\ell^{-1} \sin nk \Big|_0^{2\pi} = \Delta_\ell^{-1} \cos nk \Big|_0^{2\pi} = 0$ for all $n = 1, 2, 3, \dots, N$,

where $\ell = \frac{\ell}{N}$.

Example 3.3. Let $\ell = \frac{\pi}{N}, I = [0, 2\pi]$. Then $S_\ell = \left\{ u_n(k) = \frac{e^{ink}}{\sqrt{2\pi}} \right\}$, $n = 0, 1, 2, \dots, N$, is an orthonormal system of complex valued functions on I of period 2π .

Proof. Now, from Definition 2.5, we have

$$\|\phi_n\|_\ell^2 = (\phi_n, \phi_n)_\ell = \ell \Delta_\ell^{-1} \frac{e^{ink}}{\sqrt{2\pi}} \frac{e^{-ink}}{\sqrt{2\pi}} \Big|_0^{2\pi} = 1 \quad (25)$$

and

$$(\phi_n, \phi_m)_\ell = \ell \Delta_\ell^{-1} \frac{e^{ink}}{\sqrt{2\pi}} \frac{e^{-imk}}{\sqrt{2\pi}} \Big|_0^{2\pi} = \frac{\ell}{2\pi} \frac{e^{i(n-m)k} - 1}{e^{i(n-m)\ell} - 1} \Big|_0^{2\pi} = 0$$

textif $m \neq n \in \{1, 2, 3, \dots, N\}$. (26)

Therefore, orthonormality of S_ℓ follows from (25) and (26). \square

Theorem 3.4. *Let $\{\phi_0, \phi_1, \phi_2, \dots\}$ be an orthonormal system on I . Define two sequences of functions $\{s_n\}$ and $\{t_n\}$ on I as follows :*

$$\{s_n(k)\} = \left\{ \sum_{q=0}^n c_q \phi_q(k) \right\}, \quad \{t_n(k)\} = \left\{ \sum_{q=0}^n b_q \phi_q(k) \right\}, \quad (27)$$

where $c_q = (u, \phi_q)_\ell$ for $q = 0, 1, 2, \dots$ and b_0, b_1, b_2, \dots are arbitrary complex numbers. Then for each n , we have

$$\|u - s_n\|_{(\ell)} \leq \|u - t_n\|_{(\ell)}. \quad (28)$$

Moreover, the equality holds in (28) if and only if $b_q = c_q$ for $q = 0, 1, 2, \dots, n$.

Proof. First, we shall prove that

$$\|u - t_n\|_{(\ell)}^2 = \|u\|_{(\ell)}^2 - \sum_{q=0}^n |c_q|^2 + \sum_{q=0}^n |b_q - c_q|^2. \quad (29)$$

From the linearity of Δ_ℓ^{-1} and (17), we have

$$\|u - t_n\|_{(\ell)}^2 = (u - t_n, u - t_n)_\ell = (u, u)_\ell - (u, t_n)_\ell - (t_n, u)_\ell + (t_n, t_n)_\ell. \quad (30)$$

Using the linearity of Δ_ℓ^{-1} , the orthonormality of ϕ_n and (17), we obtain

$$(t_n, t_n)_\ell = \sum_{q=0}^n |b_q|^2, \quad (u, t_n)_\ell = \sum_{q=0}^n b_q \hat{c}_q, \quad (t_n, u)_\ell = \sum_{q=0}^n \hat{b}_q c_q,$$

where \hat{c}_q and \hat{b}_q are complex conjugates of c_q and b_q respectively.

Now (29) is derived from (30) and the following relation

$$\sum_{q=0}^n |b_q - c_q|^2 = \sum_{q=0}^n (b_q - c_q)(\hat{b}_q - \hat{c}_q) = \sum_{q=0}^n |b_q|^2 - \sum_{q=0}^n b_q \hat{c}_q - \sum_{q=0}^n \hat{b}_q c_q + \sum_{q=0}^n |c_q|^2.$$

Taking $b_q = c_q$ in (29), we get

$$\|u - s_n\|_{(\ell)}^2 = \|u\|_{(\ell)}^2 - \sum_{q=0}^n |c_q|^2. \quad (31)$$

Now, (28) follows from (29), (31) and $\sum_{k=0}^n |b_k - c_k|^2 \geq 0$. □

4. Discrete Fourier Series of a Function Relative to an Orthonormal System

Definition 4.1. Let $S_\ell = \{\phi_0, \phi_1, \phi_2, \dots, \phi_M\}$ be an orthonormal system on I , $\ell = \frac{b-a}{2N}$ and assume that u is a complex valued bounded function on I . The notation

$$u(k)_\ell \approx \sum_{n=0}^M c_n \phi_n(k) \quad (32)$$

will mean that the numbers c_0, c_1, c_2, \dots are given by the formula

$$c_n = (u, \phi_n)_\ell = \ell \Delta_\ell^{-1} (u(k) \phi_n^*(k)) \Big|_a^b, \quad n = 0, 1, 2, \dots \quad (33)$$

The series in (32) is called the Discrete Fourier Series of u relative to S_ℓ and the numbers c_0, c_1, c_2, \dots are called the Discrete Fourier Coefficients of u relative to S_ℓ .

Example 4.2. If $I = [0, 2\pi]$, $\ell = \frac{\pi}{N}$ and S_ℓ is the orthonormal system of trigonometric functions described in (18), then the series obtained by (32) is called discrete Fourier series generated by u . In this case, we can write (32) in the form

$$u(k)_\ell \approx \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos nk + b_n \sin nk), \quad (34)$$

the coefficients being given below.

$$a_0 = \frac{\ell}{\pi} \Delta_\ell^{-1}(u(k)) \Big|_0^{2\pi} = \frac{\ell}{\pi} \sum_{r=1}^{\lfloor \frac{2\pi}{\ell} \rfloor} u(2\pi - r\ell), \quad (35)$$

$$a_n = \frac{\ell}{\pi} \Delta_\ell^{-1}(u(k) \cos nk) \Big|_0^{2\pi} = \frac{\ell}{\pi} \sum_{r=1}^{\lfloor \frac{2\pi}{\ell} \rfloor} u(2\pi - r\ell) \cos(2\pi - r\ell), \quad (36)$$

and

$$b_n = \frac{\ell}{\pi} \Delta_\ell^{-1} u(k) \sin nk \Big|_0^{2\pi} = \frac{\ell}{\pi} \sum_{r=1}^{\lfloor \frac{2\pi}{\ell} \rfloor} u(2\pi - r\ell) \sin(2\pi - r\ell). \quad (37)$$

The coefficients described in (35)-(37) can be obtained either by closed form or summation form of $\Delta_\ell^{-1} u(k)$ depending on $u(k)$ and when $N \rightarrow \infty$ the Discrete Fourier Series converges to Fourier Series.

To obtain orthonormal system and Discrete Fourier Series we develop certain results of Δ_ℓ^{-1} on trigonometric functions with $u(k)$.

Lemma 4.3. *Let $\ell \neq 0, k > 0$ and $p\ell \neq m2\pi$. Then we have*

$$\Delta_\ell^{-1} \sin(pk + \beta) = \frac{\sin(p(k - \ell) + \beta) - \sin(pk + \beta)}{2(1 - \cos p\ell)} \quad (38)$$

and

$$\Delta_\ell^{-1} \cos(pk + \beta) = \frac{\cos(p(k - \ell) + \beta) - \cos(pk + \beta)}{2(1 - \cos p\ell)}. \quad (39)$$

Proof. Replacing pk by $pk + \beta$ in (2.4) and (12) completes the proof of the lemma. \square

Definition 4.4. If $\sum_{r \in \mathbb{Z}} |u(r\ell)|^2 < \infty$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, then we say that $f \in L_\ell(-\infty, \infty)$ and we denote $\sum_{r \in \mathbb{Z}} |u(r\ell)|^2 = \Delta_\ell^{-1} u(k) \Big|_{-\infty}^{\infty}$.

Remark 4.5. Here, we take $\Delta_\ell^{-1} \sin nk \Big|_0^{2\pi} = \Delta_\ell^{-1} \cos nk \Big|_0^{2\pi} = 0$, when $n\ell = 2m\pi$.

Theorem 4.6. If $u(k) \in L(I)$ and $\ell = \frac{b-a}{2N}$, then we have

$$\lim_{p \rightarrow \infty} \ell \Delta_\ell^{-1} u(k) \cos pk \Big|_a^b = 0. \quad (40)$$

Proof. $u(k) \in L(I)$ implies $|u(k)| \leq M$ for all $k \in I$.

$$\text{Since, } \Delta_\ell^{-1} \cos pk = \frac{\cos p(k-\ell) - \cos pk}{2(1 - \cos p\ell)},$$

$$\begin{aligned} \left| \lim_{p \rightarrow \infty} \ell \Delta_\ell^{-1} u(k) \cos pk \Big|_a^b \right| &\leq \ell M \left| \lim_{p \rightarrow \infty} \Delta_\ell^{-1} \cos pk \Big|_a^b \right| \\ &\leq \ell M \lim_{p \rightarrow \infty} \left| \frac{4}{2(1 - \cos p\ell)} \right| \\ &\leq \ell M \frac{2}{\frac{p^2 \ell^2}{2!} + \left(\frac{p^6 \ell^6}{6!} - \frac{p^4 \ell^4}{4!} \right) + \left(\frac{p^{10} \ell^{10}}{10!} - \frac{p^8 \ell^8}{8!} \right) \dots} \end{aligned} \quad (41)$$

Now (40) follows by taking $p \rightarrow \infty$. □

Theorem 4.7. If $u(k) \in L_\ell(-\infty, \infty)$, then we have

$$\ell \Delta_\ell^{-1} u(k) \left(\frac{1 - \cos pk}{k} \right) \Big|_{-\infty}^\infty = \ell \Delta_\ell^{-1} \left(\frac{u(k) - u(-k)}{k} \right) \Big|_0^\infty. \quad (42)$$

Proof. Since $\frac{1 - \cos pk}{k} = 0$ at $k = 0$ and it is bounded for all k ,

$$\ell \Delta_\ell^{-1} u(k) \left(\frac{1 - \cos pk}{k} \right) \Big|_{-\infty}^\infty$$

exists as $u \in L_\ell(-\infty, \infty)$.

By Theorem (4.6), as u is bounded, we have

$$\ell \Delta_\ell^{-1} u(k) \cos pk \Big|_0^\infty = 0. \quad (43)$$

Now, we have

$$\begin{aligned} \ell \Delta_\ell^{-1} u(k) \left(\frac{1 - \cos pk}{k} \right) \Big|_{-\infty}^\infty &= \ell \Delta_\ell^{-1} u(k) \left(\frac{1 - \cos pk}{k} \right) \Big|_0^\infty \\ &\quad + \ell \Delta_\ell^{-1} u(k) \left(\frac{1 - \cos pk}{k} \right) \Big|_{-\infty}^0 \end{aligned}$$

$$\begin{aligned}
&= \ell \Delta_\ell^{-1} u(k) \left(\frac{1 - \cos pk}{k} \right) \Big|_0^\infty \\
&\quad + \ell \Delta_\ell^{-1} u(-k) \left(\frac{1 - \cos pk}{-k} \right) \Big|_0^\infty \\
&= \ell \Delta_\ell^{-1} [u(k) - u(-k)] \left(\frac{1 - \cos pk}{k} \right) \Big|_0^\infty.
\end{aligned}$$

From (43) we get the required result. \square

Theorem 4.8. Let $s_\ell = \{\phi_0, \phi_1, \phi_2, \dots, \phi_M\}$ be a system of discrete orthonormal functions defined on I , assume that, f is bounded complex-valued function defined on I and suppose that $u(k)_\ell \approx \sum_{n=0}^M c_n \phi_n(k)$. Then for $n \leq M$,

(a) The series $\sum_{n=0}^M |c_n|^2$ converges and satisfies the inequality

$$\sum_{n=0}^M |c_n|^2 \leq \|u\|_{(\ell)}^2 \quad (\text{Discrete Bessel's inequality}), \quad (44)$$

(b) The equation

$$\sum_{n=0}^M |c_n|^2 = \|u\|_{(\ell)}^2 \quad (\text{Discrete Parseval's formula}) \quad (45)$$

holds if and only if $\|u - s_M\|_{(\ell)} = 0$, where $\{s_n\}$ is the sequence of partial sums defined by $s_n(k) = \sum_{q=0}^n c_q \phi_q(k)$.

Proof. We take $b_q = c_q$ in (29) and observe that the left member is non-negative. Therefore $\sum_{q=0}^n |c_q|^2 \leq \|u\|_{(\ell)}^2$. This establishes (a).

To prove (b), we again put $b_k = c_k$ in (29) to obtain

$$\|u - s_n\|_{(\ell)}^2 = \|u\|_{(\ell)}^2 - \sum_{q=0}^n |c_q|^2.$$

Part (b) follows at once from this equation. \square

Theorem 4.9. Assume that $\{\phi_0, \phi_1, \dots, \phi_n, \dots, \phi_M\}$ is a system of discrete orthonormal functions on I , N is very large and $\ell = \frac{b-a}{2N}$ is very small. Let $\{c_n\}$ be any sequence of complex numbers such that $\sum |c_q|^2$ converges. Then there is a function u bounded on I such that:

(a) $(u, \phi_q)_{(\ell)} = c_q$ for each $q \geq 0$, and

(b) $\|u\|_{\ell}^2 = \sum_{q=0}^M |c_q|^2$.

Proof. Since $\{\phi_q\}$ is discrete orthonormal, we have

$$\|c_q \phi_q\|_{(\ell)}^2 \leq |c_q|^2 \|\phi_q\|_{(\ell)}^2 \leq |c_q|^2 \quad (46)$$

Take $u(k) = \sum_{q=0}^M c_q \phi_q(k), k \in I$.

Now

$$(u, \phi_q)_{\ell} = (c_q \phi_q, \phi_q)_{(\ell)} = c_q (\phi_q, \phi_q)_{(\ell)} = c_q \|\phi_q\|_{(\ell)} = c_q.$$

Proof of Part (b) follows from (45) □

5. Conclusion

When $\ell \rightarrow 0$, the Discrete Fourier Series and Discrete Fourier Transforms become usual Fourier Series and the Fourier Transforms. If $\int(\cdot)dx$ is not exist, then we can replace $\int(\cdot)$ by $\ell \Delta_{\ell}^{-1}(\cdot)$ and we can get several applications using discrete Fourier transform and its series using summation form of Δ_{ℓ}^{-1} .

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