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FINITE AND INFINITE MULTI-SERIES TYPE SOLUTIONS OF GENERALIZED MIXED DIFFERENCE EQUATION

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Abstract: In this paper, we obtain the solutions of a generalized α_i-mixed difference equation in closed, finite and infinite multi-series forms. By equating closed form with multi-series type solutions of the α_i-mixed difference equation, we obtain the values of certain types of infinite and finite multi-series. Suitable examples are provided to illustrate the main results.

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1. Introduction

In 1984, Jerzy Popenda [5] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, Miller and Rose [9] introduced the discrete analogue of the Riemann-Liouville fractional derivative

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and proved some properties of the fractional difference operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [1, 2]. In 2011, M.Maria Susai Manuel, et.al, [8, 11] extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ defined on $u(k)$ as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$, where $\alpha \neq 0$, $\ell > 0$ are fixed and $k \in [0, \infty)$ is a variable. The results derived in [11] coincide with the results in [7] when $\alpha = 1$.

An equation involving both Δ and Δ_α is called a mixed difference equation. Oscillatory behaviour of solutions for a few mixed difference equations were discussed in [3, 4, 6, 12]. An equation involving Δ_ℓ and $\Delta_{\alpha(\ell)}$ is called as the generalized mixed difference equation.

The higher order generalized α_i-difference equation

$$\Delta_{\alpha_1(\ell_1)}(\Delta_{\alpha_2(\ell_2)}(\cdots \Delta_{\alpha_n(\ell_n)}(v(k))\cdots)) = u(k), \quad k \in [0, \infty),\ \ell_i > 0\ \alpha_i \neq 0 \quad (1)$$

becomes a generalized mixed difference equation if $\alpha_i = 1$ for some i and $n \geq 2$. The equation (1) has three types of solutions which are closed, finite and infinite multi-series forms.

2. Preliminaries

In this section, we present some basic definitions, results and examples on α_i-mixed difference equation. Throughout this paper, we assume that $k \in [0, \infty)$, $\ell_i > 0$, $\alpha_i \neq 0$ and we denote $\Delta_{\alpha_i(\ell_i)} v(k) = v(k + \ell_i) - \alpha_i v(k)$, $\hat{\ell}_i(k) = k - \left[\frac{k}{\ell_i}\right]\ell_i$.

Definition 2.1. [11] If there exists a function $v(k)$ such that $\Delta_{\alpha_i(\ell_i)} v(k) = u(k)$, then $v(k) = \Delta^{-1}_{\alpha_i(\ell_i)} u(k)$ is a solution of the first order generalized α_i-difference equation $v(k + \ell_i) - \alpha_i v(k) = u(k)$.

Example 2.2. The first order difference equation $v(k+\ell_i) - \alpha_i v(k) = a^{sk}$ has a closed form solution $\Delta^{-1}_{\alpha_i(\ell_i)} a^{sk} = \dfrac{a^{sk}}{a^{s\ell_i} - \alpha_i}$ when $a^{s\ell_i} - \alpha_i \neq 0$. Here $u(k) = a^{sk}$ and $v(k) = \dfrac{a^{sk}}{a^{s\ell_i} - \alpha_i}$. The solution in infinite series form of the first order difference equation is given in Example 2.6.

Lemma 2.3. *If $a^{s\ell_i} - \alpha_i \neq 0$ for $i = 1, 2, ..., n$, then*

$$\prod_{i=1}^{n} \Delta^{-1}_{\alpha_i(\ell_i)} a^{sk} = \frac{a^{sk}}{\prod\limits_{i=1}^{n} (a^{s\ell_i} - \alpha_i)} \tag{2}$$

is a closed form solution of the equation (1) when $u(k) = a^{sk}$ and hence $\Delta^n_{\alpha(\ell)} v(k) = a^{sk}$ has a closed form solution

$$\Delta^{-n}_{\alpha(\ell)} a^{sk} = \frac{a^{sk}}{(a^{s\ell} - \alpha)^n}. \tag{3}$$

Proof. Since $\Delta^{-1}_{\alpha_i(\ell_i)} v(k)$ is linear, $a^{s\ell_i} - \alpha_i$ is constant and $\Delta^{-1}_{\alpha_i(\ell_i)} a^{sk} = \dfrac{a^{sk}}{a^{s\ell_i} - \alpha_i}$, the relation (2) follows by operating $\Delta^{-1}_{\alpha_1(\ell_1)}$, $\Delta^{-1}_{\alpha_2(\ell_2)}$,...,$\Delta^{-1}_{\alpha_n(\ell_n)}$ on a^{sk}.

Now (3) is obtained by taking $\alpha_i = \alpha$ and $\ell_i = \ell$ for $i = 1, 2, ..., n$ in (2).

Lemma 2.4. *[11] The first order generalized α-difference equation*

$$v(k + \ell) - \alpha v(k) = u(k), \ k \in [0, \infty), \ \ell \in (0, \infty) \tag{4}$$

has a solution in the finite summation form as

$$v(k) = \Delta^{-1}_{\alpha(\ell)} u(k) - \alpha^{\left[\frac{k}{\ell}\right]} \Delta^{-1}_{\alpha(\ell)} u(\hat{\ell}(k)) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1} u(k - r\ell), \tag{5}$$

where $\hat{\ell}(k) = k - \left[\frac{k}{\ell}\right]\ell$, $\left[\frac{k}{\ell}\right]$ is the integer part of $\frac{k}{\ell}$.

Lemma 2.5. *If $\lim\limits_{r_i \to \infty} \dfrac{1}{\alpha_i^{r_i}} \Delta^{-1}_{\alpha_i(\ell_i)} u(k + r_i\ell_i) = 0$, then the α_i-difference equation*

$$\Delta_{\alpha_i(\ell_i)} v(k) = u(k), \quad k \in [0, \infty), \ \ell_i > 0, \tag{6}$$

has a solution in the infinite series form as

$$\Delta^{-1}_{\alpha_i(\ell_i)} u(k) = \frac{-1}{\alpha_i} \sum_{r_i=0}^{\infty} \alpha_i^{-r_i} u(k + r_i\ell_i). \tag{7}$$

Proof. Taking $\alpha = \alpha_i$ and $\ell = \ell_i$ in (4), we get

$$v(k + \ell_i) - \alpha_i v(k) = u(k)$$

which yields

$$v(k) = -\frac{1}{\alpha_i} u(k) + \frac{1}{\alpha_i} v(k + \ell_i). \tag{8}$$

Replacing k by $k + l_i$ in (8), we obtain

$$v(k + \ell_i) = -\frac{1}{\alpha_i} u(k + \ell_i) + \frac{1}{\alpha_i} v(k + 2\ell_i),$$

which yields

$$v(k) = -\frac{1}{\alpha_i} u(k) - \frac{1}{\alpha_i^2} u(k + \ell_i) + \frac{1}{\alpha_i^2} v(k + 2\ell_i). \tag{9}$$

From (9), $\Delta^{-1}_{\alpha_i(\ell_i)} u(k) = v(k)$, $\lim\limits_{\alpha_i \to \infty} \dfrac{1}{\alpha_i^{r_i}} v(k + r_i \ell_i) = 0$ and by replacing k by $k + 2\ell_i$, $k + 3\ell_i$,... in (8), we arrive (7).

The following example illustrates the existence of the solution in the infinite series form.

Example 2.6. If $\ell_i > 0$, $\alpha_i \geq 1$, $a > 1$ and $s < 0$, then we have

$$\frac{\alpha_i a^{sk}}{\alpha_i - a^{s\ell_i}} = \sum_{r_i=0}^{\infty} \alpha_i^{-r_i} a^{s(k + r_i \ell_i)} fork \in (-\infty, \infty). \tag{10}$$

Proof. The proof follows by equating the infinite series form solution given in Lemma 2.5 and the closed form solution in Example 2.2.

3. Main Results

In this section, we obtain the values of certain types of finite and infinite alpha multi-series by equating the closed form with finite and infinite multi-series solutions of the generalized α_i-difference equation (1). Suitable examples are presented for second order mixed difference equation.

Definition 3.1. The solution of the form

$$v(k) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{u(k + \sum\limits_{i=1}^{n} r_i \ell_i)}{\alpha_1^{r_1} \alpha_2^{r_2} \cdots \alpha_n^{r_n}}$$

is called an infinite multi-series solution of the equation (1) and the summation of the form

$$v(k) = \sum_{r_1=0}^{[\frac{k}{\ell_1}]} \sum_{r_2=0}^{[\frac{k-r_1\ell_1}{\ell_2}]} \cdots \sum_{r_n=0}^{[\frac{k-r_1\ell_1-\ldots-r_{n-1}\ell_{n-1}}{\ell_n}]} \alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_n^{r_n} u(k-r_1\ell_1-\ldots-r_n\ell_n)$$

is called a finite multi-series solution of the equation (1).

If there exists a function $v(k)$, other than the finite and infinite multi-series solutions, such that $\Delta_{\alpha_1(\ell_1)}\Delta_{\alpha_2(\ell_2)}\cdots\Delta_{\alpha_n(\ell_n)}v(k) = u(k)$, then $v(k) = \Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{\alpha_2(\ell_2)}\cdots\Delta^{-1}_{\alpha_n(\ell_n)}u(k)$ is called a closed form solution of the α_i-difference equation (1).

Example 3.2. The first order α-difference equation $\Delta_{\alpha(\ell)}v(k) = \frac{1}{k}, k > 0$ has neither closed form nor infinite series solution in any closed interval but it has a finite series solution $v(k) = \sum_{r=0}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1}\frac{1}{k-r\ell}$ for $k \in (\ell,\infty), \ell > 0$.

Theorem 3.3. *(Infinite alpha multi-series) If for $t = 1, 2, ..., n$,*

$$\lim_{r_1\to\infty}\left\{\frac{1}{\alpha_1^{r_1}}\prod_{i=1}^{t}\Delta^{-1}_{\alpha_{i+1}(\ell_{i+1})}u(k+r_1\ell_1)\right\} = 0,$$

then the summation of the form

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{u(k+\sum_{i=1}^{n}r_i\ell_i)}{\alpha_1^{r_1}\alpha_2^{r_2}\ldots\alpha_n^{r_n}} = \prod_{i=1}^{n}(-\alpha_i)\Delta^{-1}_{\alpha_i(\ell_i)}u(k) \tag{11}$$

is an infinite multi-series solution of the equation (1).

Proof. From the Lemma 2.5, we have

$$u(k) + \frac{1}{\alpha_2}u(k+\ell_2) + \frac{1}{\alpha_2^2}u(k+2\ell_2) + \ldots\infty = -\alpha_2\Delta^{-1}_{\alpha_2(\ell_2)}u(k). \tag{12}$$

Dividing (12) by $\alpha_1^{r_1}$ and replacing k by $k+r_1\ell_1$, we get

$$\frac{1}{\alpha_1^{r_1}}\left\{u(k+r_1\ell_1) + \frac{1}{\alpha_2}u(k+r_1\ell_1+l_2) + \frac{1}{\alpha_2^2}u(k+r_1\ell_1+2\ell_2) + \ldots + \infty\right\}$$

$$= \frac{1}{\alpha_1^{r_1}}(-\alpha_2)\Delta^{-1}_{\alpha_2(\ell_2)}u(k+r_1\ell_1). \quad (13)$$

Adding (13) for each $r_1 = 0, 1, 2, ...$ and then applying Lemma 2.5, we get

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{u(k+r_1\ell_1+r_2\ell_2)}{\alpha_1^{r_1}\alpha_2^{r_2}} = (-\alpha_1)(-\alpha_2)\Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{\alpha_2(\ell_2)}u(k). \quad (14)$$

Replacing r_1 by r_2, r_2 by r_3, ℓ_1 by ℓ_2 and ℓ_2 by ℓ_3, in (14), we obtain

$$\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{u(k+r_2\ell_2+r_3\ell_3)}{\alpha_2^{r_2}\alpha_3^{r_3}} = (-\alpha_2)(-\alpha_3)\Delta^{-1}_{\alpha_2(\ell_2)}\Delta^{-1}_{\alpha_3(\ell_3)}u(k). \quad (15)$$

Dividing (15) by $\alpha_1^{r_1}$ and replacing k by $k + r_1\ell_1$, we obtain

$$\frac{1}{\alpha_1^{r_1}}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{u(k+r_1l_1+r_2l_2+r_3l_3)}{\alpha_2^{r_2}\alpha_3^{r_3}} = \frac{\alpha_2\alpha_3}{\alpha_1^{r_1}}\Delta^{-1}_{\alpha_2(l_2)}\Delta^{-1}_{\alpha_3(l_3)}u(k+r_1l_1). \quad (16)$$

Adding (16) for each $r_1 = 0, 1, 2, ...$ and then applying Lemma 2.5, we arrive

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{u(k+r_1l_1+r_2l_2+r_3l_3)}{\alpha_1^{r_1}\alpha_2^{r_2}\alpha_3^{r_3}} = -\alpha_1\alpha_2\alpha_3\Delta^{-1}_{\alpha_1(l_1)}\Delta^{-1}_{\alpha_2(l_2)}\Delta^{-1}_{\alpha_3(l_3)}u(k).$$

Hence (11) follows by continuing this process.

Corollary 3.4. *If $\alpha_i > 1$ and $s < 0$, then the equation (1) for $u(k) = 2^{sk}$ has both infinite alpha multi-series and closed form solutions*

$$\prod_{i=1}^{n}(-\alpha_i)\Delta^{-1}_{\alpha_i(\ell_i)}2^{sk} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{(2)^{s(k+\sum_{i=1}^{n}r_i\ell_i)}}{\alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_n^{r_n}}$$

$$= \prod_{i=1}^{n}\Big(\frac{\alpha_i}{\alpha_i - 2^{s\ell_i}}\Big)2^{sk}. \quad (17)$$

Proof. *The proof follows by taking $u(k) = 2^{sk}$ and $n = 2$ in (11) and (2).*

Example 3.5. Taking $n = 2, k = 10, \ell_1 = 4, \ell_2 = 5, \alpha_1 = 2, \alpha_2 = 3$ and $s = -1$ in (17), we get

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(2)^{-(10+4r_1+5r_2)}}{(2)^{r_1}(3)^{r_2}} = \frac{(2)(3)(2)^{-10}}{(2^{-4}-2)(2^{-5}-3)} = 1.018675722 \times 10^{-3}.$$

Remark 3.6. For some i, by putting $\alpha_i = 1$ in (11), we get solution of mixed α_i-difference equation.

Corollary 3.7. *(Infinite multi-series) If $\lim_{r_i\to\infty} u(k + r_i\ell_i) = 0$, then we have*

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} u(k+\sum_{i=1}^{n} r_i\ell_i) = (-1)^n \prod_{i=1}^{n} \Delta_{\ell_i}^{-1} u(k), \tag{18}$$

which is an infinite multi-series solution of the equation (1) for each $\alpha_i = 1$.

Proof. The proof follows by taking each $\alpha_i = 1$ in (11).

Example 3.8. For $s > 0, a > 1, \ell_i > 0$ and $k \in (-\infty, \infty)$, we have

$$\prod_{i=1}^{n} \frac{\Delta_{\ell_i}^{-1} a^{-sk}}{(-1)^n} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} a^{-s(k+\sum\limits_{i=1}^{n} r_i\ell_i)} = \frac{a^{-sk}}{\prod\limits_{i=1}^{n}(1-a^{-s\ell_i})}. \tag{19}$$

Proof. The Proof follows by replacing $u(k)$ by a^{-sk} in (18) and using (2).

Corollary 3.9. *(Infinite series) If $\lim_{r\to\infty} u(k + r\ell) = 0$ and $(x)^{(n)}$ denotes the polynomial factorial $x(x-1)\dots(x-(n-1))$, then we have*

$$\sum_{r=0}^{\infty} \frac{(r+(n-1))^{(n-1)}}{(n-1)} u(k+r\ell) = (-1)^n \Delta_{\ell}^{-n} u(k), \tag{20}$$

which is an infinite series solution of the difference equation $\Delta_\ell^n v(k) = u(k)$.

Proof. Taking $\ell_i = \ell$ in (18), we get

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} u(k+\sum_{i=1}^{n} r_i\ell) = (-1)^n \Delta_{\ell}^{-n} u(k). \tag{21}$$

By rearranging the terms, we obtain

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty} u(k+\sum_{i=1}^{n} r_i\ell) = \sum_{r=0}^{\infty} \frac{(r+(n-1))^{(n-1)}}{(n-1)} u(k+r\ell). \tag{22}$$

By Substituting (22) in (21), we get (20).

The following example gives the value of an infinite series.

Example 3.10. If $a > 1, \ell > 0$ and $s < 0$, then for $k \in (-\infty, \infty)$ we have

$$\sum_{r=0}^{\infty} \frac{(r + (n-1))^{(n-1)}}{(n-1)} a^{s(k+r\ell)} = \frac{a^{sk}}{(1 - a^{s\ell})^n}. \tag{23}$$

In particular taking $k = 0$ in (23), we get

$$\sum_{r=0}^{\infty} \frac{(r + (n-1))^{(n-1)}}{(n-1)} a^{s(r\ell)} = \frac{1}{(1 - a^{s\ell})^n}. \tag{24}$$

Proof. By taking $u(k) = a^{sk}$ in (20) and on putting $\alpha = 1$ in (3), we get (23).

Theorem 3.11. *If* $\lim_{r_i \to \infty} \frac{1}{\alpha_i^{r_i}} \Delta^{-1}_{\alpha_i(\ell_i)} u(k + r_i \ell_i) = 0$ *for* $i = 1, 2$, *then we have*

$$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \frac{1}{\alpha_1^{r_1 - r_2} \alpha_2^{r_2}} u(k + (r_1 - r_2)\ell_1 + r_2 \ell_2) = \alpha_1 \alpha_2 \Delta^{-1}_{\alpha_1(\ell_1)} \Delta^{-1}_{\alpha_2(\ell_2)} u(k). \tag{25}$$

In particular, the mixed difference equation $\Delta_{\ell_2} \Delta_{\alpha_1(\ell_1)} v(k) = u(k)$ *has a solution in infinite series form as*

$$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} \frac{u(k + (r_1 - r_2)\ell_1 + r_2 \ell_2)}{\alpha_1^{r_1 - r_2 + 1}} = \Delta^{-1}_{\alpha_1(\ell_1)} \Delta^{-1}_{(\ell_2)} u(k). \tag{26}$$

Proof. Adding (13) for each $r_1 = 0, 1, 2, ...$ in the proof of Theorem 3.3 and grouping the terms, we arrive

$$u(k) + \left(\frac{u(k+\ell_1)}{\alpha_1} + \frac{u(k+\ell_2)}{\alpha_2} \right) + \left(\frac{u(k+2\ell_1)}{\alpha_1^2} + \frac{u(k+\ell_1+\ell_2)}{\alpha_1 \alpha_2} + \frac{u(k+2\ell_2)}{\alpha_2^2} \right)$$
$$+ \left(\frac{u(k+3\ell_1)}{\alpha_1^3} + \frac{u(k+2\ell_1+\ell_2)}{\alpha_1^2 \alpha_2} + \frac{u(k+\ell_1+2\ell_2)}{\alpha_1 \alpha_2^2} + \frac{u(k+3\ell_2)}{\alpha_2^3} \right) + \ldots + \infty$$
$$= (\alpha_1)(\alpha_2) \Delta^{-1}_{\alpha_1(\ell_1)} \Delta^{-1}_{\alpha_2(\ell_2)} u(k),$$

which gives (25). Now (26) is obtained by putting $\alpha_2 = 1$ in (25).

Theorem 3.11 yields the following infinite series form solution for the mixed difference equation $\Delta_{\ell_2} \Delta_{\alpha_1(\ell_1)} v(k) = u(k)$, $\ell_1, \ell_2 > 0$. Next, we present finite multi-series solution of equation (1).

Theorem 3.12. *(Finite multi-series formula) If $k \in (\sum \ell_i, \infty)$, then the equation (1) has a solution in finite multi-series form as*

$$\sum_{i=1}^{n}\left\{\sum_{(r\ell)_{1\to i}}^{[k]}\prod_{j=1}^{i}\alpha_j^{r_j}\alpha_{i+1}^{\left[\frac{k-\sum_{j=1}^{i} r_j\ell_j}{\ell_{i+1}}\right]+1}\prod_{j=i+1}^{n}\Delta_{\alpha_j(\ell_j)}^{-1}u\left(\hat{\ell}_{i+\ell}\left(k-\sum_{t=1}^{i}r_t\ell_t+\ell_{i+1}\right)+\sum_{t=i+2}^{n}\ell_t\right)\right\}$$

$$=\prod_{i=1}^{n}\Delta_{\alpha_i(\ell_i)}^{-1}u\left(k+\sum_{t=1}^{n}\ell_t\right)-\alpha_1^{\left[\frac{k}{\ell_1}\right]+1}\prod_{i=1}^{n}\Delta_{\alpha_i(\ell_i)}^{-1}u\left(\hat{\ell}_1(k+\ell_1)+\sum_{t=2}^{n}\ell_t\right), \quad (27)$$

where

$$\sum_{(r\ell)_{1\to i}}^{[k]}=\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}\cdots\sum_{r_i}^{\left[\frac{k-r_1\ell_1-r_2\ell_2-\cdots-r_{i-1}\ell_{i-1}}{\ell_1}\right]},$$

$$\hat{\ell}_i(k)=k-[\frac{k}{\ell_i}]\ell_i,$$

for $i=1,2,...,n,\ \hat{\ell}_{n+1}(k)=k,\ \ell_{n+1}=0$ *and the operator* $\prod_{j=n+1}^{n}\Delta_{\alpha_j(\ell_j)}^{-1}=1.$

Proof. From (5), we have

$$u(k)+\alpha_1u(k-\ell_1)+\alpha_1^2u(k-2\ell_1)+...+\alpha_1^{\left[\frac{k}{\ell_1}\right]}u(\hat{\ell_1}(k))$$
$$=\Delta_{\alpha_1(\ell_1)}^{-1}u(k+\ell_1)-\alpha_1^{\left[\frac{k}{\ell_1}\right]+1}\Delta_{\alpha_1(\ell_1)}^{-1}u(\hat{\ell_1}(k)). \quad (28)$$

Since $\hat{\ell_1}(k+\ell_1)=\hat{\ell_1}(k)$, replacing $u(k)$ by $\Delta_{\alpha_2}^{-1}u(k+\ell_2)$, $u(k-\ell_1)$ by

$$\Delta_{\alpha_2(\ell_2)}^{-1}u(k-\ell_1+\ell_2)...$$

and $u(\hat{\ell_1}(k))$ by $\Delta_{\alpha_2(\ell_2)}^{-1}u(\hat{\ell_1}(k)+\ell_2)$ in (28), we find

$$\Delta_{\alpha_2(\ell_2)}^{-1}u(k+\ell_2)+\alpha_1\Delta_{\alpha_2(\ell_2)}^{-1}u(k+\ell_2-\ell_1)+...+\alpha_1^{\left[\frac{k}{\ell_1}\right]}\Delta_{\alpha_2(\ell_2)}^{-1}u(\hat{\ell_1}(k)+\ell_2)$$

$$= \Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{\alpha_2(\ell_2)}u(k+\ell_2+\ell_1) - \alpha_1^{\left[\frac{k}{\ell_1}\right]+1}\Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{\alpha_2(\ell_2)}u(\hat{\ell_1}(k)+\ell). \quad (29)$$

Again, replacing ℓ_1 by ℓ_2 and α_1 by α_2 in (28), we have

$$u(k) + \alpha_2 u(k-\ell_2) + \alpha_2^2 u(k-2\ell_2) + ... + \alpha_2^{\left[\frac{k}{\ell_2}\right]} u(\hat{\ell_2}(k))$$
$$= \Delta^{-1}_{\alpha_2(\ell_2)}u(k+\ell_2) - \alpha_2^{\left[\frac{k}{\ell_2}\right]+1}\Delta^{-1}_{\alpha_2(\ell_2)}u(\hat{\ell_2}(k+\ell_2)). \quad (30)$$

Replacing k by $k - r_1\ell_1$ in (30) and multiplying both sides by $\alpha_1^{r_1}$ for $r_1 = 1, 2, 3, ... \left[\dfrac{k}{\ell_1}\right]$, we obtain

$$\alpha_1^{r_1}\Big\{u(k-r_1\ell_1) + \alpha_2 u(k-r_1\ell_1-\ell_2)\alpha_2^2 u(k-r_1\ell_1-2\ell_2) + \cdots$$
$$+ \alpha_2^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]} u(k-r_1\ell_1-\ell_2)\Big\}$$

$$= \alpha_1^{r_1}\Big\{\Delta^{-1}_{\alpha_2(\ell_2)}u(k-r_1\ell_1+\ell_2)$$
$$- \alpha^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]+1}\Delta^{-1}_{\alpha_2(\ell_2)}u(\hat{\ell_2}(k-r_1\ell_1+\ell_2)\Big\}. \quad (31)$$

Adding (31) for each $r_1 = 0, 1, 2, ... \left[\frac{k}{\ell_1}\right]$ and using (29), we arrive

$$\Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{\alpha_2(\ell_2)}u(k+\ell_2+\ell_1) = \sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]} \alpha_1^{r_1}\alpha_2^{r_2}u(k-r_2\ell_2-r_1\ell_1)$$

$$+ \alpha_1^{\left[\frac{k}{\ell_1}\right]+1}\Delta^{-1}_{\alpha_2(\ell_2)}u(\ell_2+\hat{\ell_1}(k))$$
$$+ \sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]} \alpha_2^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]+r_1+1}\Delta^{-1}_{\alpha_2(\ell_2)}u\left(\hat{\ell_2}(k-r_1\ell_1+\ell_2\right). \quad (32)$$

Continuing this process, we get (27).

The following example illustrates the existence of infinite and finite multi-series solutions of a second order mixed difference equation.

Example 3.13. Consider the generalized mixed α_1-difference equation $\Delta_{\ell_2}\Delta_{\alpha_1(\ell_1)}v(k) = u(k)$, $\alpha_1 > 1, \ell_1, \ell_2 > 0$ and $k \in [0, \infty)$, which is the same as

$$v(k + \ell_1 + \ell_2) - v(k + \ell_1) - \alpha_1 v(k + \ell_2) + \alpha_1 v(k) = u(k). \tag{33}$$

From (26), the equation (33) has an infinite multi-series solution

$$\Delta_{\ell_2}^{-1}\Delta_{\alpha_1(\ell_1)}^{-1}u(k) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1} \frac{u(k + (r_1 - r_2)\ell_1 + r_2\ell_2)}{\alpha_1^{r_1-r_2+1}}. \tag{34}$$

Now, we discuss $\Delta_{\ell_2}^{-1}\Delta_{\alpha_1(\ell_1)}^{-1}u(k)$ when $u(k) = k$.

Since

$$\Delta_{\alpha_1(\ell_1)}\left\{\frac{k}{(1 - \alpha_1)} - \frac{\ell_1}{(1 - \alpha_1)^2}\right\} = k, \tag{35}$$

we get

$$\Delta_{\alpha_1(\ell_1)}^{-1}k = \frac{k}{(1 - \alpha_1)} - \frac{\ell_1}{(1 - \alpha_1)^2}. \tag{36}$$

Similarly, since

$$\Delta_{\alpha_1(\ell_1)}\Delta_{\ell_2}\left\{\frac{k_{\ell_2}^{(2)}}{2\ell_2(1 - \alpha_1)} - \frac{\ell_1 k}{\ell_2(1 - \alpha_1)^2}\right\} = k, \tag{37}$$

the mixed difference equation (33) for $u(k) = k$ has a closed form solution

$$\Delta_{\ell_2}^{-1}\Delta_{\alpha_{1(\ell_1)}}^{-1}k = \frac{k_{\ell_2}^{(2)}}{2\ell_2(1 - \alpha_1)} - \frac{\ell_1 k}{\ell_2(1 - \alpha_1)^2}, \tag{38}$$

Taking $u(k) = k$ in (34), we get

$$\Delta_{\ell_2}^{-1}\Delta_{\alpha_{1(\ell_1)}}^{-1}k = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1} \frac{k + (r_1 - r_2)\ell_1 + r_2\ell_2}{\alpha_1^{r_1-r_2+1}}, \tag{39}$$

which is not convergent. Hence, (33) does not have an infinite multi-series solution.

Next we consider finite multi-series solution of (33) when $u(k) = k$.

Taking $n = 2, \alpha_2 = 1$ and $u(k) = k$ in Theorem 3.12, the mixed difference equation (33) for $u(k) = k$ has a finite multi-series solution of the form

$$\Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{(\ell_2)}(k+\ell_2+\ell_1)=\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\sum_{r_2=0}^{\left[\frac{k-r_1\ell_1}{\ell_2}\right]}\alpha_1^{r_1}\left(k-r_2\ell_2-r_1\ell_1\right)$$

$$+\alpha_1^{\left[\frac{k}{\ell_1}\right]+1}\Delta^{-1}_{\alpha_1(\ell_1)}\Delta^{-1}_{\ell_2}(\ell_2+\hat{\ell}_1(k))+\sum_{r_1=0}^{\left[\frac{k}{\ell_1}\right]}\Delta^{-1}_{\alpha_2(\ell_2)}\left(\hat{\ell}_2(k-r_1\ell_1+\ell_2)\right), \quad (40)$$

where $\Delta^{-1}_{\alpha_{1(\ell_1)}}k$ and $\Delta^{-1}_{\ell_2}\Delta^{-1}_{\alpha_{1(\ell_1)}}k$ are given in (36) and (38). Substituting (38) in (40), we can obtain a finite multi-series solution of the mixed difference equation (33).

Similarly one can obtain closed, infinite and finite multi-series solutions of the difference equation (1) for various functions $u(k)$.

4. Conclusion

In general, closed form solution of the equation (1) is not possible always. When closed form solution is not possible we can go for either infinite summation form solution or finite summation form solution according to our need. If infinite series solution is not convergent, then we can find the finite series solution always. Theorem 3.11 gives an infinite multi-series solution and the Theorem (3.12) shows that the equation (1) has a finite multi-series solution for any function $u(k)$.

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