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COMPOSITION OPERATORS BETWEEN WEIGHTED HARDY TYPE SPACES

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Abstract: Let φ be an analytic self-map of the open unit disc $\mathbb{D}$ in the finite complex plane $\mathbb{C}$. Let C_φ , D be the composition and differentiation operators defined by $C_\varphi f = f \circ \varphi$, $Df = f'$ respectively. In this paper, we shall consider the operators $C_\varphi D$ and DC_φ defined by $C_\varphi D f = f' \circ \varphi$ and $DC_\varphi f = (f \circ \varphi)'$ respectively and we shall discuss the boundedness and compactness of the operators $C_\varphi D$ and DC_φ on weighted Hardy spaces by using the orthonormal basis of the weighted Hardy spaces.

1. Introduction

Throughout this paper, by $\mathbb{D}$ we shall denote the open unit disc of the finite complex plane $\mathbb{C}$; by $\partial\mathbb{D}$ the boundary of $\mathbb{D}$; by $H(\mathbb{D})$ the set of all complex valued analytic functions on $\mathbb{D}$ and by φ, the analytic self-map of $\mathbb{D}$.

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that $\beta_0 = 1$ and $\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$. Then the weighted Hardy spaces $H^2(\beta)$ is the Banach space of all analytic functions f on the open unit disk $\mathbb{D}$:

$$H^2(\beta) = \{f : z \to \sum_{n=0}^{\infty} a_n z^n \quad s.t \quad \|f\|^2_{H^2(\beta)} = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty\}$$

where $\|.\|_{H^2(\beta)}$ is a norm on $H^2(\beta)$.

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If $\beta \equiv 1$, then $H^2(\beta)$ becomes the classical Hardy space $H^2(D)$.

Also, $H^2(\beta)$ is a Hilbert space w.r.t the inner product defined as

$$< f, g >= \sum_{n=0}^{\infty} a_n.\bar{b_n}\beta_n^2$$

where $f, g \in H^2(\beta)$. For a detailed discussion on $H^2(\beta)$, we refer [11] and references therein.

Associated with φ, the classical linear operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $f \to fo\varphi$ and this operator is called the composition operator induced by self-map φ. Let ψ be an analytic function from the open unit disc $\mathbb{D}$ to $\mathbb{C}$, then associated with ψ the multiplication operator $M_\psi f$ is defined by ψf. It has been known that the composition operator C_φ is bounded on almost all spaces of analytic functions see, for example [1,2,3], and D is usually unbounded on spaces of analytic functions. Recently, the above defined operatos has received the attention of many researcher see, for example [5,7,8,9,14]. In [5], Hibschweiles and Portony defined the product $C_\varphi D$ and DC_φ and studied the boundedness and compactness of these operators between Bergman and Hardy spaces by using the carleson-type measure, whereas in [9], the author studied the boundedness and compactness of $C_\varphi D$ and DC_φ between Hardy type spaces.

This paper is organised as follows. In the second section, we shall discuss the boundedness of the operators $C_\varphi D$ and DC_φ on weighted Hardy spaces $H^2(\beta)$. In the third section, we shall study the compactness of the operators $C_\varphi D$ and DC_φ on weighted Hardy spaces $H^2(\beta)$ and in the final section, we shall give necessary and sufficient condition for the operators $C_\varphi D$ and DC_φ to be the Hilbert-Schmidt operator on weighted Hardy spaces $H^2(\beta)$.

2. Boundedness of $C_\varphi D$ and DC_φ

In this section, we shall establish the necessary and sufficient condition for the operators $C_\varphi D$ and DC_φ to be bounded on weighted Hardy spaces $H^2(\beta)$.

Recall that a linear operator T on a Hilbert space X is bounded if it takes every bounded set in X into itself.

Theorem 1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic self-map such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then $C_\varphi \mathbb{D} : H^2(\beta) \to H^2(\beta)$ is bounded iff*

$$\|\varphi^{n-1}\|_{H^{\mathbf{2}}(\beta)} \leq \frac{M}{n}\beta_n.$$

Proof. First, suppose that $C_\varphi D : H^2(\beta) \to H^2(\beta)$ is bounded. Then $\exists\ M > 0$ such that

$$\|C_\varphi Df\|_{H^2(\beta)} \leq M\|f\|_{H^2(\beta)} \quad \forall\ \ f \in H^2(\beta). \tag{2.1}$$

Let $f(z) = z^n \ \ \forall\ \ z \in D$.

Then

$$\|f\|^2_{H^2(\beta)} = \sum_{n=0}^{\infty} |a_n|^2\beta_n^2 = \sum_{n=0}^{\infty} \beta_n^2 = \sum_{n=0}^{\infty} \|z^n\|^2 = \sum_{n=0}^{\infty} \|z\|^{2n} = \frac{1}{1 - \|z\|^2} < \infty$$

Thus, $f \in H^2(\beta)$.

From equation (2.1), we have

$$\|C_\varphi Dz^n\|_{H^2(\beta)} \leq M\|z^n\|_{H^2(\beta)} = M\beta_n.$$

This implies that

$$\|n\varphi^{n-1}\|_{H^2(\beta)} \leq M\beta(n),$$

that is

$$\|\varphi^{n-1}\|_{H^2(\beta)} \leq \frac{M}{n}\beta_n.$$

Conversely, suppose that

$$\|\varphi^{n-1}\|_{H^2(\beta)} \leq \frac{M}{n}\beta_n.$$

To prove that $C_\varphi D$ is bounded, let $f \in H^2(\beta)$ be such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\beta).$$

Then

$$\begin{aligned}
\|C_\varphi Df\|^2_{H^2(\beta)} &= \|\sum_{n=0}^{\infty} a_n n\varphi^{n-1}\|^2 \\
&= \sum_{n=0}^{\infty} |a_n|^2 n^2 \|\varphi^{n-1}\|^2 \\
&\leq n^2 \frac{M^2}{n^2} \sum_{n=0}^{\infty} |a_n|^2\beta_n^2 \\
&= M^2\|f\|^2_{H^2(\beta)}
\end{aligned}$$

Thus, $\|C_\varphi Df\|^2_{H^2(\beta)} \leq M\|f\|_{H^2(\beta)} \quad \forall \quad f \in H^2(\beta)$.

Therefore, $C_\varphi D$ is bounded on $H^2(\beta)$. $\square$

Corollary 2. *If $\beta \equiv 1$ and $\varphi(z) = \frac{z}{2}$, then $C_\varphi D : H^2(\beta) \to H^2(\beta)$ is bounded.*

Proof. Since
$$\|\frac{n\varphi^{n-1}}{\beta_n}\| \leq 1 \;\; \forall \;\; n = 0, 1, 2....$$
Therefore, from the theorem (2.1), it follows that the operator $C_\varphi D$ is bounded. □

Theorem 3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic map and $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then $DC_\varphi : H^2(\beta) \to H^2(\beta)$ is bounded iff*
$$\|\varphi^{n-1}\varphi'\| \leq \frac{M}{n}\beta_n.$$

Proof. Suppose that $DC_\varphi : H^2(\beta) \to H^2(\beta)$ is bounded. Then $\exists \;\; M > 0$ such that
$$\|DC_\varphi f\|_{H^2(\beta)} \leq M\|f\|_{H^2(\beta)} \quad \forall \;\; f \in H^2(\beta) \qquad (2.2).$$
Let $f(z) = z^n$, then clearly as in theorem (2.1), we have $f \in H^2(\beta)$.
From equation (2.2), we have
$$\|DC_\varphi z^n\|_{H^2(\beta)} \leq M\|z^n\|_{H^2(\beta)} = M\beta_n.$$
This implies that
$$\|n\varphi^{n-1}\varphi'\|_{H^2(\beta)} \leq M\beta_n.$$
that is
$$\|\varphi^{n-1}\varphi'\|_{H^2(\beta)} \leq \frac{M}{n}\beta_n.$$
Conversely, assume that $\|\varphi^{n-1}\varphi'\|_{H^2(\beta)} \leq \frac{M}{n}\beta_n$.
To prove that DC_φ is bounded. Let $f \in H^2(\beta)$ be such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$.
Then
$$\begin{aligned} \|DC_\varphi f\|^2_{H^2(\beta)} &= \|\sum_{n=1}^{\infty} a_n n\varphi^{n-1}\varphi'\|^2_{H^2(\beta)} \\ &= \sum_{n=1}^{\infty} |a_n|^2 n^2 \|\varphi^{n-1}\varphi'\|^2_{H^2(\beta)} \\ &\leq n^2\frac{M^2}{n^2}\sum_{n=0}^{\infty} |a_n|^2\beta_n^2 \end{aligned}$$

$$= \quad M^2\|f\|^2_{H^{\mathbf{2}}(\beta)}.$$

This implies that

$$\|DC_\varphi f\|^2_{H^\beta} \leq M\|f\|^2_{H^{\mathbf{2}}(\beta)}$$

and so the operator DC_φ is bounded. $\square$

Corollary 4. *If $\beta \equiv 1$ and if $\varphi(z) = \frac{z}{2}$, then $DC_\varphi : H^2(\beta) \to H^2(\beta)$ is bounded.*

Proof. Since

$$\|\frac{n\varphi^{n-1}\varphi'}{\beta_n}\|_{H^{\mathbf{2}}(\beta)} \leq 1 \quad \forall \quad n = 0, 1, 2, 3.....$$

Therefore from the theorem (2.2), it follows that the operator DC_φ is bounded. $\square$

3. Compactness of the Operators $C_\varphi D$ and DC_φ

In this section, we shall study the compactness of the operators $C_\varphi D$ and DC_φ on weighted Hardy spaces $H^2(\beta)$. For this, we shall need the following Lemma.

Lemma 5. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic self-map of $\mathbb{D}$. Then the composition operators $C_\varphi D$(or DC_φ) : $H^2(\beta) \to H^2(\beta)$ is compact iff for any bounded sequence $\{f_n\}_{n=0}^\infty$ converges locally uniformaly on $\mathbb{D}$, we have*

$$\|C_\varphi D f_n (or \quad DC_\varphi f_n)\|_{H^{\mathbf{2}}(\beta)} \to 0.$$

Proof. The proof of this lemma can be written by using the similar argument as in [2, p.128] so, we omit its proof. $\square$

Theorem 6. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic map such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then $C_\varphi D : H^2(\beta) \to H^2(\beta)$ is compact iff*

$$\frac{n}{\beta_n}\|\varphi^{n-1}\| \to 0 \quad as \quad n \to \infty.$$

Proof. Suppose that $C_\varphi D : H^2(\beta) \to H^2(\beta)$ is compact and $\{\frac{z^n}{\beta_n}\}_{n=0}^\infty$ converges uniformally to zero.

Then, by using the Lemma (3.1), we have

$$\|C_\varphi D\{\frac{z^n}{\beta_n}\}\| \to 0 \quad as \quad n \to \infty,$$

that is $\frac{n}{\beta_n}\|\varphi^{n-1}\| \to 0 \quad as \quad n \to \infty$.

Conversely, suppose that

$$\frac{n}{\beta_n}\|\varphi^{n-1}\| \to 0 \quad as \quad n \to \infty.$$

Then for any $\varepsilon > 0$, there exists positive integer m such that $\frac{n}{\beta_n}\|\varphi^{n-1}\| < \varepsilon \quad \forall \quad n \geq m$.

Let $f \in H^2(\beta)$ be such that $f = \sum_{n=1}^{\infty} a_n z^n$.

Define

$$T_k f = \sum_{n=0}^{k} a_n C_\varphi D z^n = \sum_{n=0}^{k} a_n n \varphi^{n-1}.$$

Then T_k is finite rank operator on $H^2(\beta)$.

$$\begin{aligned} Now \qquad \|(C_\varphi D - T_k)f\|^2_{H^{\mathbf{2}}(\beta)} &= \|\sum_{n=k+1}^{\infty} a_n.C_\varphi D z^n\|^2_{H^{\mathbf{2}}(\beta)} \\ &= \|\sum_{n=k+1}^{\infty} a_n.n\varphi^{n-1}\|^2_{H^{\mathbf{2}}(\beta)} \\ &= \sum_{n=k+1}^{\infty} |a_n|^2.n^2\|\varphi^{n-1}\|^2_{H^{\mathbf{2}}(\beta)} \\ &\leq \sum_{n=0}^{\infty} n^2|a_n|^2\|\varphi^{n-1}\|^2_{H^{\mathbf{2}}(\beta)} \\ &\leq \sum_{n=0}^{\infty} |a_n|^2\varepsilon^2\beta_n^2 \\ &= \varepsilon^2\|f\|^2_{H^{\mathbf{2}}(\beta)}. \end{aligned}$$

Thus, $\|C_\varphi D - T_k\| < \varepsilon \quad \forall \quad n \geq m$.

This proves that $C_\varphi D$ is compact operator. □

Theorem 7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic map such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then $DC_\varphi : H^2(\beta) \to H^2(\beta)$ is compact iff*

$$\frac{n}{\beta_n}\|\varphi^{n-1}\| \to 0 \quad as \quad n \to \infty.$$

proof. *By using the similar argument as in theorem (3.2), we can write the proof of this theorem and so we omit its details.*

4. Necessary and Sufficient Conditions for Operators $C_\varphi D$ and DC_φ to be Hilbert-Schmidt Operators on $H^2(\beta)$

In this section, we shall give necessary and sufficient condition for the operator $C_\varphi D$(*or* DC_φ) to be Hilbert-Schmidt operator on $H^2(\beta)$.

Recall that a linear operator T on a Hilbert space H is said to be Hilbert-Schmidt operator if $\sum_{n=0}^{\infty}\|Te_n\|^2<\infty$ for some orthonormal basis $\{e_n : n\geq 0\}$ of Hilbert spaces H.

Theorem 8. *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D$ is a Hilbert-Schmidt operator on $H^2(\beta)$ iff*

$$\sum_{n=0}^{\infty}(n+1)\|\varphi^n\|^2_{H^2(\beta)}<\infty.$$

Proof. Assume that $C_\varphi D$ is Hilbert-Schmidt operator . Then for any orthonormal basis $\{z^n :\geq 0\}$ of $H^2(\beta)$, we have

$$\sum_{n=0}^{\infty}\|C_\varphi Dz^n\|^2_{H^2(\beta)}<\infty.$$

This implies that

$$\begin{aligned}\sum_{n=1}^{\infty}\|n\varphi^{n-1}\|^2_{H^2(\beta)} &= \sum_{n=1}^{\infty}n^2\|\varphi^{n-1}\|^2_{H^2(\beta)}\\ &= \sum_{n=0}^{\infty}(n+1)^2\|\varphi^n\|^2_{H^2(\beta)}\\ &< \infty\end{aligned}$$

Hence, $\sum_{n=0}^{\infty}(n+1)^2\|\varphi^n\|^2_{H^2(\beta)}<\infty$.

Conversely, suppose that $\sum_{n=0}^{\infty}(n+1)^2\|\varphi^n\|^2_{H^2(\beta)}<\infty$.

To prove that $C_\varphi D$ is Hilbert-Schmidt operator on $H^2(\beta)$. Let $\{z^n : n\geq 0\}$ be an orthonormal basis, then

$$\begin{aligned}\sum_{n=0}^{\infty}\|C_\varphi Dz^n\|^2_{H^2(\beta)} &= \sum_{n=1}^{\infty}\|n\varphi^{n-1}\|^2_{H^2(\beta)}\\ &= \sum_{n=0}^{\infty}(n+1)^2\|\varphi^n\|^2_{H^2(\beta)}<\infty.\end{aligned}$$

This proves that $C_\varphi D$ is Hilbert- Schmidt operator on $H^2(\beta)$.

By using the similar argument, we can prove the following theorem and so, we omit the detail of its proof. □

Theorem 9. *Let φ be an analytic self-map of $\mathbb{D}$. Then DC_φ is Hilbert-Schmidt operator on $H^2(\beta)$ iff*

$$\sum_{n=0}^{\infty}(n+1)^2\|\varphi^n.\varphi'\|^2_{H^{\mathbf{2}}(\beta)} < \infty.$$

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