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A RECURRENCE RELATION WITH COMBINATORIAL IDENTITIES

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Abstract: In the present paper, we consider a pair of recurrence relations whose simultaneous solution involves two parameters k, n. We also find generating function of the sequence. Identities related to the Fibonacci and Lucas numbers are given.

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1. Introduction

The study of linear recurrence relations has a long history. Theon of Smyrna (130 AD) studied a coupled pair of linear recurrence relations to approximate $\sqrt{2}$, [2]. Riordan [10] gave a summary of methods to investigate and find combinatorial identities. For a comprehensive catalogue of combinatorial and

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other results see Gould [3]. One can also find well-known combinatorial identities involving the Fibonacci and Lucas numbers in Koshy [8]. The following result, G. Pólya and G. Szegö [9], (pp. 6, No.39),

$$\sum_{k=0}^{n}(-1)^{n-k}2^{2k}\binom{n+k+1}{2k+1} = n+1, \tag{1.1}$$

is significant because of its derivation. From historical perspective the derivation of (1.1) follows from finding the coefficient of x^{2n+1} in

$$\sum_{k=0}^{n}\frac{1}{2}(1+2x)^{n+k+1}(-x^2)^{n-k}. \tag{1.2}$$

From [9], pp. 159 (or by application of MAPLE software,) (1.2) simplifies to,

$$\frac{1}{2}(1+2x)^{n+1}((1+2x)^{n+1}-(-x^2)^{n+1})(1+x)^{-2}. \tag{1.3}$$

By inspection, in (1.3), it suffices to consider the term $\frac{1}{2}(1+2x)^{2n+2}(1+x)^{-2}$, which, upon division, has remainder $(1-(2n+2)(2x+2))/2$. Thus, the coefficient of $\frac{1}{2}(1+2x)^{2n+2}(1+x)^{-2}$ is $n+1$.

This nontrivial derivation depends on several analytical insights. Riordan [10] has also shown that (1.1) satisfies that recurrence relation

$$f_n = 2f_{n-1} - f_{n-2},\ f_0 = 1,\ f_1 = 2.$$

Thus, one can see how ideas from analysis, combinatorics and recurrence relations are used to derive (1.1).

The objective of this paper is to investigate a pair of (nonhomogeneous) second order recurrence relations in one unknown $w_{k,n}$ involving two subscripts n and k. We then outline a methodology to derive new representations of Fibonacci numbers and give two results. Recurrence relations and generating functions are essentially the focus of the present paper. The paper is organized as follows. In Section 2 we find the generating function for $w_{k,n}$. Several other formulas for $w_{k,n}$ are given, including the general solution using two approaches. In Section 3 we will present some representations of the Fibonacci numbers by applying the main results.

2. Main Results

The generating function of the standard linear recurrence relation

$$-w_n - w_{n+1} + w_{n+2} = 0$$

that generates the Fibonacci numbers with $F_0 = 0,\ F_1 = 1$, is known to be

$$f(x) = \frac{1}{1 - x - x^2}. \tag{2.1}$$

Consider now the linear recurrence relation with initial conditions

$$-w_n - bw_{n+1} + cw_{n+2} = 0$$

with initial conditions $w_0 = 1/c,\ w_1 = 1/c + b/c^2$. Then, by using standard methods [11], [4], one can show that its generating function has the form

$$f(x) = \frac{1 + x}{c - bx - x^2}.$$

Similarly the generating function of the nonhomogeneous linear recurrence equation

$$-w_n - bw_{n+1} + cw_{n+2} = 1, w_0 = 1/c,\ w_1 = 1/c + b/c^2 \text{ is}$$

$$f(x) = \frac{1}{(c - bx - x^2)(1 - x)}.$$

The following theorems show how to generalize this result for a double indexed recurrence relation.

Theorem 1. *Let $w_{0,0} = 1/c$ and $w_{0,1} = 1/c + b/c^2$ for $c \neq 0$ and $c, b \in R$. Let $w_{0,n}$ be defined recursively by*

$$-w_{0,n} - bw_{0,n+1} + cw_{0,n+2} = 1,\ n = 0, 1, 2, \ldots$$

and let

$$w_{k,n} = \sum_{i=0}^{n} w_{k-1,i},\ k = 1, 2, ... \tag{2.2}$$

Then

$$-w_{k,n} - bw_{k,n+1} + cw_{k,n+2} = \binom{n+2+k}{k}. \tag{2.3}$$

This result first appeared in [5] and later in [6]. The generating function $g(x)$ for (2.3) is given by

Theorem 2. *Let $w_{k,n}$ be as in (2.3). Then the generating function $g(x)$ is given by*

$$g(x) = \sum_{n=0}^{\infty} w_{k,n} x^n = \frac{1}{(c - bx - x^2)(1-x)^{k+1}}, \ k = 0, 1, 2, \dots \ . \tag{2.4}$$

Proof. Let

$$S(x) = \sum_{n=0}^{\infty} w_{k,n} x^n. \tag{2.5}$$

After employing (2.5) in (2.3), multiplying by x^n and summing, we obtain

$$-\frac{c}{x^2}\left(w_{k,0} + w_{k,1}x\right) + \frac{c}{x^2}S(x) + \frac{b}{x}w_{k,0} - \frac{b}{x}S(x) - S(x) = \sum_{n=0}^{\infty} \binom{n+k+2}{k} x^n$$

or,

$$S(x)\left(\frac{c}{x^2} - \frac{b}{x} - 1\right) = \frac{c}{x^2}\left(\frac{1}{c} + \left(\frac{k+1}{c} + \frac{b}{c^2}\right)x\right) - \frac{b}{cx} + \sum_{n=0}^{\infty} \binom{n+k+2}{k} x^n$$

or,

$$S(x)\left(c - bx - x^2\right) = 1 + (k+1)x + \sum_{n=0}^{\infty} \binom{n+k+2}{k} x^{n+2} = \frac{1}{(1+x)^{k+1}}.$$

from which the result follows. □

Consider now the polynomials $p_1(x) = -x^2 - bx + c$ and $p_2(x) = cx^2 - bx - 1$. $p_2(x)$ is the characteristic polynomial of (2.3). The zeros of $p_1(x)$, $p_2(x)$ and the relations among them reveal some interesting facts. Let the zeors of $p_1(x)$ be β_1 and β_2 that of $p_2(x)$ be α_1 and α_2. Then,

$$\beta_1 = \frac{-b + \sqrt{b^2 + 4c}}{2}, \ \beta_2 = \frac{-b - \sqrt{b^2 + 4c}}{2}, \ p_1(x) = -(x - \beta_1)(x - \beta_2). \tag{2.6}$$

$$\alpha_1 = \frac{b + \sqrt{b^2 + 4c}}{2c}, \ \alpha_2 = \frac{b - \sqrt{b^2 + 4c}}{2c}, \ \ p_2(x) = c(x - \alpha_1)(x - \alpha_2). \tag{2.7}$$

$$\frac{1}{p_1(x)} = \frac{1}{-(x - \beta_1)(x - \beta_2)} = \frac{1}{\beta_2 - \beta_1}\left[\frac{1}{x - \beta_1} - \frac{1}{x - \beta_2}\right], \tag{2.8}$$

and

$$\frac{1}{p_2(x)} = \frac{1}{c(x-\alpha_1)(x-\alpha_2)} = \frac{1}{c(\alpha_2-\alpha_1)}\left[\frac{1}{x-\alpha_2} - \frac{1}{x-\alpha_1}\right]. \tag{2.9}$$

This leads then to

Proposition 1.

$$\frac{x}{p_1(x)} = \frac{1}{c(\alpha_1-\alpha_2)}\left[\frac{1}{1-x\alpha_1} - \frac{1}{1-x\alpha_2}\right] = \frac{1}{c}\sum_{n=0}^{\infty} x^n \frac{\alpha_1^n - \alpha_2^n}{\alpha_1-\alpha_2}. \tag{2.10}$$

Proof. The proof follows from (2.6-2.9). □

Using Proposition 1 and Theorem 2 we get,

Proposition 2.

$$g(x) = \frac{1}{c}\sum_{n=1}^{\infty} \frac{\alpha_1^n - \alpha_2^n}{\alpha_1-\alpha_2}\,\frac{x^{n-1}}{(1-x)^{k+1}}.$$

Proof. In Proposition 1, divide by x, sum from $n=1$ and use the definiton of $p_1(x)$. □

From Theorem 1 and Proposition 2 it follows that,

Corollary 1. *Let* $a=b=1$ *in* Theorem 1. *Define* $w_{k,n} = w_{k,n}(1,1)$, *then*

$$\sum_{n=0}^{\infty} w_{k,n}(1,1)x^n = \sum_{n=1}^{\infty} F_n \frac{x^{n-1}}{(1-x)^{k+1}}.$$

Proof. This follows from Propostion 1 and the definition of the n-th Fibonacci number $F_n = (\alpha_1^n - \alpha_2^n)/(\alpha_1-\alpha_2)$, where $\alpha_1 = (1+\sqrt{5})/2$, $\alpha_2 = (1-\sqrt{5})/2$. □

Next, a closed form for $w_{k,n}$ is found using two approaches: analytic and inductive approach.

Theorem 3.

$$\begin{aligned} w_{k,n} &= \frac{1}{c}\sum_{j=0}^{n}\sum_{j-2s\geq 0}\binom{j-s}{s}\frac{b^{j-2s}}{c^{j-s}}\binom{k+n-j}{n-j} \qquad (2.11)\\ &= \frac{1}{c}\sum_{j=0}^{n}\sum_{s=0}^{\lfloor j/2\rfloor}\binom{j-s}{s}\frac{b^{j-2s}}{c^{j-s}}\binom{k+n-j}{k}. \end{aligned}$$

where $\lfloor j/2 \rfloor$ denotes the 'floor' of $j/2$.

Proof. Analytic Proof.

$$\frac{1}{(1-x)^{k+1}} = 1 + \binom{k+1}{1}x + \binom{k+2}{2}x^2 + \ldots + \binom{k+n}{k}x^n + \ldots. \quad (2.12)$$

Using (2.12), the coefficient of x^n in Propostion 2 is,

$$\sum_{j=0}^{n} \binom{k+n-j}{k} \frac{\alpha_1^{j+1} - \alpha_2^{j+1}}{\alpha_1 - \alpha_2},$$

The result follows by expanding $(\alpha_1^{j+1} - \alpha_2^{j+1})/(\alpha_1 - \alpha_2)$. □

Proof. Inductiive Proof. Solving for $w_{k,n+2}$ from (2.3) yields,

$$w_{k,n+2} = \frac{1}{c}\binom{k+n+2}{k} + \frac{b}{c}w_{k,n+1} + \frac{1}{c}w_{k,n}. \quad (2.13)$$

The conditions,

$$w_{k,0} = \frac{1}{c}, \quad (2.14)$$

and

$$w_{k,1} = \frac{1}{c}\left[\binom{k+1}{1} + \frac{b}{c}\right] \quad (2.15)$$

follow from (2.2).

Taking $n = 0$ in (2.13) and employing (2.14), (2.15) one obtains,

$$w_{k,2} = \frac{1}{c}\left[\binom{k+2}{2} + \frac{b}{c}\binom{k+1}{1} + \left(\frac{b^2}{c^2} + \frac{1}{c}\right)\right]. \quad (2.16)$$

Similarly,

$$w_{k,3} = \frac{1}{c}\left[\binom{k+3}{3} + \frac{b}{c}\binom{k+2}{2} + \left(\frac{b^2}{c^2} + \binom{1}{1}\frac{1}{c}\right)\binom{k+1}{1} + \left(\frac{b^3}{c^3} + \binom{2}{1}\frac{1}{c}\cdot\frac{b}{c}\right)\right]. \quad (2.17)$$

Inspection of (2.16),(2.17) (eventually) yields a pattern. By induction, assume for all $v \leq n$,

$$w_{k,v} = \frac{1}{c}\sum_{j=0}^{v}\sum_{j-2s\geq 0}\binom{j-s}{s}\frac{b^{j-2s}}{c^{j-2s}}\cdot\frac{1}{c^s}\binom{k+v-j}{k} \quad (2.18)$$

.

Replacing v by $n+1, n \geq 1$ in (2.18) and employing (2.3) gives,

$$\begin{aligned}
w_{k,n+1} &= \frac{1}{c}\binom{k+n+1}{k} + \frac{b}{c}w_{k,n} + \frac{1}{c}w_{k,n-1} + \qquad (2.19)\\
&= \frac{1}{c}\binom{k+m+1}{k} + \frac{1}{c}\cdot\frac{b}{c}\binom{k+n}{k}\\
&+ \frac{1}{c}\sum_{j=1}^{n}\sum_{j-2s\geq 0}\binom{j-s}{s}\frac{b^{j+1-2s}}{c^{j+1-2s}}\cdot\frac{1}{c^s}\binom{k+n-j}{k}\\
&+ \frac{1}{c}\sum_{j=0}^{n-1}\sum_{j-2s\geq 0}\binom{j-s}{s}\frac{b^{j-2s}}{c^{j-2s}}\cdot\frac{1}{c^{s+1}}\binom{k+n-1-j}{k}\\
&= \frac{1}{c}\binom{k+n+1}{k} + \frac{b}{c}\cdot\frac{1}{c}\binom{k+n}{k}\\
&+ \frac{1}{c}\sum_{j=0}^{n-1}\sum_{j+1-2s\geq 0}\binom{j+1-s}{s}\frac{b^{j+2-2s}}{c^{j+2-2s}}\cdot\frac{1}{c^s}\binom{k+n-j-1}{k}\\
&+ \frac{1}{c}\sum_{j=0}^{n-1}\sum_{j+2-2s\geq 0}\binom{j+1-s}{s-1}\frac{b^{j+2-2s}}{c^{j+2-2s}}\cdot\frac{1}{c^s}\binom{k+n-1-j}{k}.
\end{aligned}$$

Note that in the last term $s \geq 1$. The useful identity,

$$\binom{j+1-s}{s} + \binom{j+1-s}{s-1} = \binom{j+2-s}{s},$$

can simplify equation (2.19) to obtain,

$$\begin{aligned}
w_{k,m+1} &= \frac{1}{c}\binom{k+n+1}{k} + \frac{b}{c}\cdot\frac{1}{c}\binom{k+n}{k} \qquad (2.20)\\
&+ \frac{1}{c}\sum_{j=0}^{n-1}\sum_{j+2-2s\geq 0}\binom{j+2-s}{s}\frac{b^{j+2-2s}}{c^{j+2-2s}}\cdot\frac{1}{c^s}\binom{k+n-j-1}{k}\\
&= \frac{1}{c}\binom{k+n+1}{k} + \frac{b}{c}\cdot\frac{1}{c}\binom{k+n}{k}\\
&+ \frac{1}{c}\sum_{j=2}^{n+1}\sum_{j-2s\geq 0}\binom{j-s}{s}\frac{b^{j-2s}}{c^{j-2s}}\cdot\frac{1}{c^s}\binom{k+n+1-j}{k}.
\end{aligned}$$

Combining terms in (2.20) gives

$$w_{k,n+1} = \frac{1}{c}\sum_{j=0}^{n+1}\sum_{s=0}^{\lfloor j/2\rfloor}\binom{j-s}{s}\frac{b^{j-2s}}{c^{j-s}}\binom{k+n+1-j}{k}. \tag{2.21}$$

The proof is complete by noting that when j is odd, $s = 0, 1, \ldots, j-1$. □

From Theorem 3 one gets

Corollary 2.

$$w_{k,n} = w_{k,n}(1,1) = \sum_{j=0}^{n} F_{j+1}\binom{k+n-j}{k}. \tag{2.22}$$

With $w_{k,n}$ given by 2.11 and employing Proposition 2 it follows that,

Theorem 4.

$$\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2} = \begin{cases} \sum_{i=0}^{k+1} w_{k,n-i}(-1)^i\binom{k+1}{i}, & \textit{if } n \geq k+1; \\ \sum_{i=0}^{n} w_{k,n-i}(-1)^i\binom{k+1}{i}, & \textit{if } n \leq k. \end{cases} \tag{2.23}$$

Proof. In Proposition 2, multiply both sides by $(1-x)^{k+1}$, use definition of $g(x)$ and expand to find power of x^n. □

Corollary 3.

$$F_{n+1} = \begin{cases} \sum_{i=0}^{k+1} w_{k,n-i}(1,1)(-1)^i\binom{k+1}{i}, & \textit{if } n \geq k+1; \\ \sum_{i=0}^{n} w_{k,n-i}(1,1)(-1)^i\binom{k+1}{i}, & \textit{if } n \leq k. \end{cases} \tag{2.24}$$

The case $b = 0$ is also considered to get the following,

Corollary 4. *If $b = 0$ in Theorem 2, then*

$$a_{k,n} = \frac{1}{c}\sum_{s=0}^{\lfloor n/2\rfloor}\frac{1}{c^s}\binom{k+n-2s}{k}. \tag{2.25}$$

Proof. The result follows directly from (2.11). □

Several examples involving the golden section, the Fibonacci and the Lucas numbers are presented in next section.

3. Combinatorial Identities

In this section we present some examples for the representation of a linear combination of the Fibonacci and Lucas numbers that involve a polynomial and a double sum combinatorial term. These representations are derived by employing the method of undetermined coefficients for specific values of k in (2.3).

Case 1: k=0.

$$a_{1,n} = D + c_1\alpha^n + c_2\beta^n, \tag{3.1}$$

for some real constants D, c_1, c_2. This yields, upon substitution into (2.3),

$$-D - bD + cD \;=\; \binom{n+2}{n+2}, \tag{3.2}$$

which yields,

$$D = -\frac{1}{b+1-c}. \tag{3.3}$$

By solving the characteristic equation of (2.3),

$$-1 - bx + cx^2 = 0, \tag{3.4}$$

we get

$$\alpha = \frac{b+\sqrt{b^2+4c}}{2c}, \quad \beta = \frac{b-\sqrt{b^2+4c}}{2c}. \tag{3.5}$$

Let $b = c = 1$. This gives, $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Employing (3.5), (3.3), in (3.1) and using the initial conditions for $n = 0$, 1, yields

$$D = -1,\; c_1 = \frac{2\sqrt{5}}{5} + 1,\; c_2 = -\frac{2\sqrt{5}}{5} + 1. \tag{3.6}$$

Employing (3.6) in (3.1), and equating it to (2.11) yields

$$\begin{aligned} &\left(\frac{2\sqrt{5}}{5}+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-2\sqrt{5}}{5}+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad (3.7)\\ &= 2F_n + L_n = \frac{4}{2}F_n + \frac{2}{2}L_n = \frac{1}{2}\left(L_3F_n + F_3L_n\right)\\ &= 1 + \sum_{j=0}^{n}\sum_{s=0}^{\lfloor j/2\rfloor}\binom{j-s}{s} = 1 + \sum_{j=0}^{n} F_{j+1}. \end{aligned}$$

which is known already.

Case 2: k=1

$$\left(\frac{11\sqrt{5}}{10}+\frac{5}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{-11\sqrt{5}}{10}+\frac{5}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{3.8}$$
$$=\frac{11}{2}F_n+\frac{5}{2}L_n=\frac{1}{2}\left(L_5F_n+F_5L_n\right)$$
$$=n+4+\sum_{j=0}^{n}\sum_{s=0}^{\lfloor j/2\rfloor}\binom{j-s}{s}\binom{n-j+1}{n-j},$$

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