UNIQUENESS OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract: In this paper, we study the zero distributions on the derivatives of $q$-shift difference polynomials of meromorphic functions with zero order and obtain two theorems that extend results of [3].

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1. Introduction

In this paper, a meromorphic functions $f$ means meromorphic in the complex plane. If no poles occur, then $f$ reduces to an entire function. Throughout of this paper, we denote by $\rho(f)$ and $\rho_2(f)$ the order of $f$ and the hyper order of $f$ (Laine, 1993 and Yang and Yi, 2003). In addition, if $f - a$ and $g - a$ have the same zeros, then we say that $f$ and $g$ share the value $a$ IM(ignoring multiplicities). If $f - a$ and $g - a$ have the same zeros, then we say that $f$ and $g$ share the value $a$ CM(counting multiplicities). We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory(Halburd Korhonen and Tohge; Laine, 1993 and Yang and Yi, 2003).

Given a meromorphic function $f(z)$, recall that $\alpha(z) \neq 0, \infty$ is a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(T(r, f))$, and $r \to \infty$ outside of a
possible exceptional set of finite logarithmic measure.

Recently, K. Liu, X. Liu and T. B. Cao (2012) proved the following.

**Theorem A.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental entire function of \( \rho_2(f) < 1 \). For \( n \geq t(k+1)+1 \), then \( \left[ P(f) f(z+c) \right]^{(k)} - \alpha(z) \) has infinitely many zeros.

**Theorem B.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \), not a periodic function with period \( c \). If \( n \geq (t+1)(k+1)+1 \), then \( \left[ f(z)^n (\Delta_c f)^s \right]^{(k)} - \alpha(z) \) has infinitely many zeros.

**Theorem C.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \). For \( n \geq t(k+1)+5 \), then \( \left[ P(f) f(z+c) \right]^{(k)} - \alpha(z) \) has infinitely many zeros.

**Theorem D.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \). If \( n \geq (t+2)(k+1)+3+s \), then \( \left[ P(f)(\Delta_c f)^s \right]^{(k)} - \alpha(z) \) has infinitely many zeros.

**Theorem E.** (Liu, Liu and Coa, 2012) Let \( f \) and \( g \) be a transcendental entire function of \( \rho_2(f) < 1 \), \( n \geq 2k + m + 6 \). If \( \left[ f^n(f^m-1)f(z+c) \right]^{(k)} \) and \( \left[ g^n(g^m-1)g(z+c) \right]^{(k)} \) share the 1 CM, then \( f = tg \), where \( t^{n+1} = t^m = 1 \).

**Theorem F.** (Liu, Liu and Coa, 2012) The conclusion of Theorem E is also valid, if \( n \geq 5k + 4m + 12 \). and \( \left[ f^n(f^m-1)f(z+c) \right]^{(k)} \) and \( \left[ g^n(g^m-1)g(z+c) \right]^{(k)} \) share the 1 IM.

In 2013, Harina P. Waghamore and Tanuja A. extend Theorem E and Theorem F to meromorphic functions.

**Theorem G.** (Harina P.W and Tanuja A, 2013) Let \( f \) and \( g \) be a transcendental meromorphic function with zero order. If \( n \geq 4k + m + 8 \), \( \left[ f^n(f^m-1)f(qz+c) \right]^{(k)} \) and \( \left[ g^n(g^m-1)g(qz+c) \right]^{(k)} \) share the 1 CM, then \( f = tg \), where \( t^{n+1} = t^m = 1 \).

**Theorem H.** (Harina P.W and Tanuja A, 2013) Let \( f \) and \( g \) be a transcendental meromorphic function with zero order. If \( n \geq 5k + 4m + 17 \), \( \left[ f^n(f^m-1)f(qz+c) \right]^{(k)} \) and \( \left[ g^n(g^m-1)g(qz+c) \right]^{(k)} \) share the 1 IM, then \( f = tg \), where \( t^{n+1} = t^m = 1 \).
In this paper, we extend Theorem G and Theorem H to difference polynomials and obtain the following results.

**Theorem 1.** Let $f$ and $g$ be a transcendental meromorphic (resp. entire) function with zero order. If $n \geq 4k + 8(n \geq 2k + 6)$, $[P(f)f(qz + c)]^{(k)}$ and $[P(g)g(qz + c)]^{(k)}$ share the 1 CM, then:

1. $f \equiv tg$ for a constant $t$ such that $t^d = 1$.
2. $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c)$.

**Theorem 2.** Let $f$ and $g$ be a transcendental meromorphic (resp. entire) function with zero order. If $n \geq 10k + 14(n \geq 5k + 12)$, $[P(f)f(qz + c)]^{(k)}$ and $[P(g)g(qz + c)]^{(k)}$ share the 1 IM, then the conclusion of theorem 1 still holds.

### 2. Some Lemmas

In this section, we present some definitions and lemmas which will be needed in the sequel.

**Lemma 2.1.** (Halburd, Korhonen and Tohge, Theorem 5.1) Let $f(z)$ be a transcendental meromorphic function of $\rho_1(f) < 1$, $\varsigma < 1$, $\epsilon$ is enough small number. Then

$$m(r, \frac{f(z + c)}{f(z)}) = o\left(\frac{T(r, f)}{r^{1-\varsigma-\epsilon}}\right) = S(r, f),$$

for all $r$ outside of a set of finite logarithmic measure. Combining the proof of (Luo and Lin, 2011, Lemma 5) with Lemma 2.1, we can get the following Lemma 2.2.

**Lemma 2.2.** Let $f(z)$ be a transcendental entire function of $\rho_2(f) < 1$. If $F = P(f)f(z + c)$, then

$$T(r, F) = T(P(f)f(z)) + S(r, f) = (n + 1)T(r, f) + S(r, f).\quad (2.2)$$

**Lemma 2.3.** (Liu, Liu and Cao, 2012, Lemma 2.5) Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$. If $F = P(f)f(z + c)$, then

$$(n - 1)T(r, f) + S(r, f) \leq T(r, F) \leq (n + 1)T(r, f) + S(r, f).\quad (2.3)$$
Lemma 2.4. (Zhang and Korhonen, 2010, Theorem 1.1) Let $f(z)$ be a transcendental meromorphic function of zero order. Then
\[ T(r, f(qz)) = T(r, f(z)) + S(r, f) \]
on a set of logarithmic density 1.

The following lemma has little modifications of the original version (Theorem 2.1 of Chiang and Feng, 2008).

Lemma 2.5. Let $f(z)$ be a transcendental meromorphic function of finite order. Then
\[ T(r, f(z + c)) = T(r, f) + S(r, f). \] (2.4)

Combining Lemma 2.4 with Lemma 2.5, we get the following result easily.

Lemma 2.6. Let $f(z)$ be a transcendental meromorphic function of zero order. Then
\[ T(r, f(qz + c)) = T(r, f(z)) + S(r, f) \] (2.5)
on a set of logarithmic density 1.

Lemma 2.7. (Yang and Hua, 1997, Lemma 3) Let $F$ and $G$ be non constant meromorphic functions. If $F$ and $G$ share 1 CM, then one of the following three cases holds:

(i) $\max \{T(r, F), T(r, G)\} \leq N_2 \left(r, \frac{1}{F}\right) + N_2(r, F) + N_2 \left(r, \frac{1}{G}\right) + N_2(r, G) + S(r, F) + S(r, G)$.

(ii) $F = G$.

(iii) $F.G = 1$.

Lemma 2.8. (Xu an Yi, 2007, Lemma 2.3) Let $F$ and $G$ be non constant meromorphic function sharing the value 1 IM. Let
\[ H = \frac{F''}{F'} - 2 \frac{F'}{F - 1} - \frac{G''}{G} + 2 \frac{G'}{G - 1}. \]
If $H \neq 0$, then
\[ T(r, F) + T(r, G) \leq 2 \left( N_2 \left(r, \frac{1}{F}\right) + N_2(r, F) + N_2 \left(r, \frac{1}{G}\right) + N_2(r, G) \right) \]
\[+ 3 \left( N(r, F) + \overline{N} \left( r, \frac{1}{F} \right) + \overline{N}(r, G) + \overline{N} \left( r, \frac{1}{G} \right) \right) + S(r, F) + S(r, G). \]  
(2.6)

**Lemma 2.9.** Let \( f(z) \) be a meromorphic function, and \( p, k \) be positive integers. Then

\[T(r, f^{(k)}) \leq T(r, f) + kN(r, f) + S(r, f). \]  
(2.7)

\[N_p \left( r, \frac{1}{f^{(k)}} \right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f). \]  
(2.8)

\[N_p \left( r, \frac{1}{f^{(k)}} \right) \leq k\overline{N}(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f). \]  
(2.9)

**Lemma 2.10.** Let \( f \) and \( g \) be a transcendental meromorphic function of zero order. If \( n \geq k + 6 \) and

\[[P(f)f(qz + c)]^{(k)} = [P(g)g(qz + c)]^{(k)} \]  
(2.10)

then \( f = tg \), where \( t^{n+1} = t^m = 1 \), and \( f \) and \( g \) satisfy the algebraic equation

\[R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c). \]

**Proof.** From (2.10), we have

\[P(f)f(qz + c) = P(g)g(qz + c) + Q(z). \]

Where \( Q(z) \) is a polynomial of degree atmost \( k = 1 \). If \( Q(z) \neq 0 \), then we have

\[\frac{P(f)f(z + c)}{Q(z)} = \frac{P(g)g(qz + c)}{Q(z)} + 1. \]
From the second main theorem of Nevanlinna and by Lemma 2.2, we have

\[(n + 1)T(r, f) = T \left(r, \frac{P(f) f(qz + c)}{Q(z)} \right) + S(r, f) \]
\[\leq N \left( r, \frac{P(f) f(qz + c)}{Q(z)} \right) + N \left( r, \frac{Q(z)}{P(f) f(qz + c)} \right) \]
\[+ N \left( r, \frac{Q(z)}{P(g) g(qz + c)} \right) + S(r, f) \]
\[\leq N(r, P(f)) + N(r, f(qz + c)) + N \left( r, \frac{1}{P(f)} \right) \]
\[+ N \left( r, \frac{1}{f(qz + c)} \right) + N \left( r, \frac{1}{g(z)} \right) + N \left( r, \frac{1}{g(qz + c)} \right) + S(r, f) + S(r, g) \]
\[\leq 4T(r, f) + 2T(r, g) + S(r, f) + S(r, g). \quad (2.11) \]

Similarly as above, we have

\[(n + 1)T(r, g) \leq 4T(r, g) + 2T(r, f) + S(r, f) + S(r, g). \quad (2.12) \]

Thus, we get

\[(n + 1)[T(r, f) + T(r, g)] \leq 6[T(r, f) + T(r, g)] + S(r, f) + S(r, g). \quad (2.13) \]

which is in contradiction with \( n \geq k + 6 \). Hence, we get \( Q(z) \equiv 0 \), which implies that

\[P(f) f(qz + c) = P(g) g(qz + c). \quad (2.14) \]

Set \( h(z) = \frac{f(z)}{g(z)} \), we break the rest of the proof into two cases.

**Case 1.** Suppose \( h(z) \) is a constant. Then by substituting \( f = gh \) into (2.14), we obtain

\[g(qz + c) [a_n g^n (h^{n+1} - 1) + a_{n-1} g^{n-1} (h^n - 1) + \ldots + a_0 (h - 1)] \equiv 0 \quad (2.15)\]

where \( a_n (\neq 0), a_{n-1}, \ldots, a_0 \) are complex constants. By the fact that \( g \) is a transcendental entire functions, we have \( g(qz + c) \neq 0 \). Hence, we obtain

\[[a_n g^n (h^{n+1} - 1) + a_{n-1} g^{n-1} (h^n - 1) + \ldots + a_0 (h - 1)] \equiv 0. \quad (2.16)\]

Equation (2.16) implies that \( h^{n+1} = 1 \) and \( h^{i+1} = 1 \) when \( a_i \neq 0 \) for \( i = 0, 1, \ldots, n - 1 \). Therefore \( h^d = 1 \), where \( d = GCD(\lambda_0, \lambda_1, \ldots, \lambda_n) \).
Case 2. Suppose that \( h \) is not a constant, then we know by (2.14) that \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(w_1, w_2) = p(w_1)w_1(qz + c) - p(w_2)w_2(qz + c) \).

Lemma 2.11. Let \( f \) and \( g \) be transcendental entire function of finite order. If \( n \geq k + 4 \), and \( [P(f)g(qz + c)]^{(k)} = [P(g)g(qz + c)]^{(k)} \) then the condition of Lemma 2.10 holds.

Proof. Substituting \( N(r, f) = N(r, g) = 0 \) and proceeding as in the proof of Lemma 2.10, we get Lemma 2.11.

3. Proof of the Theorem

Proof of Theorem 1.1. Let \( F = [P(f)g(qz + c)]^{(k)} \) and \( G = [P(g)g(qz + c)]^{(k)} \). Thus \( F \) and \( G \) share the value 1 CM. From (2.7) and \( f \) is a transcendental meromorphic function, then

\[
T(r, F) \leq T(r, P(f)g(qz + c)) + kN(r, f) + S(r, P(f)g(qz + c)) \tag{3.1}
\]

combining (3.1) with Lemma 2.2, we have \( S(r, F) = S(r, f) \). We also have \( S(r, G) = S(r, g) \), from the same reason as above, from (2.8) we obtain

\[
N_2(r, \frac{1}{F}) = N_2\left(r, \frac{1}{[P(f)g(qz + c)]^{(k)}}\right) \\
\leq T(r, F) - T(r, P(f)g(qz + c)) \\
+ N_{k+2}\left(r, \frac{1}{P(f)g(qz + c)}\right) + S(r, f). \tag{3.2}
\]

Thus, from Lemma 2.2 and (3.2) we get

\[
(n + 1)T(r, f) = T(r, P(f)g(qz + c)) + S(r, f) \\
\leq T(r, F) - N_2(r, \frac{1}{F}) \\
+ N_{k+2}\left(r, \frac{1}{P(f)g(qz + c)}\right) + S(r, f). \tag{3.3}
\]

From (2.9), we obtain

\[
N_2(r, \frac{1}{F}) \leq N_{k+2}\left(r, \frac{1}{P(f)g(qz + c)}\right) + S(r, f) \\
\leq (k + 2)N(r, \frac{1}{f}) + N\left(r, \frac{1}{f(qz + c)}\right) + kN(r, f) + S(r, f) \tag{3.4}
\]

\leq (2k + 3)T(r, f) + S(r, f).
Similarly as above, we have

\[(n+1)T(r,g) \leq T(r,G) - N_2\left(\frac{1}{G}\right) + N_{k+2}\left(\frac{1}{P(g)g(qz+c)}\right) + S(r,g)\]  

(3.5)

\[N_2\left(\frac{1}{G}\right) \leq (2k+3)T(r,g) + S(r,g).\]  

(3.6)

If the (i) of Lemma 2.7 is satisfied implies that

\[\max\{T(r,F)T(r,G)\} \leq N_2\left(\frac{1}{F}\right) + N_2\left(\frac{1}{G}\right) + N_2(r,F)
\]

\[+ N_2(r,G) + S(r,F) + S(r,G).\]

Thus, combining above with (3.3)-(3.6) we obtain

\[(n+1)\{T(r,f) + T(r,g)\} \leq 2[N(r,f) + N(r,g)] + 2N_{k+2}\left(\frac{1}{P(f)f(qz+c)}\right)
\]

\[+ 2N_{k+2}\left(\frac{1}{P(g)g(qz+c)}\right) + S(r,f) + S(r,g)
\]

\[\leq 2(2k+4)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g).
\]

(3.7)

Which is in contradiction with \(n \geq 4k + 8\). Hence \(F = G\) or \(FG = 1\). From Lemma 2.10, we get \(f = tg\) for \(t^m = t^{n+1} = 1\) and \(f\) and \(g\) satisfy the algebraic equation \(R(f,g) = 0\), where \(R(w_1, w_2) = P(w_1)w_1(qz+c) - p(w_2)w_2(qz+c)\).

**Proof of Theorem 1.2.** Let

\[F = [P(f)f(qz+c)]^{(k)}, \quad G = [P(g)g(qz+c)]^{(k)}.\]

Let \(H\) be defined as in Lemma 2.8. Assume that \(H \neq 0\), from (2.6) we get

\[T(r,F) + T(r,G) \leq 2\left[N_2\left(\frac{1}{F}\right) + N_2\left(\frac{1}{G}\right) + N_2(r,F) + N_2(r,G)\right]
\]

\[+ 3\left[N(r,F) + N(r,G) + N\left(\frac{1}{F}\right) + N\left(\frac{1}{G}\right)\right]
\]

\[+ S(r,F) + S(r,G).\]  

(3.7)
Combining above with (3.3)-(3.6) and (2.9), we obtain
\[
(n + 1)[T(r, f) + T(r, g)] \leq T(r, F) + T(r, G) - N_2(r, \frac{1}{F}) - N_2(r, \frac{1}{G})
\]
\[
+ N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right) + N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right)
\]
\[
+ S(r, f) + S(r, g)
\]
\[
\leq 2(N_2(r, F) + N_2(r, G)) + 2N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right)
\]
\[
+ 2N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right) + 3 \left[ \mathcal{N}(r, \frac{1}{F}) + \mathcal{N}(r, \frac{1}{G}) \right]
\]
\[
+ S(r, f) + S(r, g)
\]
\[
\leq 2(2k + 4)\{T(r, f) + T(r, g)\} + 3(2k + 2)\{T(r, f) + T(r, g)\}
\]
\[
+ S(r, f) + S(r, g)
\]
\[
\leq (10k + 14)\{T(r, f) + T(r, g)\}
\]
which is a contradiction with \( n \geq 10k + 14 \). Thus we get \( H \equiv 0 \). The following proof is trivial, we give the complete proof. By integration for \( H \) twice, we obtain
\[
F = \frac{(b + 1)G + (a - b - 1)}{bG + (a - b)}, \quad G = \frac{(a - b - 1) - (a - b)F}{Fb - (b + 1)}
\] (3.8)
which implies that \( T(r, F) = T(r, G) + O(1) \). Since
\[
T(r, F) \leq T(r, P(f)f(qz + c)) + S(r, f)
\]
\[
\leq (n + 1)T(r, f) + S(r, f),
\]
then \( S(r, F) = S(r, f) \). So \( S(r, G) = S(r, g) \) is. We distinguish into three cases as follows.

**Case 1.** \( b \neq 0, -1 \). If \( a - b - 1 \neq 0 \), then by (3.8), we get
\[
\mathcal{N}(r, \frac{1}{F}) = \mathcal{N} \left( r, \frac{1}{F - \frac{a-b-1}{b+1}} \right)
\] (3.9)
By the Nevanlinna second main theorem, (2.8) and (2.9), we have
\[
(n + 1)T(r, g) \leq T(r, G) + N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right)
\]
\[
- N \left( r, \frac{1}{G} \right) + S(r, g)
\]
\[
\leq (k + 1)T(r, g) + (2k + 2)T(r, f) + S(r, f) + S(r, g)
\] (3.10)
Similarly, we get
\[(n + 1)T(r, f) \leq (k + 1)T(r, f) + (2k + 2)T(r, g) + S(r, f) + S(r, g). \quad (3.11)\]

Thus from (3.10) and (3.11), then
\[(n + 1)\{T(r, f) + T(r, g)\} \leq (3k + 3)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).\]
Which is in contradiction with \(n \geq 10k + 14\). Thus \(a - b - 1 = 0\), then
\[F = \frac{(b + 1)G}{bG + 1}\]
using the same method as above, we get
\[(n + 1)T(r, g) \leq T(r, G) + N_k \left( r, \frac{1}{P(g)g(qz + c)} \right)
- N(r, \frac{1}{G}) + S(r, g)
\leq N_k \left( r, \frac{1}{P(g)g(qz + c)} \right) + \mathcal{N} \left( r, \frac{1}{G + \frac{1}{b}} \right) + S(r, g)
\leq (k + 1)T(r, g) + S(r, g).
Which is a contradiction.

**Case 2.** \(b = 0, a \neq 1\). From (3.8), we have
\[F = \frac{G + a - 1}{a}.\]
Similarly, we also can get a contradiction. Thus \(a = 1\) follows, it implies that \(F = G\).

**Case 3.** \(b = -1, a \neq -1\). From (3.8), we obtain
\[F = \frac{a}{a + 1 - G}.\]
Similarly, we can get a contradiction, \(a = -1\) follows. Thus, we get \(FG = 1\).

From Lemma 2.10, we get \(f = tg\) for \(t^m = \frac{t^{n+1}}{1} = 1\), and \(f\) and \(g\) satisfy the algebraic expression \(R(f, g) = 0\), where
\[R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c)\]
Thus, we have completed the proofs.
References


