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UNIQUENESS OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract: In this paper, we study the zero distributions on the derivatives of q-shift difference polynomials of meromorphic functions with zero order and obtain two theorems that extend results of [3].

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1. Introduction

In this paper, a meromorphic functions f means meromorphic in the complex plane. If no poles occur, then f reduces to an entire function. Throughout of this paper, we denote by $\rho(f)$ and $\rho_2(f)$ the order of f and the hyper order of f (Laine, 1993 and Yang and Yi, 2003). In addition, if $f-a$ and $g-a$ have the same zeros, then we say that f and g share the value a IM(ignoring multiplicities). If $f-a$ and $g-a$ have the same zeros, then we say that f and g share the value a CM(counting multiplicities). We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory(Halburd Korhonen and Tohge; Laine, 1993 and Yang and Yi, 2003).

Given a meromorphic function $f(z)$, recall that $\alpha(z) \neq 0, \infty$ is a small function with respect to $f(z)$, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ is used to denote any quantity satisfying $S(r,f) = o(T(r,f))$, and $r \to \infty$ outside of a

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possible exceptional set of finite logarithmic measure.

Recently, K. Liu, X. Liu and T. B. Cao (2012) proved the following.

Theorem A. (Liu, Liu and Coa, 2012) *Let f be a transcendental entire function of $\rho_2(f) < 1$. For $n \geq t(k+1)+1$, then $[P(f)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem B. (Liu, Liu and Coa, 2012) *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, not a periodic function with period c. If $n \geq (t+1)(k+1)+1$, then $[f(z)^n(\Delta_c f)^s]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem C. (Liu, Liu and Coa, 2012) *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$. For $n \geq t(k+1)+5$, then $[P(f)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem D. (Liu, Liu and Coa, 2012) *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$. If $n \geq (t+2)(k+1)+3+s$, then $[P(f)(\Delta_c f)^s]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem E. (Liu, Liu and Coa, 2012) *Let f and g be a transcendental entire function of $\rho_2(f) < 1$, $n \geq 2k+m+6$. If $[f^n(f^m-1)f(z+c)]^{(k)}$and $[g^n(g^m-1)g(z+c)]^{(k)}$share the 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

Theorem F. (Liu, Liu and Coa, 2012) *The conclusion of Theorem E is also valid, if $n \geq 5k+4m+12$. and $[f^n(f^m-1)f(z+c)]^{(k)}$and $[g^n(g^m-1)g(z+c)]^{(k)}$ share the 1 IM.*

In 2013, Harina P. Waghamore and Tanuja A. extend Theorem E and Theorem F to meromorphic functions.

Theorem G. (Harina P.W and Tanuja A, 2013) *Let f and g be a transcendental meromorphic function with zero order. If $n \geq 4k+m+8$, $[f^n(f^m-1)f(qz+c)]^{(k)}$and $[g^n(g^m-1)g(qz+c)]^{(k)}$share the 1 CM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

Theorem H. (Harina P.W and Tanuja A, 2013) *Let f and g be a transcendental meromorphic function with zero order. If $n \geq 5k+4m+17$, $[f^n(f^m-1)f(qz+c)]^{(k)}$and $[g^n(g^m-1)g(qz+c)]^{(k)}$share the 1 IM, then $f = tg$, where $t^{n+1} = t^m = 1$.*

In this paper, we extend Theorem G and Theorem H to difference polynomials and obtain the following results.

Theorem 1. *Let f and g be a transcendental meromorphic(resp. entire) function with zero order. If $n \geq 4k+8(n \geq 2k+6)$, $[P(f)f(qz+c)]^{(k)}$ and $[P(g)g(qz+c)]^{(k)}$ share the 1 CM, then:*

1. $f \equiv tg$ for a constant t such that $t^d = 1$.

2. f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1,w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$.

Theorem 2. *Let f and g be a transcendental meromorphic(resp. entire) function with zero order. If $n \geq 10k+14(n \geq 5k+12)$, $[P(f)f(qz+c)]^{(k)}$ and $[P(g)g(qz+c)]^{(k)}$ share the 1 IM, then the conclusion of theorem 1 still holds.*

2. Some Lemmas

In this section, we present some definitions and lemmas which will be needed in the sequel.

Lemma 2.1. (Halburd, Korhonen and Tohge, Theorem 5.1) *Let $f(z)$ be a transcendental meromorphic function of $\rho_1(f) < 1$, $\varsigma < 1$, ϵ is enough small number. Then*

$$m(r, \frac{f(z+c)}{f(z)}) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\epsilon}}\right) = S(r,f), \tag{2.1}$$

for all r outside of a set of finite logarithmic measure.Combining the proof of (Luo and Lin, 2011, Lemma 5) with Lemma 2.1, we can get the following Lemma 2.2.

Lemma 2.2. *Let $f(z)$ be a transcendental entire functionof $\rho_2(f) < 1$. If $F = P(f)f(z+c)$, then*

$$T(r,F) = T(P(f)f(z)) + S(r,f) = (n+1)T(r,f) + S(r,f). \tag{2.2}$$

Lemma 2.3. *(Liu, Liu and Cao, 2012, Lemma 2.5) Let $f(z)$ be a transcendental meromorphic function of $\rho_2(f) < 1$. If $F = P(f)f(z+c)$, then*

$$(n-1)T(r,f) + S(r,f) \leq T(r,F) \leq (n+1)T(r,f) + S(r,f). \tag{2.3}$$

Lemma 2.4. *(Zhang and Korhonen, 2010, Theorem 1.1) Let $f(z)$ be a transcendental meromorphic function of zero order. Then*

$$T(r, f(qz)) = T(r, f(z)) + S(r, f)$$

on a set of logarithmic density 1.

The following lemma has little modifications of the original version (Theorem 2.1 of Chiang and Feng, 2008).

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function of finite order. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f). \tag{2.4}$$

combining Lemma 2.4 with Lemma 2.5, we get the following result easily.

Lemma 2.6. *Let $f(z)$ be a transcendental meromorphic function of zero order. Then*

$$T(r, f(qz+c)) = T(r, f(z)) + S(r, f) \tag{2.5}$$

on a set of logarithmic density 1.

Lemma 2.7. (Yang and Hua, 1997, Lemma 3) *Let F and G be non constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(i) $\max\{T(r,F), T(r,G)\} \leq N_2\left(r, \frac{1}{F}\right) + N_2(r,F) + N_2\left(r, \frac{1}{G}\right) + N_2(r,G) + S(r,F) + S(r,G)$.

(ii) $F = G$.

(iii) $F.G = 1$.

Lemma 2.8. (Xu an Yi, 2007, Lemma 2.3) *Let F and G be non constant meromorphic function sharing the value 1 IM. Let*

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}.$$

If $H \neq 0$, then

$$T(r,F) + T(r,G) \leq 2\left(N_2\left(r, \frac{1}{F}\right) + N_2(r,F) + N_2\left(r, \frac{1}{G}\right) + N_2(r,G)\right)$$

$$+3\left(\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)\right)$$
$$+S(r,F)+S(r,G). \tag{2.6}$$

Lemma 2.9. *Let $f(z)$ be a meromorphic function, and p, k be positive integers. Then*

$$T(r,f^{(k)})\leq T(r,f)+k\overline{N}(r,f)+S(r,f). \tag{2.7}$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.8}$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq k\overline{N}(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.9}$$

Lemma 2.10. *Let f and g be a transcendental meromorphic function of zero order. If $n\geq k+6$ and*

$$[P(f)f(qz+c)]^{(k)}=[P(g)g(qz+c)]^{(k)} \tag{2.10}$$

then $f=tg$, where $t^{n+1}=t^m=1$, and f and g satisfy the algebraic equation

$$R(w_1,w_2)=P(w_1)w_1(qz+c)-P(w_2)w_2(qz+c).$$

Proof. From (2.10), we have

$$P(f)f(qz+c)=P(g)g(qz+c)+Q(z).$$

Where $Q(z)$ is a polynomial of degree atmost $k=1$. If $Q(z)\neq 0$, then we have

$$\frac{P(f)f(z+c)}{Q(z)}=\frac{P(g)g(qz+c)}{Q(z)}+1$$

From the second main theorem of Nevanlinna and by Lemma 2.2, we have

$$\begin{aligned}(n+1)T(r,f) &= T\left(r, \frac{P(f)f(qz+c)}{Q(z)}\right) + S(r,f)\\ &\le \overline{N}\left(r, \frac{P(f)f(qz+c)}{Q(z)}\right) + \overline{N}\left(r, \frac{Q(z)}{P(f)f(qz+c)}\right)\\ &+ \overline{N}\left(r, \frac{Q(z)}{P(g)g(qz+c)}\right) + S(r,f)\\ &\le \overline{N}(r,P(f)) + \overline{N}(r,f(qz+c)) + \overline{N}\left(r, \frac{1}{P(f)}\right) \qquad (2.11)\\ &\quad + \overline{N}\left(r, \frac{1}{f(qz+c)}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g(qz+c)}\right) + S(r,f) + S(r,g)\\ &\le 4T(r,f) + 2T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Similarly as above, we have

$$(n+1)T(r,g) \le 4T(r,g) + 2T(r,f) + S(r,f) + S(r,g). \qquad (2.12)$$

Thus, we get

$$(n+1)[T(r,f) + T(r,g)] \le 6[T(r,f) + T(r,g)] + S(r,f) + S(r,g). \qquad (2.13)$$

which is in contradiction with $n \ge k+6$. Hence, we get $Q(z) \equiv 0$, which implies that

$$P(f)f(qz+c) = P(g)g(qz+c). \qquad (2.14)$$

Set $h(z) = \frac{f(z)}{g(z)}$, we break the rest of the proof into two cases.

Case 1. Suppose $h(z)$ is a constant. Then by substituting $f = gh$ into (2.14), we obtain

$$g(qz+c)[a_n g^n(h^{n+1} - 1) + a_{n-1}g^{n-1}(h^n - 1) + ... + a_0(h-1)] \equiv 0 \qquad (2.15)$$

where $a_n(\neq 0), a_{n-1}, .., a_0$ are complex constants. By the fact that g is a transcendental entire functions, we have $g(qz+c) \not\equiv 0$. Hence, we obtain

$$[a_n g^n(h^{n+1} - 1) + a_{n-1}g^{n-1}(h^n - 1) + ... + a_0(h-1)] \equiv 0. \qquad (2.16)$$

Equation (2.16) implies that $h^{n+1} = 1$ and $h^{i+1} = 1$ when $a_i \neq 0$ for $i = 0, 1, ..., n-1$. Therefore $h^d = 1$, where $d = GCD(\lambda_0, \lambda_1, ..., \lambda_n)$.

Case 2. Suppose that h is not a constant, then we know by (2.14) that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(w_1,w_2)=p(w_1)w_1(qz+c)-p(w_2)w_2(qz+c)$.

Lemma 2.11. *Let f and g be transcendental entire function of finite order. If $n \geq k+4$, and $[P(f)f(qz+c)]^{(k)}=[P(g)g(qz+c)]^{(k)}$ then the condition of Lemma 2.10 holds.*

Proof. Substituting $\overline{N}(r,f)=\overline{N}(r,g)=0$ and proceeding as in the proof of Lemma 2.10, we get Lemma 2.11.

3. Proof of the Theorem

Proof of Theorem 1.1. Let $F=[P(f)f(qz+c)]^{(k)}$ and $G=[P(g)g(qz+c)]^{(k)}$. Thus F and G share the value 1 CM. From (2.7) and f is a transcendental meromorphic function, then

$$T(r,F)\leq T(r,P(f)f(qz+c))+k\overline{N}(r,f)+S(r,P(f)f(qz+c)) \tag{3.1}$$

combining (3.1) with Lemma 2.2, we have $S(r,F)=S(r,f)$. We also have $S(r,G)=S(r,g)$, from the same reason as above, from (2.8) we obtain

$$\begin{aligned} N_2(r.\frac{1}{F}) &= N_2\left(r,\frac{1}{[P(f)f(qz+c)]^{(k)}}\right)\\ &\leq T(r,F)-T(r,P(f)f(qz+c))\\ &\quad+N_{k+2}\left(r,\frac{1}{P(f)f(qz+c)}\right)+S(r,f). \end{aligned} \tag{3.2}$$

Thus, from Lemma 2.2 and (3.2) we get

$$\begin{aligned} (n+1)T(r,f) &= T(r,P(f)f(qz+c))+S(r,f)\\ &\leq T(r,F)-N_2(r,\frac{1}{F})\\ &\quad+N_{k+2}\left(r,\frac{1}{P(f)f(qz+c)}\right)+S(r,f) \end{aligned} \tag{3.3}$$

From (2.9), we obtain

$$\begin{aligned} N_2(r.\frac{1}{F}) &\leq N_{k+2}\left(r,\frac{1}{P(f)f(qz+c)}\right)+S(r,f)\\ &\leq (k+2)N(r,\frac{1}{f})+N\left(r,\frac{1}{f(qz+c)}\right)+k\overline{N}(r,f)+S(r,f)\\ &\leq (2k+3)T(r,f)+S(r,f). \end{aligned} \tag{3.4}$$

Similarly as above, we have

$$(n+1)T(r,g) \leq T(r,G) - N_2(r,\frac{1}{G}) + N_{k+2}\left(r, \frac{1}{P(g)g(qz+c)}\right) + S(r,g) \quad (3.5)$$

$$N_2(r.\frac{1}{G}) \leq (2k+3)T(r,g) + S(r,g). \quad (3.6)$$

If the (i) of Lemma 2.7 is satisfied implies that

$$max\{T(r,F)T(r,G)\} \leq N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G).$$

Thus, combining above with (3.3)-(3.6) we obtain

$$\begin{aligned}(n+1)\{T(r,f)+T(r,g)\} &\leq 2[N(r,f)+N(r,g)] + 2N_{k+2}\left(r, \frac{1}{P(f)f(qz+c)}\right) \\ &+ 2N_{k+2}\left(r, \frac{1}{P(g)g(qz+c)}\right) + S(r,f) + S(r,g) \\ &\leq 2(2k+4)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).\end{aligned}$$

Which is in contradiction with $n \geq 4k+8$. Hence $F = G$ or $FG = 1$. From Lemma 2.10, we get $f = tg$ for $t^m = t^{n+1} = 1$ and f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1,w_2) = P(w_1)w_1(qz+c) - p(w_2)w_2(qz+c)$.

Proof of Theorem 1.2. Let

$$F = [P(f)f(qz+c)]^{(k)}, \quad G = [P(g)g(qz+c)]^{(k)}.$$

Let H be defined as in Lemma 2.8. Assume that $H \neq 0$, from (2.6) we get

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\left[N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_2(r,F) + N_2(r,G)\right] \\ &+ 3\left[\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right)\right] \\ &+ S(r,F) + S(r,G).\end{aligned} \quad (3.7)$$

Combining above with (3.3)-(3.6) and (2.9), we obtain

$$\begin{aligned}(n+1)[T(r,f)+T(r,g)] &\le T(r,F)+T(r,G)-N_2(r,\frac{1}{F})-N_2(r,\frac{1}{G})\\ &+N_{k+2}\left(r,\frac{1}{P(f)f(qz+c)}\right)+N_{k+2}\left(r,\frac{1}{P(g)g(qz+c)}\right)\\ &+S(r,f)+S(r,g)\\ &\le 2\left(N_2(r,F)+N_2(r,G)\right)+2N_{k+2}\left(r,\frac{1}{P(f)f(qz+c)}\right)\\ &+2N_{k+2}\left(r,\frac{1}{P(g)g(qz+c)}\right)+3\left[\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})\right]\\ &+S(r,f)+S(r,g)\\ &\le 2(2k+4)\{T(r,f)+T(r,g)\}+3(2k+2)\{T(r,f)+T(r,g)\}\\ &+S(r,f)+S(r,g)\\ &\le (10k+14)\{T(r,f)+T(r,g)\}\end{aligned}$$

which is a contradiction with $n \ge 10k+14$. Thus we get $H \equiv 0$. The following proof is trivial, we give the complete proof. By integration for H twice, we obtain

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)},\quad G=\frac{(a-b-1)-(a-b)F}{Fb-(b+1)} \tag{3.8}$$

which implies that $T(r,F)=T(r,G)+O(1)$. Since

$$\begin{aligned}T(r,F)&\le T(r,P(f)f(qz+c))+S(r,f)\\ &\le (n+1)T(r,f)+S(r,f),\end{aligned}$$

then $S(r,F)=S(r,f)$. So $S(r,G)=S(r,g)$ is. We distinguish into three cases as follows.

Case 1. $b \ne 0,-1$. If $a-b-1 \ne 0$, then by (3.8), we get

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{F-\frac{a-b-1}{b+1}}\right) \tag{3.9}$$

By the Nevanlinna second main theorem, (2.8) and (2.9), we have

$$\begin{aligned}(n+1)T(r,g) &\le T(r,G)+N_{k+2}\left(r,\frac{1}{P(g)g(qz+c)}\right)\\ &-N\left(r,\frac{1}{G}\right)+S(r,g)\\ &\le (k+1)T(r,g)+(2k+2)T(r,f)+S(r,f)+S(r,g)\end{aligned} \tag{3.10}$$

Similarly, we get

$$(n+1)T(r,f) \leq (k+1)T(r,f) + (2k+2)T(r,g) + S(r,f) + S(r,g). \quad (3.11)$$

Thus from (3.10) and (3.11), then

$$(n+1)\{T(r,f)+T(r,g)\} \leq (3k+3)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).$$

Which is in contradiction with $n \geq 10k+14$. Thus $a-b-1=0$, then

$$F = \frac{(b+1)G}{bG+1} \quad (3.12)$$

using the same method as above, we get

$$\begin{aligned}(n+1)T(r,g) &\leq T(r,G) + N_k\left(r, \frac{1}{P(g)g(qz+c)}\right) \\ &\quad - N(r, \frac{1}{G}) + S(r,g) \\ &\leq N_k\left(r, \frac{1}{P(g)g(qz+c)}\right) + \overline{N}\left(r, \frac{1}{G+\frac{1}{b}}\right) + S(r,g) \\ &\leq (k+1)T(r,g) + S(r,g).\end{aligned}$$

Which is a contradiction.

Case 2. $b=0$, $a \neq 1$. From (3.8), we have

$$F = \frac{G+a-1}{a}.$$

Similarly, we also can get a contradiction. Thus $a=1$ follows, it implies that $F=G$.

Case 3. $b=-1$, $a \neq -1$. From (3.8), we obtain

$$F = \frac{a}{a+1-G}.$$

Similarly, we can get a contradiction, $a=-1$ follows. Thus, we get $F.G=1$.

From Lemma 2.10, we get $f=tg$ for $t^m = t^{n+1} = 1$, and f and g satisfy the algebraic expression $R(f,g)=0$, where

$$R(w_1,w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$$

Thus, we have completed the proofs.

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