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# MULTIPLE VALUES AND UNIQUENESS OF MEROMORPHIC FUNCTIONS ON ANNULI

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**Abstract:** The purpose of this paper is to deal with the multiple values and deficiencies on the uniqueness problem of meromorphic functions on annuli, and extend some uniqueness theorems of meromorphic functions dealing with multiple values and deficient values to meromorphic functions on annuli.

**AMS Subject Classification:** 30D35
**Key Words:** Nevanlinna theory, the annulus

## 1. Introduction

We assume that the reader is familiar with Nevanlinna's theory of meromorphic functions (see [13]). The uniqueness of meromorphic functions in the complex plane $\mathbb{C}$ is an important subject in the value distribution theory. The uniqueness of meromorphic functions with shared values on $\mathbb{C}$ attracted many investigations (see book [13]). Here we shall mainly study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. By Doubly connected mapping theorem [9] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0,\ R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

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The uniqueness theory of meromorphic function is an interesting problem, recently Khrystiyanyn and Kondratyuk [6, 7 ] proposed the Nevenlinna theory of meromorphic functions on annuli, see also an important paper [3]. We will show the basic notions of the Nevanlinna theory on annuli in the next section. In this paper, we mainly study the uniqueness problem of meromorphic functions on annuli, and extend some uniqueness theorems of meromorphic functions dealing with multiple values and deficient values to meromorphic functions on annili.

## 2. Basic Notations in the Nevanlinna Theory on Annuli

Let $f$ be a meromorphic function on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$. We recall classical notations of Nevanlinna theory as follows

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log R,$$

$$m(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\theta})| d\theta, \qquad T(R, f) = N(R, f) + m(R, f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t, f)$ is the counting function of poles of the function $f$ in $\{z : |z| \leq t\}$. Here we show the notations of the Nevanlinna theory on annuli. Let

$$N_1(R, f) = \int_{\frac{1}{R}}^1 \frac{n_1(t, f)}{t} dt, \qquad N_2(R, f) = \int_1^R \frac{n_2(t, f)}{t} dt,$$

$$m_0(R, f) = m(R, f) + m\left(\frac{1}{R}, f\right), \qquad N_0(R, f) = N_1(R, f) + N_2(R, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of the poles of the function $f$ in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. The Nevanlinna charecteristic of $f$ on the annulus $\mathbb{A}$ is defined by

$$T_0(R, f) = m_0(r, f) + N_0(R, f) - 2m(1, f).$$

**Definition .** [4] Let $f(z)$ be a non-constant meromorphic function on the annulus $\mathbb{A}(R_0) = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 < +\infty$. The function $f$ is called a transcedental or admissible meromorphic function on the annulus $\mathbb{A}(R_0)$ provided that

$$\lim_{R\to\infty} sup \frac{T_0(R, f)}{logR} = \infty, \qquad 1 < R < R_0 = +\infty \tag{2.1}$$

or

$$\lim_{R\to R_0} sup\frac{T_0(R,f)}{-log(R_0-R)} = \infty, \quad 1 < R < R_0 < +\infty \tag{2.2}$$

respectively.

Thus for a transcedental or admissible meromorphic function on the annulus $\mathbb{A}$, $S(R,f) = o(T_0(R,f))$ holds for all $1 < R < R_0$ except for the set $\triangle_R$ or the set $\triangle'_R$ mentioned in Theorem 2.1, respectively.

**Definition .** Let $f(z)$ be a meromorphic functions on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$. Let $a$ be any arbitrary complex number. The Valiron deficiency of $f(z)$ on the annulus $\mathbb{A}$ with respect to the value $'a'$ will be defined by

$$\Delta_0(a,f) = \overline{\lim_{r\to\infty}} \frac{m_0(R,\frac{1}{f-a})}{T_0(R,f)} = 1 - \underline{\lim_{r\to\infty}} \frac{N_0(R,\frac{1}{f-a})}{T_0(R,f)}.$$

We denote the deficiency of $a \in \overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ with respect to a meromorphic function $f$ on the annulus $\mathbb{A}$ by

$$\delta_0(a,f) = \delta_0(0,f-a) = \underline{\lim_{r\to R_0}} \frac{m_0\left(R,\frac{1}{f-a}\right)}{T_0(R,f)} = 1 - \overline{\lim_{r\to R_0}} \frac{N_0\left(R,\frac{1}{f-a}\right)}{T_0(R,f)},$$

and denote the reduced deficiency by

$$\Theta_0(a,f) = \Theta_0(0,f-a) = 1 - \overline{\lim_{r\to R_0}} \frac{\overline{N_0}\left(R,\frac{1}{f-a}\right)}{T_0(R,f)},$$

where

$$\begin{aligned} \overline{N_0}\left(R,\frac{1}{f-a}\right) &= \overline{N_1}\left(R,\frac{1}{f-a}\right) + \overline{N_2}\left(R,\frac{1}{f-a}\right) \\ &= \int_{\frac{1}{R}}^{1} \frac{\overline{n_1}\left(t,\frac{1}{f-a}\right)}{t}dt + \int_1^R \frac{\overline{n_2}\left(t,\frac{1}{f-a}\right)}{t}dt \end{aligned}$$

in which each zero of the function $f-a$ is counted only once.

We use $\overline{n_1}^{k)}\left(t,\frac{1}{f-a}\right)$ $(or\ \ \overline{n_1}^{(k}\left(t,\frac{1}{f-a}\right))$ to denote the counting function of poles of the function $\frac{1}{f-a}$ with the multiplicities $\leq k$ $(or\ \ > k)$ in $\{z : t < |z| \leq 1\}$, each point counted only once. Similarly, we can give the notations $\overline{N_1}^{k)}(t,f), \overline{N_1}^{(k}(t,f), \overline{N_2}^{k)}(t,f), \overline{N_2}^{(k}(t,f), \overline{N_0}^{k)}(t,f), \overline{N_0}^{(k}(t,f)$.

**Lemma 2.1.** [**3**] Let $f$ be a non constant meromorphic function on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$. Let $'a'$ be an arbitrary complex number, and $k$ be a positive integer. Then

$$\begin{aligned}(i)\ \overline{N_0}\left(R, \frac{1}{f-a}\right) &\leq \frac{k}{k+1}\overline{N_0}^{k)}\left(R, \frac{1}{f-a}\right) + \frac{1}{k+1}\overline{N_0}\left(R, \frac{1}{f-a}\right),\\ (ii)\ \overline{N_0}\left(R, \frac{1}{f-a}\right) &\leq \frac{k}{k+1}\overline{N_0}^{k)}\left(R, \frac{1}{f-a}\right) + \frac{1}{k+1}T_0(R,f) + O(1).\end{aligned}$$

**Theorem 2.1.** [**3**](The Second Fundamental Theorem in the annulus). Let $f$ be a non constant meromorphic function on the annulus

$$\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\},$$

where $1 < R < R_0 \leq +\infty$. Let $a_1, a_2, ..., a_q$ be $q$ distinct complex numbers in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, let $k_1, k_2, ..., k_q$ be $q$ positive integers, and let $\lambda \geq 0$. Then:

$$\begin{aligned}(i)\ (q-2)T_0(R,f) &< \sum_{j=1}^{q} N_0\left(R, \frac{1}{f-a_j}\right) - N_0^{(1)}(R,f) + S(R,f),\\ (ii)\ (q-2)T_0(R,f) &< \sum_{j=1}^{q} \overline{N_0}\left(R, \frac{1}{f-a_j}\right) + S(R,f),\end{aligned}$$

$$\begin{aligned}(iii)\ (q-2)T_0(R,f) < &\sum_{j=1}^{q} \frac{k_j}{k_j+1}\overline{N_0}^{k_j)}\left(R, \frac{1}{f-a_j}\right)\\ &+\sum_{j=1}^{q} \frac{1}{k_j+1}N_0\left(R, \frac{1}{f-a_j}\right) + S(R,f),\end{aligned}$$

$$(iv)\ \left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right)T_0(R,f) < \sum_{j=0}^{q}\frac{k_j}{k_j+1}\overline{N_0}^{k_j)}\left(R, \frac{1}{f-a_j}\right)+S(R,f).$$

where

$$N_0^{(1)}(R,f) = N_0\left(R, \frac{1}{f'}\right) + 2N_0(R,f) - N_0(R,f'),$$

and

1. In the case, $R_0 = +\infty$,

$$S(R,f) = O\left(\log(RT_0(R,f))\right)$$

for $R \in (1,+\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\alpha-1}dR < +\infty$;

2. In the case, $R_0 < +\infty$,

$$S(R,f) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for $R \in (1,R_0)$ except for the set $\triangle'_R$ such that $\int_{\triangle'_R} \frac{dR}{(R_0 - R^{\alpha-1})} < +\infty$.

## 3. Multiple Values and Uniqueness of Meromorphic Functions on Annuli

Let $f$ be a meromorphic function on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$, and $'a'$ be a complex number in the extende complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Write $E(a,f) = \{z \in \mathbb{A} : f(z) - a = 0\}$, where each zero with multiplicity $m$ is counted $m$ times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,f)$. We use $\overline{E}_{k)}(a,f)$ to denote the set of zeros of $f - a$ with multiplicities no greater than $k$, in which each zero is counted only once.

We now show our main results below which is an analog of a result on the plane $\mathbb{C}$ obtained by H. X. Yi [12](see Theorem 3.19 and 3.20 in [13]).

**Theorem 3.1.** Let $f_1$ and $f_2$ be two transcedental or admissible meromorphic functions on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$. Let $a_j\,(j = 1,2,...,q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$, and $k_j\,(j = 1,2,...,q)$ be positive integers or $\infty$ satisfying

$$k_1 \geq k_2 \geq ... \geq k_q \tag{3.1}$$

and

$$\overline{E}_{k_j)}(a_j,f_1) = \overline{E}_{k_j)}(a_j,f_2), \qquad (j = 1,2,..,q). \tag{3.2}$$

Set

$$A_i = \frac{\delta_0(a_1,f_i) + \delta_0(a_2,f_i)}{k_3 + 1} + \sum_{j=3}^{q} \frac{\delta_0(a_j,f_i)}{k_j + 1} \quad (i = 1,2).$$

If

$$min\{A_1, A_2\} \geq 2 - \sum_{j=3}^{q} \frac{k_j}{k_j+1}, \tag{3.3}$$

and

$$max\{A_1, A_2\} > 2 - \sum_{j=3}^{q} \frac{k_j}{k_j+1}, \tag{3.4}$$

then $f_1(z) \equiv f_2(z)$.

*Proof.* We may assume, without loss of generality, that all $a_j$ $(j = 1, 2, ..., q)$ are finite, otherwise, a suitable Mobius transformation will be done. From Theorem 2.1, we have

$$\begin{aligned}(q-2)\, T_0(R, f_1) < &\sum_{j=1}^{q} \frac{k_j}{k_j+1} \overline{N_0}^{k_j)} \left(R, \frac{1}{f_1 - a_j}\right) \\ &+ \sum_{j=1}^{q} \frac{1}{k_j+1} N_0 \left(R, \frac{1}{f_1 - a_j}\right) + S(R, f_1). \end{aligned} \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned}(q-2)\, T_0(R, f_1) < &\frac{k_3}{k_3+1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(R, \frac{1}{f_1 - a_j}\right) \\ &+ \sum_{j=1}^{2} \left(\frac{k_j}{k_j+1} - \frac{k_3}{k_3+1}\right) N_0 \left(R, \frac{1}{f_1 - a_j}\right) \\ &+ \sum_{j=1}^{q} \frac{1}{k_j+1} N_0 \left(R, \frac{1}{f_1 - a_j}\right) + S(R, f_1) \\ = &\frac{k_3}{k_3+1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(R, \frac{1}{f_1 - a_j}\right) \\ &+ \sum_{j=1}^{2} \left(\frac{k_j}{k_j+1} - \frac{k_3}{k_3+1}\right) (1 - \delta_0(a_j, f_1)) T_0(R, f_1) \\ &+ \sum_{j=1}^{q} \frac{1}{k_j+1} (1 - \delta_0(a_j, f_1)) T_0(R, f_1) + S(R, f_1) \\ = &\frac{k_3}{k_3+1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(R, \frac{1}{f_1 - a_j}\right) \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&+\sum_{j=1}^{2}\left(\frac{k_j}{k_j+1}+\frac{1}{k_j+1}\right)(1-\delta_0(a_j,f_1))T_0(R,f_1)\\
&-\sum_{j=1}^{2}\frac{k_3}{k_3+1}(1-\delta_0(a_j,f_1))T_0(R,f_1)\\
&+\sum_{j=3}^{q}\frac{1}{k_j+1}(1-\delta_0(a_j,f_1))T_0(R,f_1)+S(R,f_1)\\
=&\frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R,\frac{1}{f_1-a_j}\right)+2T_0(R,f_1)\\
&-\sum_{j=1}^{2}\delta_0(a_j,f_1)T_0(R,f_1)-\frac{2\,k_3}{k_3+1}T_0(R,f_1)\\
&+\sum_{j=1}^{2}\frac{k_3}{k_3+1}\delta_0(a_j,f_1)T_0(R,f_1)\\
&+\sum_{j=3}^{q}\frac{1}{k_j+1}(1-\delta_0(a_j,f_1))T_0(R,f_1)+S(R,f_1)\\
=&\frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R,\frac{1}{f_1-a_j}\right)+2T_0(R,f_1)\\
&-\sum_{j=1}^{2}\frac{\delta_0(a_j,f_1)}{k_3+1}T_0(R,f_1)\\
&-\frac{2k_3}{k_3+1}T_0(R,f_1)+\sum_{j=3}^{q}\frac{1}{k_j+1}T_0(R,f_1)\\
&-\sum_{j=3}^{q}\frac{1}{k_j+1}(\delta_0(a_j,f_1))T_0(R,f_1)+S(R,f_1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left(\frac{\delta_0(a_1,f_1)+\delta_0(a_2,f_1)}{k_3+1}+\sum_{j=3}^{q}\frac{\delta_0(a_j,f_1)}{k_j+1}+\frac{2k_3}{k_3+1}+\sum_{j=3}^{q}\frac{k_j}{k_j+1}-2\right)T_0(R,f_1)\\
&\qquad<\frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R,\frac{1}{f_1-a_j}\right)
\end{aligned}
$$

$$+ S(R, f_1) \left( A_1 + \frac{2k_3}{k_3+1} + \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 \right) T_0(R, f_1)$$

$$< \frac{k_3}{k_3+1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left( R, \frac{1}{f_1 - a_j} \right) + S(R, f_1). \quad (3.7)$$

If $f_1(z) \not\equiv f_2(z)$, then from (3.2), we have

$$\sum_{j=1}^{q} \overline{N_0}^{k_j)} \left( R, \frac{1}{f_1 - a_j} \right) \leq N_0 \left( R, \frac{1}{f_1 - f_2} \right) \leq T_0(R, f_1) + T_0(R, f_2) + O(1). \quad (3.8)$$

From (3.7) and (3.8), we get

$$\left( A_1 + \frac{k_3}{k_3+1} + \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 \right) T_0(R, f_1) < \frac{k_3}{k_3+1} T_0(R, f_2) + S(R, f_1). \quad (3.9)$$

Combining (3.3) and (3.9)

$$T_0(R, f_1) < T_0(R, f_2) + S(R, f_1). \quad (3.10)$$

Similarly,

$$\left( A_2 + \frac{2k_3}{k_3+1} + \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 \right) T_0(R, f_2) < \frac{k_3}{k_3+1} T_0(R, f_1) + S(R, f_2) \quad (3.11)$$

and

$$T_0(R, f_2) < T_0(R, f_1) + S(R, f_2). \quad (3.12)$$

From (3.9) and (3.12) we have

$$\left( A_1 + \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 \right) T_0(R, f_1) < S(R, f_1). \quad (3.13)$$

while (3.10) and (3.11) mean

$$\left( A_2 + \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 \right) T_0(R, f_2) < S(R, f_2). \quad (3.14)$$

However according to (3.4), we know that one of the inequalities (3.13), (3.14) is a contradiction.

Hence $f_1(z) \equiv f_2(z)$. □

As a consequence of Theorem 3.1, we get the following corollary

**Corollary 3.1.** Let $f_1$ and $f_2$ be two transcedental or admissible meromorphic functions on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$. Let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers. Then:

(i) If $q = 6$ and $\overline{E}_{1)}(a_j, f_1) = \overline{E}_{1)}(a_j, f_2),\ (j = 1, 2, .., 6)$,

$$\sum_{j=1}^{6} max\{\delta_0(a_j, f_1), \delta_0(a_j, f_2)\} > 0,$$

then $f_1(z) \equiv f_2(z)$;

(ii) If $q = 5$ and $\overline{E}_{2)}(a_j, f_1) = \overline{E}_{2)}(a_j, f_2),\ (j = 1, 2, .., 5)$,

$$\sum_{j=1}^{5} max\{\delta_0(a_j, f_1), \delta_0(a_j, f_2)\} > 0,$$

then $f_1(z) \equiv f_2(z)$;

(iii) If $q = 5$ and $\overline{E}_{1)}(a_j, f_1) = \overline{E}_{1)}(a_j, f_2),\ (j = 1, 2, .., 5)$,

$$\sum_{j=1}^{5} min\{\delta_0(a_j, f_1), \delta_0(a_j, f_2)\} \geq 1,$$

$max\{\delta_0(a_j, f_1), \delta_0(a_j, f_2)\} > 1$, then $f_1(z) \equiv f_2(z)$.

**Theorem 3.2.** Let $f_1$ and $f_2$ be two transcedental or admissible meromorphic functions on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$. Let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$, and $k_j\,(j = 1, 2, ..., q)$ be positive integers or $\infty$ satisfying (3.1) and (3.2).Set

$$B_i = \frac{\Delta_0(a_1, f_i) + \Delta_0(a_2, f_i)}{k_3 + 1} + \sum_{j=1}^{q} \frac{\Delta_0(a_j, f_i)}{k_j + 1} \quad (i = 1, 2).$$

If

$$\sum_{j=3}^{q} \frac{k_j}{k_j + 1} = 2. \tag{3.15}$$

and

$$max\{B_1, B_2\} > 0. \tag{3.16}$$

Then $f_1(z) \equiv f_2(z)$.

*Proof.* We may assume, without loss of generality, that all $a_j$ $(j = 1, 2, ..., q)$ are finite, otherwise, a suitable Mobius transformation will be done. From (3.6), we have

$$\begin{aligned}
&(q-2)\,T_0(R, f_1) \\
&< \frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R, \frac{1}{f_1-a_j}\right) + \sum_{j=1}^{2}\left(\frac{k_j}{k_j+1} - \frac{k_3}{k_3+1}\right)N_0\left(R, \frac{1}{f_1-a_j}\right) \\
&+ \sum_{j=1}^{q}\frac{1}{k_j+1}N_0\left(R, \frac{1}{f_1-a_j}\right) + S(R, f_1) \\
&= \frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R, \frac{1}{f_1-a_j}\right) + \sum_{j=1}^{2}N_0\left(R, \frac{1}{f_1-a_j}\right) \\
&- \sum_{j=1}^{2}\frac{k_3}{k_3+1}N_0\left(R, \frac{1}{f_1-a_j}\right) + \sum_{j=3}^{q}\frac{1}{k_j+1}N_0\left(R, \frac{1}{f_1-a_j}\right) + S(R, f_1) \\
&= \frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R, \frac{1}{f_1-a_j}\right) + \sum_{j=1}^{2}\left(1 - \frac{k_3}{k_3+1}\right)N_0\left(R, \frac{1}{f_1-a_j}\right) \\
&+ \sum_{j=3}^{q}\frac{1}{k_j+1}N_0\left(R, \frac{1}{f_1-a_j}\right) + S(R, f_1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(q-2)\,T_0(R, f_1) \quad < \quad & \frac{k_3}{k_3+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R, \frac{1}{f_1-a_j}\right) \\
& + \frac{1}{k_3+1}\sum_{j=1}^{2}N_0\left(R, \frac{1}{f_1-a_j}\right) \\
& + \sum_{j=3}^{q}\frac{1}{k_j+1}N_0\left(R, \frac{1}{f_1-a_j}\right) + S(R, f_1) \qquad (3.17)
\end{aligned}$$

If $f_1(z) \not\equiv f_2(z)$, then from (3.2), we have

$$\begin{aligned}
\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(R, \frac{1}{f_1-a_j}\right) &\le N_0\left(R, \frac{1}{f_1-f_2}\right) \\
&\le T_0(R, f_1) + T_0(R, f_2) + O(1). \qquad (3.18)
\end{aligned}$$

From (3.17) and (3.18), we get

$$
\begin{aligned}
(q-2)\,T_0(R,f_1) &< \frac{k_3}{k_3+1}\left(T_0(R,f_1)+T_0(R,f_2)\right)\\
&\quad+\frac{1}{k_3+1}\sum_{j=1}^{2}N_0\left(R,\frac{1}{f_1-a_j}\right)\\
&\quad+\sum_{j=3}^{q}\frac{1}{k_j+1}N_0\left(R,\frac{1}{f_1-a_j}\right)+S(R,f_1) \qquad (3.19)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(q-2)\,T_0(R,f_2) &< \frac{k_3}{k_3+1}\left(T_0(R,f_1)+T_0(R,f_2)\right)\\
&\quad+\frac{1}{k_3+1}\sum_{j=1}^{2}N_0\left(R,\frac{1}{f_2-a_j}\right)\\
&\quad+\sum_{j=3}^{q}\frac{1}{k_j+1}N_0\left(R,\frac{1}{f_2-a_j}\right)+S(R,f_2) \qquad (3.20)
\end{aligned}
$$

From (3.19) and (3.20), we get

$$
\begin{aligned}
\left(q-2-\frac{2k_3}{k_3+1}\right)\sum_{i=1}^{2}T_0(R,f_i) &< \frac{1}{k_3+1}\sum_{i=1}^{2}\sum_{j=1}^{2}N_0\left(R,\frac{1}{f_i-a_j}\right)\\
&\quad+\sum_{i=1}^{2}\sum_{j=3}^{q}\frac{1}{k_j+1}N_0\left(R,\frac{1}{f_i-a_j}\right)\\
&\quad+S(R,f_1)+S(R,f_2) \qquad (3.21)\\
&< \left(\frac{2}{k_3+1}+\sum_{j=3}^{q}\frac{1}{k_j+1}+O(1)\right)\sum_{i=1}^{2}T_0(R,f_i).
\end{aligned}
$$

From (3.15) and (3.21), we get

$$
\left(q-2-\frac{2k_3}{k_3+1}\right)=\left(\frac{2}{k_3+1}+\sum_{j=3}^{q}\frac{1}{k_j+1}\right) \qquad (3.22)
$$

It follows from (3.21) and (3.22) that $B_1=B_2=0$.

This contradicts to (3.16). Hence $f_1(z)\equiv f_2(z)$. □

As a consequence of Theorem 3.2, we get the following corollary

**Corollary 3.2.** Let $f_1$ and $f_2$ be two transcedental or admissible meromorphic functions on the annulus $\mathbb{A} = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $1 < R < R_0 \leq +\infty$. Let $a_j$ $(j = 1, 2, ..., q)$ be $q$ distinct complex numbers. Then:

(i) If $q = 6$ and $\overline{E}_{1)}(a_j, f_1) = \overline{E}_{1)}(a_j, f_2)$, $(j = 1, 2, .., 6)$,

$$\sum_{j=1}^{6} max\{\Delta_0(a_j, f_1), \Delta_0(a_j, f_2)\} > 0,$$

then $f_1(z) \equiv f_2(z)$;

(ii) If $q = 5$ and $\overline{E}_{2)}(a_j, f_1) = \overline{E}_{2)}(a_j, f_2)$, $(j = 1, 2, .., 5)$,

$$\sum_{j=1}^{5} max\{\Delta_0(a_j, f_1), \Delta_0(a_j, f_2)\} > 0,$$

then $f_1(z) \equiv f_2(z)$.

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