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PRIME RADICAL IN MATRIX SEMIRING $\begin{pmatrix} \boldsymbol{R} & \boldsymbol{\Gamma} \\ \boldsymbol{S} & \boldsymbol{L} \end{pmatrix}$

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Abstract: In this paper we obtain the prime radical of both sided $\Gamma-$semiring (Nobusawa $\Gamma-$ semiring or Γ_N-semiring). Relation between the prime radicals of a Γ_N-semiring S and that of the matrix semiring $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$ is obtained.

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Key Words: Γ_N-semiring, operator semiring, prime radical, matrix semiring

1. Introduction

In 1934, H. S. Vandiver[11] introduced the notion of semiring as an algebraic structure. The set of nonnegative integers is the most trivial example of semiring. In 1894, Richard Dedekind observed that the ideals of a commutative ring does not form a ring. This was the first nontrivial example of semirings. The concept of Γ was first introduced by Nobusawa[5] when he introduced the notion of $\Gamma-$ring. Rao introduced the notion of Γ-semiring[7]. Suppose (S,+) and (Γ,+) are two additive commutative monoids. If there exits a mapping $S\times\Gamma\times S \to S$, (with $(m,\gamma, n)\to m\gamma n \in S$), satisfying the following conditions: for all m, n, p $\in$ S and for all γ, $\mu \in \Gamma$, $m\gamma(n + p) = m\gamma n+m\gamma p$; $(m + n)\gamma p = m\gamma p + n\gamma p$; $m(\gamma+\mu)n = m\gamma n+m\mu n$; $m\gamma(n\mu p) = (m\gamma n)\mu p$; $m\gamma 0=0\gamma m=0$, 0 is the zero element of S and $m\theta n=0$, θ is the zero element of Γ then S is called a Γ-semiring. If S is $\Gamma-$semiring, Γ is S-semiring and

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$(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$, $(\alpha a\beta)b\gamma = \alpha(a\beta b)\gamma = \alpha a(\beta b\gamma)$, for all s_1, s_2 $\in$S, $s_1\alpha s_2$=$s_1\beta s_2$ implies $\alpha = \beta$ then S is a Nobusawa Γ-semiring or simply Γ_N-semiring[10]. Let F be the free additive commutative semigroup[4] generated by S×Γ. Then the relation ρ on F, defined by $\sum_{i=1}^{m}(x_i, \alpha_i)\rho\sum_{j=1}^{n}(y_j, \beta_j)$ if and only if $\sum_{i=1}^{m} x_i\alpha_i a = \sum_{j=1}^{n} y_j\beta_j a$ for all a ∈S (m, n∈ Z^+) is a congruence on F. Congruence class containing $\sum_{i=1}^{m}(x_i, \alpha_i)$ is denoted by $\sum_{i=1}^{m}[x_i, \alpha_i]$. Then F/$\rho$ is an additive commutative semigroup. Now F/ρ forms a semiring with the multiplication defined by $(\sum_{i=1}^{m}[x_i, \alpha_i])(\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j]$. This semiring is denoted by L and called the left operator semiring [2] of the Γ-semiring S. Dually the right operator semiring R of the Γ-semiring S has been defined. By using a Nobusawa Γ−semiring S and its operator semirings L and R, we define the matrix semiring[9] S_2 or $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$. Olson et al [6] studied a lot on radicals of semiring. Dutta et al studied the prime radical[1] of one sided Γ−semiring using its operator semirings[2]. In this paper we investigate the prime radical of a Nobusawa Γ−semiring S and that of the matrix semiring$\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$. Readers are referred to [2, 3, 8, 10] for the requisite preliminaries on semiring and Γ−semiring.

2. Prime Radical in Γ_N-Semiring

It is well known that prime radical[1] $\mathcal{P}$(S) of a semiring (Γ-semiring) S is the intersection of all prime ideals of S and prime radical $\mathcal{P}(I)$ of an ideal I of S is the intersection of all prime ideals of S containing I. Dutta et al showed that if $\mathcal{P}(L)$ and $\mathcal{P}(R)$ are the prime radicals of the left and right operator semirings L and respectively of a Γ−semiring S then $\mathcal{P}(S) = \mathcal{P}(L)^+$, $\mathcal{P}(S)^{+'} = \mathcal{P}(L)$, P$(S)$=P$(R)^*$ and $\mathcal{P}(S)^{*'} = \mathcal{P}(R)$.

Proposition 1. *[1] Let S be a Γ-semiring with R , L its right and left operator semiring respectively. Suppose Λ is the set of prime ideals of L and Ω is the set of respective ideals of R. (i) If P∈ Ω then $(P^*)^{+'} \in \Lambda$. (ii) If Q∈ Λ then $(Q^+)^{*'} \in \Omega$.*

Theorem 2. *Let $\mathcal{P}(R)$, $\mathcal{P}(L)$ be respectively the Prime radicals of R and L. Then $[S\mathcal{P}(R), \Gamma] \subseteq \mathcal{P}(L)$ and $[\Gamma, \mathcal{P}(L)S] \subseteq \mathcal{P}(R)$.*

Proof. By definition of $(\ )^+$, we obtain $[\mathcal{P}(\mathrm{L})^+, \Gamma] \subseteq \mathcal{P}(\mathrm{L})$. We have

$$\mathcal{P}(S) = \mathcal{P}(R)^* \text{ and } \mathcal{P}(S) = \mathcal{P}(L)^+.$$

Hence $\mathcal{P}(\mathrm{L})^+ = \mathcal{P}(\mathrm{R})^*$. Consequently, $[\mathcal{P}(\mathrm{R})^*, \Gamma] \subseteq \mathcal{P}(\mathrm{L})$. Since $\mathcal{P}(\mathrm{R})$ is an ideal of R, $\mathrm{R}\mathcal{P}(\mathrm{R}) \subseteq \mathcal{P}(\mathrm{R})$ i.e., $[\Gamma, \mathrm{S}]\mathcal{P}(\mathrm{R}) \subseteq \mathcal{P}(\mathrm{R})$. Hence $[\Gamma, \mathrm{S}\mathcal{P}(\mathrm{R})] \subseteq \mathcal{P}(\mathrm{R})$ whence $\mathrm{S}\mathcal{P}(\mathrm{R}) \subseteq \mathcal{P}(\mathrm{R})^*$. Consequently, $[\mathrm{S}\mathcal{P}(\mathrm{R}), \Gamma] \subseteq \mathcal{P}(\mathrm{L})$.

Now by using $\mathcal{P}(L)^+ = \mathcal{P}(R)^*$ and definition of $(\ )^*$ we obtain

$$[\Gamma, \mathcal{P}(L)^+] \subseteq \mathcal{P}(R). \tag{2}$$

Since $\mathcal{P}(L)$ is an ideal of L, $\mathcal{P}(L)L \subseteq \mathcal{P}(L)$. Hence $\mathcal{P}(L)[S,\Gamma] \subseteq \mathcal{P}(L)$ whence $[\mathcal{P}(L)S,\Gamma] \subseteq \mathcal{P}(L)$. Hence by definition of $(\ )^+$, $\mathcal{P}(L)S \subseteq \mathcal{P}(L)^+$ whence $[\Gamma, \mathcal{P}(L)S] \subseteq [\Gamma, \mathcal{P}(L)^+]$. Hence by (2) we obtain $[\Gamma, \mathcal{P}(L)S] \subseteq \mathcal{P}(R)$. □

Theorem 3. *Suppose $\mathcal{P}(\Gamma)$, $\mathcal{P}(L)$ and $\mathcal{P}(R)$ are the prime radicals of Γ, L and R respectively. Then:*

(i) $\mathcal{P}(\Gamma) =^+ (\mathcal{P}(L))$ and $\mathcal{P}(L) =^{+'} (\mathcal{P}(\Gamma))$.

(ii) $\mathcal{P}(\Gamma) =^* (\mathcal{P}(R))$ and $\mathcal{P}(R) =^{*'} (\mathcal{P}(\Gamma))$.

Proof. (i) Let Λ, Ω be the collections of all prime ideals of Γ and L respectively. Then

$$\mathcal{P}(\Gamma) = \bigcap_{\Phi\in\Lambda} \Phi = \bigcap_{\Phi\in\Lambda} {}^+({}^{+'}\Phi)({}^+({}^{+'}\Phi) = \Phi) = {}^+(\bigcap_{\Phi\in\Lambda} {}^{+'}\Phi)$$
$$\supseteq {}^+(\bigcap_{Q\in\Omega} Q)(\Phi \in \Lambda \Rightarrow\ {}^{+'}\Phi \in \Omega) =^+ \mathcal{P}(L).$$

Again

$${}^+\mathcal{P}(L) = {}^+(\bigcap_{Q\in\Omega} Q) = \bigcap_{Q\in\Omega} {}^+Q \supseteq (\bigcap_{P\in\Lambda} P)(Q \in \Omega \Rightarrow\ {}^+Q \in \Lambda) = \mathcal{P}(\Gamma).$$

Consequently, $\mathcal{P}(\Gamma) =^+ (\mathcal{P}(\mathrm{L}))$.

By applying similar argument we deduce that $\mathcal{P}(\mathrm{L}) =^{+'} (\mathcal{P}(\Gamma))$.

(ii) The proofs are similar to the above. □

Following is an immediate corollary of the above theorem.

Corollary 4. $^{+}(\mathcal{P}(L)) =^{*} (\mathcal{P}(R))$.

Theorem 5. *(i) $\mathcal{P}(^{+}P) =^{+} (\mathcal{P}(P))$ and $\mathcal{P}(^{*}Q) =^{*} (\mathcal{P}(Q))$.*

(ii) $\mathcal{P}(^{+'}\Phi)=^{+'}(\mathcal{P}(\Phi))$ and $\mathcal{P}(^{*'}\Phi)=^{*'}(\mathcal{P}(\Phi))$, where P, Q, Φ are respectively the ideals of L, R and Γ.

Proof. (i) Let $\alpha \in\mathcal{P}(^{+}\mathrm{P})$ and A be any prime ideal of L containing P. Then $^{+}P \subseteq^{+} A$. Since $^{+}$A is a prime ideal of Γ and prime radical of an ideal is the intersection of all prime ideals containing the ideal, $\mathcal{P}(^{+}\mathrm{P})\subseteq^{+}$A whence $\alpha \in^{+}$A. Hence $[\mathrm{S},\alpha]\subseteq$A. This together with the arbitrariness of A implies that $[\mathrm{S},\alpha]\subseteq \mathcal{P}(\mathrm{P})$. So $\alpha \in^{+} (\mathcal{P}(\mathrm{P}))$. Consequently, $\mathcal{P}(^{+}\mathrm{P})\subseteq^{+} (\mathcal{P}(\mathrm{P}))$.

Conversely, suppose that $\beta \in^{+} (\mathcal{P}(P))$. Then

$$[S,\beta] \subseteq \mathcal{P}(P). \tag{1}$$

Let Φ be any prime ideal of Γ containing $^{+}$P. Then $^{+'}(^{+}\mathrm{P})\subseteq {}^{+'}\Phi$, i.e., $\mathrm{P}\subseteq {}^{+'}\Phi$. Since $^{+'}\Phi$ is a prime ideal of L, we obtain $\mathcal{P}(\mathrm{P})\subseteq {}^{+'}\Phi$. Hence by using (1) we see that $[\mathrm{S},\beta]\subseteq {}^{+'}\Phi$ whence $\beta \in^{+} (^{+'}\Phi) = \Phi$. So by the arbitrariness of Φ and the definition of radical of an ideal we deduce that $\beta \in \mathcal{P}(^{+}\mathrm{P})$. Consequently, $^{+}(\mathcal{P}(\mathrm{P}))\subseteq \mathcal{P}(^{+}\mathrm{P})$. Hence $\mathcal{P}(^{+}\mathrm{P})=^{+}(\mathcal{P}(\mathrm{P}))$.

The proof of the remaining part follows similarly.

(ii) Let $l \in^{+'}(\mathcal{P}(\Phi))$. Then

$$\Gamma l \subseteq (\mathcal{P}(\Phi)). \tag{2}$$

Let D be a prime ideal of L containing $^{+'}\Phi$. Then $^{+}(^{+'}\Phi)\subseteq^{+}$D. i.e., $\Phi\subseteq^{+}$D. Since $^{+}$D is a prime ideal of S, by using definition of prime radical of an ideal we deduce that $\mathcal{P}(\Phi)\subseteq^{+}$D and hence by using (2) we obtain $\Gamma l \subseteq^{+}$D whence $l \in^{+'}(^{+}\mathrm{D})$=D. So by arbitrariness of D we deduce that $l \in \mathcal{P}(^{+'}\Phi)$. Consequently, $^{+'}(\mathcal{P}(\Phi))\subseteq\mathcal{P}(^{+'}\Phi)$.

Conversely, let $l \in\mathcal{P}(^{+'}\Phi)$ and V be a prime ideal of Γ containing Φ. Then $^{+'}\Phi \subseteq^{+'}$V. Since $^{+'}$V is a prime ideal of L, $\mathcal{P}(^{+'}\Phi)\subseteq^{+'}$V. So $l \in^{+'}$V. This implies that $\Gamma l \subseteq$V whence by the arbitrariness of V $\Gamma l \subseteq \mathcal{P}(\Phi)$. So $l \in^{+'}(\mathcal{P}(\Phi))$. Consequently, $\mathcal{P}(^{+'}\Phi)\subseteq^{+'}(\mathcal{P}(\Phi))$. Hence $\mathcal{P}(^{+'}\Phi)=^{+'}(\mathcal{P}(\Phi))$.

The proof of the remaining part follows similarly. □

Theorem 6. *Let $\mathcal{P}(S)$ and $\mathcal{P}(\Gamma)$ be the prime radicals of the Γ-semiring S and S-semiring Γ respectively. Then (i) $\mathcal{P}(\Gamma) = \Gamma(\mathcal{P}(S))$, (ii) $\mathcal{P}(S) = S(\mathcal{P}(\Gamma))$.*

Proof. (i) Let Λ, Ω be the collection of all prime ideals of Γ and S respectively. Then

$$\mathcal{P}(\Gamma) = \bigcap_{\Phi \in \Lambda} \Phi = \bigcap_{\Phi \in \Lambda} (\Gamma(S(\Phi)) = \Gamma(\bigcap_{\Phi \in \Lambda} (S(\Phi)) \supseteq \Gamma(\bigcap_{P \in \Omega} P) = \Gamma(\mathcal{P}(S)).$$

Again

$$\Gamma(\mathcal{P}(S)) = \Gamma(\bigcap_{P \in \Omega} P) = \bigcap_{P \in \Omega} \Gamma(P) \supseteq \bigcap_{\Phi \in \Lambda} \Phi(P \in \Omega \Rightarrow \Gamma(P) \in \Lambda) = \mathcal{P}(\Gamma).$$

Consequently,

$$\mathcal{P}(\Gamma) = \Gamma(\mathcal{P}(S)).$$

(ii) Analogously, we obtain (ii). □

Theorem 7. *Suppose P, Φ are respectively the ideals of S and Γ. Then (i)* $\mathcal{P}(\Gamma(\mathrm{P}))=\Gamma(\mathcal{P}(\mathrm{P}))$ and (ii) $\mathcal{P}(\mathrm{S}(\Phi))=\mathrm{S}(\mathcal{P}(\Phi))$.

Proof. (i) Let $\alpha \in \mathcal{P}(\Gamma(\mathrm{P}))$ and A be any prime ideal of S containing P. Then $\Gamma(P) \subseteq \Gamma(A)$. Since Γ(A) is a prime ideal of Γ and Γ(P) is an ideal of Γ, we deduce that $\mathcal{P}(\Gamma(\mathrm{P})) \subseteq \Gamma(\mathrm{A})$. Hence $\alpha \in \Gamma(\mathrm{A})$. Hence $\mathrm{S}\alpha\mathrm{S} \subseteq \mathrm{A}$. This together with the arbitrariness of A implies that $\mathrm{S}\alpha\mathrm{S} \subseteq \mathcal{P}(\mathrm{P})$. Hence $\alpha \in \Gamma(\mathcal{P}(\mathrm{P}))$. Consequently, $\mathcal{P}(\Gamma(\mathrm{P})) \subseteq \Gamma(\mathcal{P}(\mathrm{P}))$.

Conversely, suppose that $\beta \in \Gamma(\mathcal{P}(P))$. Then $\mathrm{S}\beta\mathrm{S} \subseteq \mathcal{P}(P)$. Let Θ be any prime ideal of Γ containing Γ(P). Then $\mathrm{S}(\Gamma(\mathrm{P})) \subseteq \mathrm{S}(\Theta)$, i.e., $\mathrm{P} \subseteq \mathrm{S}(\Theta)$. Since $\mathrm{S}(\Theta)$ is a prime ideal of S, by using definition of prime radical of an ideal, we deduce that $\mathcal{P}(\mathrm{P}) \subseteq \mathrm{S}(\Theta)$. Hence $\mathrm{S}\beta\mathrm{S} \subseteq \mathrm{S}(\Theta)$ whence $\beta \in \Gamma(S(\Theta)) = \Theta$. So by arbitrariness of Θ and by definition of prime radical of an ideal we deduce that $\beta \in \mathcal{P}(\Gamma(\mathrm{P}))$. Thus $\Gamma(\mathcal{P}(\mathrm{P})) \subseteq \mathcal{P}(\Gamma(\mathrm{P}))$. Hence $\mathcal{P}(\Gamma(\mathrm{P}))=\Gamma(\mathcal{P}(\mathrm{P}))$.

(ii) By applying similar argument we can prove (ii). □

3. Prime Radical in $\begin{pmatrix} \boldsymbol{R} & \boldsymbol{\Gamma} \\ \boldsymbol{S} & \boldsymbol{L} \end{pmatrix}$

Let S be a Nobusawa $\Gamma-$semiring and L and R be its left and right operator semirings. The matrix semiring S_2 or

$$\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix} = \left\{ \begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} : r \in R, l \in L, \gamma \in \Gamma, s \in S \right\}.$$

The addition and multiplication are defined as follows:

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 + r_2 & \gamma_1 + \gamma_2 \\ s_1 + s_2 & l_1 + l_2 \end{pmatrix}$$

and

$$\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + [\gamma_1, s_2] & r_1\gamma_2 + \gamma_1 l_2 \\ s_1 r_2 + l_1 s_2 & [s_1, \gamma_2] + l_1 l_2 \end{pmatrix}.$$

Let I be an ideal of S. Then $I_2 = \begin{pmatrix} \mathtt{I}^{*'} & \Gamma(\mathtt{I}) \\ \mathtt{I} & \mathtt{I}^{+'} \end{pmatrix}$ is an ideal of S_2. Also every ideal of S_2 is of the form $I_2 = \begin{pmatrix} \mathtt{I}^{*'} & \Gamma(\mathtt{I}) \\ \mathtt{I} & \mathtt{I}^{+'} \end{pmatrix}$, where I is an ideal of S.

Theorem 8. *[9]Let S be a Γ_N-semiring and let Γ-semiring S has right unity. Let I be a prime ideal of the Γ-semiring S. Then $I_2 = \begin{pmatrix} I^{*'} & \Gamma(I) \\ I & I^{+'} \end{pmatrix}$ is a prime ideal of the semiring S_2.*

Theorem 9. $\mathcal{P}(S_2) = \begin{pmatrix} \mathcal{P}(R) & \mathcal{P}(\Gamma) \\ \mathcal{P}(S) & \mathcal{P}(L) \end{pmatrix}$.

Proof. Let P_2 be a prime ideal of S_2. Then

$$P_2 = \begin{pmatrix} \mathtt{P}^{*'} & \Gamma(\mathtt{P}) \\ \mathtt{P} & \mathtt{P}^{+'} \end{pmatrix}.$$

Let I and J be two ideals of S such that $I\Gamma J \subseteq P$. Then

$$I_2 J_2 = \begin{pmatrix} \mathtt{I}^{*'} & \Gamma(\mathtt{I}) \\ \mathtt{I} & \mathtt{I}^{+'} \end{pmatrix} \begin{pmatrix} \mathtt{J}^{*'} & \Gamma(\mathtt{J}) \\ \mathtt{J} & \mathtt{J}^{+'} \end{pmatrix}.$$

Hence

$$\mathtt{I}^{*'} \mathtt{J}^{*'} + [\Gamma(\mathtt{I}), J] \subseteq P^{*'}, \tag{i}$$

$$\mathtt{I}^{*'} \Gamma(\mathtt{J}) + \Gamma(\mathtt{I})\mathtt{J}^{+'} \subseteq \Gamma(P), \tag{ii}$$

$$\mathtt{I}\ \mathtt{J}^{*'} + \mathtt{I}^{+'} J \subseteq P, \tag{iii}$$

$$[I, \Gamma(\mathtt{J})] + \mathtt{I}^{+'} \mathtt{J}^{+'} \subseteq P^{+'}. \tag{iv}$$

So $I_2J_2 \subseteq P_2$. Hence either $I_2 \subseteq P_2$ or $J_2 \subseteq P_2$. So I$\subseteq P$ or J$\subseteq P$.

Next let P be a prime ideal of S and A and B be two ideals of S_2 such that $AB \subseteq P_2$. Then there exist ideals I and J such that $A=I_2$ and $B=J_2$. Hence

$$IJ^{*'} + I^{+'} J \subseteq P.$$

Now

$$I\ J^{*'} + I^{+'} J = I[\Gamma, J] + [I, \Gamma]J = I\Gamma J = I\Gamma J.$$

So $I\Gamma J \subseteq P$. Hence $I \subseteq P$ or $J \subseteq P$. So $I_2 \subseteq P$ or $J_2 \subseteq P$. So P_2 is a prime ideal of S_2. Again in [2]it is shown that P is prime (a) if and only if $P^{*'}$ is prime (b) if and only if $P^{+'}$ is prime and in [10] we shown that P is prime if and only if $\Gamma(P)$ is prime.

Suppose Λ is the collection of all prime ideals in S_2. Then

$$\mathcal{P}(S_2) = \bigcap_{P_2 \in \Lambda} P_2 = \bigcap_{P_2 \in \Lambda} \begin{pmatrix} P^{*'} & \Gamma(P) \\ P & P^{+'} \end{pmatrix} = \begin{pmatrix} \bigcap\limits_{P^{*'} \in \Delta} P^{*'} & \bigcap\limits_{\Gamma(P) \in \Pi} \Gamma(P) \\ \bigcap\limits_{P \in K} P & \bigcap\limits_{P^{+'} \in \Omega} P^{+'} \end{pmatrix},$$

where Δ, Π, K and Ω are respectively the collection of all prime ideals of R, Γ, S and L □

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