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NONHOLONOMIC FRAMES FOR A FINSLER SPACE WITH GENERAL (α, β)-METRIC

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Abstract: The main purpose of this paper is to first determine the two Finsler deformations of the special (α, β)-metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$, where ϵ and k are constants. Consequently, we obtain the nonholonomic frame for Finsler space with special (α, β)-metric which is often considered as the generalization of Randers metric as well as Z. Shen's square metric. Further, we obtain some results which give nonholonomic frame for Finsler spaces with certain (α, β)metrics such as Randers metric, a special (α, β)-metric $\alpha + \frac{\beta^2}{\alpha}$, first approximate Matsumoto metric and Finsler space with square metric.

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Key Words: Finsler space with (α, β)-metric, generalized lagrange metric, nonholonomic Finsler frame

1. Introduction

In [19], G. Vranceanu introduces the concept of a nonholonomic space which is more general than a Riemannian space and generalized the parallelism of Levi-Civita and geodesic curves in that space. From another standpoint Z. Horak [11] considers a nonholonomic region as a space with a nonholonomic

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dynamical system. In [8], T. Hosokawa discusses the nonholonomic system in a space of line-elements with an affine connection. In [13], K. Yoshie introduces the theory of nonholonomic system in Finsler space.

In [9, 10], P. R. Holland studies a unified formalism that uses a nonholonomic frame on a space-time arising from consideration of a charged particle moving in an external electromagnetic field. R. S. Ingarden [12] was the first to point out that the Lorentz force law could be written as geodesic equations on a Finsler space, called Randers space. In [4, 7], R. G. Beil studies gauge transformation as a nonholonomic frame as a tangent bundle on a four dimensional base manifold. In these papers, the common Finsler idea used by these physicists is the existence of nonholonomic frame on the verticle subbundle VTM of the tangent bundle of a base manifold M. In [2, 3], P.L. Antonelli, finds such a nonholonomic frame for two important classes that are dual in the sense of Randers and Kropina spaces. Though the study of nonholonomic frames for Finsler spaces with (α, β)-metric is quite old and so many results have been obtained by authors of so many countries, still it is an important topic of research in Finsler geometry.

The main purpose of the current paper is to determine a nonholonomic frame for a Finsler space with a special (α, β)-metric $\alpha + \epsilon\beta + \delta\frac{\beta^2}{\alpha}$ (where $\epsilon, \delta \neq 0$ are constants). The paper is organized as follows:

Starting with literature survey in section one, we recall some basic definitions and results in section two. In section three, initially we determine the two Finsler deformations to the general (α, β)-metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ (where ϵ and k $\neq 0$ are constants) and obtain a corresponding frame for each of these Finsler deformations. Then, a nonholonomic frame for Finsler space with the aforesaid metric is determined as the product of these Finsler frames (see Theorem 3). Further, we find some results which give nonholonomic frame for Finsler spaces with certain (α, β)metrics such as Randers metric (see Corollory 4), a special (α, β)-metric $\alpha + \frac{\beta^2}{\alpha}$ (see Corollory 5), first approximate Matsumoto metric (see Corollory 6) and Z Shen's square metric (see Corollory 7).

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with (α, β)-metric [16]. The first Finsler space with (α, β)-metric was introduced in forties by the physicist G. Randers [18] and it is called Randers space . The other notable Finsler spaces with (α, β)-metric are Kropina space [14], Generalized Kropina Space and Matsumoto space [17].

Definition 1. A Finsler space F^n= (M, F(x,y)) is called the (α, β)-metric, if there exists a 2-homogenous function L of two variables such that the Finsler metric F: TM $\longrightarrow \Re$ is given by:

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y)), \tag{1}$$

where $\alpha^2(x, y) = a_{ij}(x)y^i y^j$, α is Riemannian metric on M, and $\beta(x, y) = b_i(x)y^i$ is a 1-form on M.

Example 1.1. If L$(\alpha, \beta) = (\alpha + \beta)^2$, then the Finsler space with metric

$$F(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

is called a Randers space.

Example 1.2. If L$(\alpha, \beta) = \frac{\alpha^4}{\beta^2}$, then the Finsler space with metric

$$F(x, y) = \frac{a_{ij}(x)y^i y^j}{b_i(x)y^i}$$

is called a Kropina space.

For a Finsler space with (α, β)-metric $F^2(x, y) = L(\alpha, \beta)$, the following Finsler invariants are well known [15]:

$$\rho = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}, \quad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}, \quad \rho_{-1} = \frac{1}{2}\frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad \rho_{-2} = \frac{1}{2\alpha^2}\Big(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial \alpha}\Big). \tag{2}$$

For a Finsler space with (α, β)-metric, we have

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \tag{3}$$

With these Finsler invariants, the metric tensor g_{ij} of a Finsler space with (α, β)-metric is given by ([1], [16])

$$g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_i y_j. \tag{4}$$

The metric tensor g_{ij} of a Lagrange space with (α, β)-metric can be arranged in the form

$$\begin{aligned} g_{ij}(x, y) \quad &= \quad \rho a_{ij}(x) + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) \\ &\quad + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_i b_j. \end{aligned} \tag{5}$$

From (5), we can see that g_{ij} is the result of two Finsler deformations

$$a_{ij} \mapsto h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-2}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j), \tag{6}$$

$$h_{ij} \mapsto g_{ij} = \rho h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \tag{7}$$

The nonholonomic Finsler frame that corresponds to the first deformation (6) is, according to the Theorem (7.9.1) in [12], given by

$$\begin{aligned} X^i_j = {} & \sqrt{\rho}\delta^i_j - \frac{1}{B^2}\Big(\sqrt{\rho} \pm \sqrt{\rho + \frac{B^2}{\rho_{-2}}}\Big)(\rho_{-1}b_i \\ & +\rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j), \end{aligned} \tag{8}$$

where $B^2 = a_{ij}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)$. The metric tensors a_{ij} and h_{ij} are related by the formula

$$h_{ij} = X^k_i X^i_j a_{kl}. \tag{9}$$

According to the Theorem (7.9.1) in [12], the nonholonomic Finsler frame that corresponds to the second deformation (7) is given by

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\Big(1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\Big)b^ib_j, \tag{10}$$

where $C^2 = h_{ij}b^ib^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$. The metric tensors h_{ij} and g_{ij} are related by the formula

$$g_{mn} = Y^i_m Y^j_n h_{ij}. \tag{11}$$

From (9) and (11), we have that $V^k_m = X^k_i Y^i_m$ with X^k_i given by (8).

Theorem 2. *([5]) Let $F^2(x,y) = L(\alpha(x,y), \beta(x,y))$ be a metric function of a Finsler space with (α,β)-metric for which the condition (3) is true. Then*

$$V^i_j = X^i_k Y^k_j \tag{12}$$

is a nonholonomic Finsler frame where X^i_k and Y^k_j are given by (8) and (10) respectively.

3. Nonholonomic Frame for Finsler Space with General (α, β)-Metric

We consider a Finsler space with general (α, β)-metric given by

$$F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}, \tag{13}$$

where ϵ and k are constants.

For the fundamental function $\mathrm{L} = (\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})^2$, the Finsler invariants (2) are given by

$$\begin{aligned}
\rho &= \frac{(\alpha^2 - k\beta^2)(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})}{\alpha^3} \\
&= \frac{\alpha^4 + \epsilon\alpha^3\beta - k\epsilon\alpha\beta^3 - k^2\beta^4}{\alpha^4}, \\
\rho_0 &= \frac{\epsilon^2\alpha^2 + 6k^2\beta^2 + 6k\epsilon\alpha\beta + 2k\alpha^2}{\alpha^2}, \\
\rho_{-1} &= \frac{\epsilon\alpha^3 - 3\epsilon k\alpha\beta^2 - 4k^2\beta^3}{\alpha^4}, \\
\rho_{-2} &= \frac{-\beta(\epsilon\alpha^3 - 3\epsilon k\alpha\beta^2 - 4k^2\beta^3)}{\alpha^6}, \\
B^2 &= \frac{(\epsilon\alpha^3 - 3\epsilon k\alpha\beta^2 - 4k^2\beta^3)^2(b^2\alpha^2 - \beta^2)}{\alpha^{10}}.
\end{aligned} \tag{14}$$

Using (14) in (8), we have

$$\begin{aligned}
X^i_k = {} & \sqrt{\frac{(\alpha^2 - k\beta^2)(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})}{\alpha^3}}\delta^i_j - \frac{\alpha^2}{(b^2\alpha^2 - \beta^2)} \\
& \Big[\sqrt{\frac{(\alpha^2 - k\beta^2)(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})}{\alpha^3}} \pm \\
& \sqrt{\frac{(\alpha^2 - k\beta^2)(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})}{\alpha^3} - \frac{(\epsilon\alpha^3 - 3\epsilon k\alpha\beta^2 - 4k^2\beta^3)(\alpha^2 b^2 - \beta^2)}{\alpha^4\beta}}\,\Big] \\
& \Big(b^i - \frac{\beta}{\alpha^2}y^i\Big)\Big(b_j - \frac{\beta}{\alpha^2}y_j\Big).
\end{aligned} \tag{15}$$

Again, using (14) in (10), we have

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\Big(1 \pm \sqrt{1 + \frac{\alpha^2\beta C^2}{\epsilon^2\alpha^2\beta + 2k^2\beta^3 + 3\epsilon k^2\alpha\beta^2 + 2k\alpha^2\beta + \epsilon\alpha^3}}\,\Big)b^i b_j, \tag{16}$$

where

$$C^2 = b^2 \frac{(\alpha^4 + \epsilon\alpha^3\beta - k\epsilon\alpha\beta^3 - k^2\beta^4)}{\alpha^4} - \frac{(b^2\alpha^2 - \beta^2)^2(\epsilon\alpha^3 - 3\epsilon k\alpha\beta^2 - 4k^2\beta^3)}{\alpha^6\beta}. \tag{17}$$

Hence we have the following:

Theorem 3. *Consider a Finsler space $F^n = (M, F)$ with L= $(\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})^2$, for which the condition (3) is true, then*

$$V_j^i = X_k^i Y_j^k, \tag{18}$$

is a nonholonomic Finsler frame, where X_k^i and Y_j^k are given by (15) and (16) respectively.

One can easily obtain that

$$\begin{aligned} V_j^i = X_k^i Y_j^k \quad = \quad & \sqrt{\rho}\delta_j^i - \frac{1}{C^2}\sqrt{\rho}\Big(1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\Big)b^i b_j \\ & - \frac{\alpha^2}{\alpha^2 b^2 - \beta^2}\Big(\sqrt{\rho} \pm \sqrt{\rho + \frac{B^2}{\rho_{-2}}}\Big)(b^i - \frac{\beta}{\alpha^2}y^i)(b_j - \frac{\beta}{\alpha^2}y_j) \\ & - \frac{1}{C^2}\Big(\frac{\alpha^2}{\alpha^2 b^2 - \beta^2}\Big)\Big(\sqrt{\rho} \pm \sqrt{\rho + \frac{B^2}{\rho_{-2}}}\Big)\Big(1 \pm \\ & \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \rho_{-1}^2}}\Big)(b^i - \frac{\beta}{\alpha^2}y^i)(b^2 - \frac{\beta^2}{\alpha^2})b_j. \end{aligned}$$

Substituting the values of Finsler invariants, we obtain the required Finsler frame.

From the theorem 3.1, we can find nonholonomic frames for certain Finsler spaces with (α, β)-metric. We have the following cases.

Case (i) If $\epsilon = 1$ and $k = 0$, we have $F = \alpha + \beta$ which is Randers metric. In this case, the Finsler invariants (2) are reduced to

$$\begin{aligned} \rho &= \frac{\alpha + \beta}{\alpha}, \\ \rho_0 &= 1, \\ \rho_{-1} &= \frac{1}{\alpha}, \\ \rho_{-2} &= \frac{-\beta}{\alpha^3}, \end{aligned}$$

$$B^2 = \frac{b^2\alpha^2 - \beta^2}{\alpha^4}.$$

The Finsler deformations of the Finsler metric are obtained as

$$X_k^i = \sqrt{\frac{\alpha+\beta}{\alpha}}\delta_j^i - \frac{\alpha^2}{\alpha^2 b^2 - \beta^2}\Big[\sqrt{\frac{\alpha+\beta}{\alpha}} \pm \sqrt{\frac{\alpha\beta + 2\beta^2 - b^2\alpha^2}{\alpha\beta}}\ \Big] \Big(b^i - \frac{\beta y^i}{\alpha^2}\Big)\Big(b_j - \frac{\beta y_j}{\alpha^2}\Big), \tag{19}$$

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\Big(1 \pm \sqrt{1 + \frac{\beta C^2}{\alpha+\beta}}\ \Big)b^i b_j, \quad and \tag{20}$$

$$C^2 = \frac{(\alpha+\beta)b^2}{\alpha} - \frac{\alpha}{\beta}\Big(b^2 - \frac{\beta^2}{\alpha^2}\Big)^2. \tag{21}$$

Hence we have the following:

Corollary 4. *Consider a Finsler space $F^n = (M, F)$ with L= $(\alpha+\beta)^2$, for which the condition (3) is true, then*

$$V_j^i = X_k^i Y_j^k, \tag{22}$$

is a nonholonomic Finsler frame, where X_k^i and Y_j^k are given by (19) and (20) respectively.

Case (ii) If $\epsilon = 0$ and $k = 1$, we have $F = \alpha + \frac{\beta^2}{\alpha}$. In this case, the Finsler invariants (2) are reduced to

$$\rho = \frac{\alpha^4 - \beta^4}{\alpha^4},$$
$$\rho_0 = \frac{2(\alpha^2 + 3\beta^2)}{\alpha^2},$$
$$\rho_{-1} = \frac{-4\beta^3}{\alpha^4},$$
$$\rho_{-2} = \frac{4\beta^4}{\alpha^6},$$
$$B^2 = \frac{16\beta^6(b^2\alpha^2 - \beta^2)}{\alpha^{10}}.$$

The Finsler deformations of the Finsler metric are obtained as:

$$X_k^i = \frac{\sqrt{\alpha^4 - \beta^4}}{\alpha^2}\delta_j^i - \frac{\Big[\sqrt{\alpha^4 - \beta^4} \pm \sqrt{\alpha^4 - 5\beta^4 + 4b^2\alpha^2\beta^2}\Big]}{(b^2\alpha^2 - \beta^2)}$$

$$\Big(b^i - \frac{\beta y^i}{\alpha^2}\Big)\Big(b_j - \frac{\beta y_j}{\alpha^2}\Big), \tag{23}$$

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\Big(1 \pm \sqrt{1 + \frac{\alpha^2 C^2}{2(\alpha^2+\beta^2)}}\Big)b^i b_j, \quad \textit{and} \tag{24}$$

$$C^2 = \frac{b^2(\alpha^4-\beta^4)}{\alpha^4} + \frac{4\beta^2(b^2\alpha^2-\beta^2)^2}{\alpha^6}. \tag{25}$$

Hence we have the following:

Corollary 5. *Consider a Finsler space F^n = (M, F) with L= $(\alpha + \frac{\beta^2}{\alpha})^2$, for which the condition (3) is true, then*

$$V^i_j = X^i_k Y^k_j, \tag{26}$$

is a nonholonomic Finsler frame, where X^i_k and Y^k_j are given by (23) and (24) respectively.

Case (iii) If $\epsilon = 1$ and $k = 1$, we have $F = \alpha + \beta + \frac{\beta^2}{\alpha}$ which is first approximate Matsumato metric. In this case, the Finsler invariants (2) are reduced to

$$\begin{aligned}
\rho &= \frac{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^4},\\
\rho_0 &= \frac{3(\alpha^2+2\alpha\beta+2\beta^2)}{\alpha^2},\\
\rho_{-1} &= \frac{(\alpha^3-3\alpha\beta^2-4\beta^3)}{\alpha^4},\\
\rho_{-2} &= \frac{\beta(\alpha^3-3\alpha\beta^2-4\beta^3)}{\alpha^6},\\
B^2 &= \frac{(\alpha^3-3\alpha\beta^2-4\beta^3)^2(b^2\alpha^2-\beta^2)}{\alpha^{10}}.
\end{aligned}$$

Here, the Finsler deformations are obtained as

$$\begin{aligned}
X^i_k &= \sqrt{\frac{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^4}}\delta^i_j - \frac{\alpha^4}{(b^2\alpha^2-\beta^2)}\\
&\quad \Big[\sqrt{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)} \pm\\
&\quad \sqrt{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2) - \frac{(\alpha^3+3\alpha\beta^2+4\beta^3)}{\beta}}\ \Big]
\end{aligned}$$

$$\Big(b^i - \frac{\beta y^i}{\alpha^2}\Big)\Big(b_j - \frac{\beta y_j}{\alpha^2}\Big), \tag{27}$$

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\Big(1 \pm \sqrt{1 + \frac{\alpha^2\beta C^2}{\alpha^3 + 3\alpha\beta(\alpha+\beta) + 2\beta^3}}\Big) b^i b_j, \quad \textit{and} \tag{28}$$

$$C^2 = \frac{b^2(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^4} - \frac{(b^2\alpha^2-\beta^2)^2(\alpha^3-3\alpha\beta^2-4\beta^3)}{\alpha^6\beta}. \tag{29}$$

Hence we have the following:

Corollary 6. *Consider a Finsler space $F^n = (M, F)$ with L$= (\alpha+\beta+\frac{\beta^2}{\alpha})^2$, for which the condition (3) is true, then*

$$V^i_j = X^i_k Y^k_j, \tag{30}$$

is a nonholonomic Finsler frame, where X^i_k and Y^k_j are given by (27) and (28) respectively.

Case (iv) If $\epsilon = 2$ and $k = 1$, we have $F = \frac{(\alpha+\beta)^2}{\alpha}$ which is known as square metric. In this case, the Finsler invariants (2) are reduced to

$$\begin{aligned}
\rho &= \frac{(\alpha^2-\beta^2)(\alpha+2\beta+\frac{\beta^2}{\alpha})}{\alpha^3} \\
&= \frac{\alpha^4+2\alpha^3\beta-2\alpha\beta^3-\beta^4}{\alpha^4}, \\
\rho_0 &= \frac{6(\alpha+\beta)^2}{\alpha^2}, \\
\rho_{-1} &= \frac{2(\alpha^3-3\alpha\beta^2-2\beta^3)}{\alpha^4}, \\
\rho_{-2} &= \frac{-2\beta(\alpha^3-3\alpha\beta^2-2\beta^3)}{\alpha^6}, \\
B^2 &= \frac{4(\alpha^3-3\alpha\beta^2-4\beta^3)^2(b^2\alpha^2-\beta^2)}{\alpha^{10}}.
\end{aligned}$$

The two Finsler deformations for this Finsler metric are obtained as

$$\begin{aligned}
X^i_k &= \sqrt{\frac{(\alpha^4+2\alpha^3\beta-2\alpha\beta^3-\beta^4)}{\alpha^4}}\delta^i_j - \frac{\alpha^2}{(b^2\alpha^2-\beta^2)} \\
&\Big[\sqrt{\frac{(\alpha^4+2\alpha^3\beta-2\alpha\beta^3-\beta^4)}{\alpha^4}} \pm \\
&\sqrt{\frac{\alpha^4\beta-8\alpha\beta^4-2b^2\alpha^5+(4+6b^2)\alpha^3\beta^2+4b^2\alpha^2\beta^3-5\beta^3}{\alpha^4\beta}}\Big]
\end{aligned}$$

$$\Big(b^i - \frac{\beta}{\alpha^2} y^i\Big)\Big(b_j - \frac{\beta}{\alpha^2} y_j\Big), \tag{31}$$

$$Y^i_j = \delta^i_j - \frac{1}{C^2}\Big(1 \pm \sqrt{1 + \frac{\alpha^2\beta C^2}{2(\alpha+\beta)^3}}\Big) b^i b_j, \quad and \tag{32}$$

$$C^2 = \frac{b^2(\alpha^4 + 2\alpha^3\beta - 2\alpha\beta^3 - \beta^4)}{\alpha^4} - \frac{2(b^2\alpha^2 - \beta^2)^2(\alpha^3 - 3\alpha\beta^2 - 2\beta^3)}{\alpha^6\beta} \tag{33}$$

Hence we have the following:

Corollary 7. *Consider a Finsler space $F^n = (M, F)$ with L= $\frac{(\alpha+\beta)^4}{\alpha^2}$, for which the condition (3) is true, then*

$$V^i_j = X^i_k Y^k_j, \tag{34}$$

is a nonholonomic Finsler frame, where X^i_k and Y^k_j are given by (31) and (32) respectively.

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