International Journal of Pure and Applied Mathematics
Volume 107 No. 4 2016, 1037-1051
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i4.21
ijpam.eu

TOTAL AND DIRECTIONAL FRACTIONAL DERIVATIVES

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Abstract: Vector calculus is an important subject in mathematics with applications in all areas of applied sciences. Till now researchers deal with the partial fractional derivative as the fractional derivative with respect to x, y,... .

In this paper we shall define total and directional fractional derivative of functions of several variables, we set some basics about fractional vector calculus then we use our definition to modify the definition of conformal fractional derivative obtained by R. Khalil et al [6].

AMS Subject Classification: 26A33

Key Words: fractional derivative, conformal fractional derivative, total fractional derivative, directional fractional derivative

1. Introduction

The subject of fractional derivative is as old as calculus. The most popular definitions of fractional derivative are:

(i) Riemann-Liouville definition [6]: If n is a positive integer and $\alpha \in$

Received: October 18, 2015
Published: May 9, 2016

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$[n-1, n)$, the α^{th} derivative of f given is by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo definition [5]. If n is a positive integer and $\alpha \in [n-1, n)$, the α^{th} derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

All definitions of fractional derivatives do not satisfy the known *product rule, quotient rule, chain rule*. In [6] Khalil etall gave a new definition of fractional derivative "*conformable fractional derivative*" of a function f . This definition seems to be a natural extension of the usual definition of derivative. Many theorems which are proved using the classical definitions are still valid using the new definition as product rule, quotient rule, chain rule. Many authors used the new definition to solve a fractional differential equations as in [3] and [4], since the computation using the new definition is more easier than using Riemann-Liouville or Caputo definition of fractional derivative. Thabet Abdeljawd in [1] define the left and right conformable fractional derivative, so the connection can be mad between the conformable fractional derivative and the classical definition.

In this paper we shall define the concepts of *directional fractional derivative* and *total fractional derivative* of functions of several variables, these definitions announce the born of fractional vector calculus. Also we set some basics about fractional vector calculus, beside we shall give a simple modification of the definition given in [6] as an application of the new definition.

2. Basics

The concept of conformable fractional derivative is recently introduced by R. Khalil etall in 2014 by imitating the usual definition of derivative.

Definition 1. [6]. Let $f:[0,\infty)\longrightarrow\mathbb{R}, t>0$, and $\alpha\in(0,1]$. Then the fractional derivative of f of order α, T_α or $f^{(\alpha)}$ is defined as:

$$f^{(\alpha)}(t)=T_\alpha(f)(t)=\lim_{\in\to 0}\frac{f\left(t+\in t^{1-\alpha}\right)-f(t)}{\in}.$$

If f is $\alpha-$ differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then $f^{(\alpha)}(0)$ is defined by

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t).$$

It is clear that $T_\alpha(t^p)=pt^{p-\alpha}$. Further this definition coincides with the classical definition of Riemann-Liouville definition and Caputo definition on polynomials (up to constant multiple), also if $\alpha=1$ we have the classical definition of derivative.

Definition 2. [6]. Let $\alpha\in(n,n+1)$, and f be an $n-$ differentiable function at $t>0$.Then $T_\alpha(f)(t)$ is defined by

$$T_\alpha(f)(t)=\lim_{\in\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\in t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\in},$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Remark 1. The form of Definition 2 given in this form in [5] . Now since $\alpha\in(n,n+1)$, then $\lceil\alpha\rceil=n+1$. So
$T_\alpha(f)(t)=\lim_{\in\to 0}\frac{f^{(n)}\left(t+\in t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(n)}(t)}{\in}=T_{n+1-\alpha}\left(f^{(n)}\right).$

As a consequence of Definition 2 one can show that $T_\alpha(f)(t)=t^{(\lceil\alpha\rceil-\alpha)f^{\lceil\alpha\rceil}(t)}$. Also T_α satisfies the following theorem.

Theorem 1. [6]. *Let $\alpha\in(0,1]$ and $f.g$ be α- differentiable at t. Then:*

1. *$T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g)$ for all $a,b\in\mathbb{R}$.*
2. *$T_\alpha(t^p)=pt^{p-1}$ for any $p\in\mathbb{R}$.*
3. *$T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.*
4. *$T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.*
5. *$T_\alpha(\lambda)=0$, for all constant function λ.*
6. *If, in addition f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}$.*

In the following example the fractional derivatives of well known functions are given.

Example 1. Let $\alpha \in (0,1]$ and $c, b \in \mathbb{R}$. Then:

1. $T_\alpha \left(e^{ct}\right) = ct^{1-\alpha} e^{ct}$.
2. $T_\alpha (\sin bt) = bt^{1-\alpha} \cos bt$.
3. $T_\alpha (\cos bxt) = -bt^{1-\alpha} \sin bt$.
4. $T_\alpha \left(\frac{1}{\alpha} t^\alpha\right) = 1$.

Using the new definition of conformable fractional derivative, most of the important Theorems in calculus as Roll's theorem, Mean Value theorem and differentiability implies continuity still valid.

3. Directional Fractional Derivative

So far researchers deal with the partial fractional derivative as the fractional derivative with respect to x or y, etc. In this section we shall generalize the definition given in [6] ,to cover the concepts of directional fractional derivative.

Let f be any function on $\mathbb{R}^n$, with domain D_f and $u \in \mathbb{R}^n$ be any vector. Put $u^\perp = \{v \in \mathbb{R}^n : v \text{ is orthogonal to } u\}$ and $\|u\|$ be the Euclidean norm of u. Finally, let $c \cdot u$ be the dot product of c and u. By $A \backslash B$ we mean the points in the set A not in the set B.

Definition 3. Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be any function, $\alpha \in (0,1]$, $u \in \mathbb{R}^n$ be a non-zero vector and c in the interior of D_f $\left(c \in D_f^\circ\right)$.

1. If $c \in D_f^\circ \backslash u^\perp$ and there exists a vector $L_u^\alpha \in \mathbb{R}^m$ that satisfies for each $\in >$ 0,there is a $\delta(\in) > 0$ such that for all $t \in \mathbb{R}$ satisfying $0 < |t| < \delta(\in)$,we have $\left\| \frac{1}{t} \left(f\left(c + t\,|c \cdot u|^{1-\alpha} u \right) - f(c) \right) - L_u^\alpha \right\| < \in$, then L_u^α is called the fractional directional derivative of f of order α in the direction of u at c and denoted by $D_u^\alpha f(c)$. If there is no such a vector then we say that f is not $\alpha-$differentiable at c in the direction of u.

2. If $c \in D_f^\circ \cap u^\perp$, then $D_u^\alpha f(c)$ is defined by $\lim\limits_{d \to c} D_u^\alpha f(d)\,, d \in D_f$ if the limit exist. If the limit does not exist we say that f not $\alpha-$differentiable at c in the direction of u.

Using the above definition it is clear that if $c \in D_f \backslash u^\perp$ and

$$\lim_{t \to 0} \frac{f\left(c + t\,|c \cdot u|^{1-\alpha} u \right) - f(c)}{t}$$

exists then

$$L_u^\alpha = D_u^\alpha f(c) = \lim_{t\to 0} \frac{f\left(c + t\,|c\cdot u|^{1-\alpha} u\right) - f(c)}{t}.$$

If $c \in D_f \cap u^\perp$, then

$$D_u^\alpha f(c) = \lim_{d\to c} D_u^\alpha f(d), \quad d \in D_f.$$

In the above definition if $c \in D_f^\circ \cap u^\perp$ then we can't use (1) to define $D_u^\alpha f(c)$, so we use (2). The case $c \in u^\perp$ corresponds to the case $t = 0$ in Definition 1. Also if f is continuous and $c \in u^\perp$, Then $D_u^\alpha f(c) = 0$.

If one is interested in the fractional directional derivative at a boundary point of D_f , that is $c \in D_f \setminus D_f^\circ$, we suggest the following definition.

Definition 4. Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function, $\alpha \in (0,1)$, and $u \in \mathbb{R}^n$ be a non-zero vector. Let c be a boundary point of D_f and $c_n \in D_f^\circ$ be such that $c_n \to c$. Then $D_u^\alpha f(c) = \lim_{n\to\infty} D_u^\alpha f(c_n)$, if the limit exists. And f not $\alpha-$differentiable at c in the direction of u if the limit does not exist.

Note that, since f is continuous the limit is independent of the choice of c_n.

As consequence of the above definitions we have.

Lemma 1. *1. If $\alpha = 1$, then the above definition coincides with classical definition.*

2. If the directional derivative of f in the direction of u at a point c , $D_u f(c)$ exists, then $D_u^\alpha f(c) = |c\cdot u|^{1-\alpha} D_u f(c)$.

Now one can easily prove the following theorem.

Theorem 2. *Let $\alpha \in (0,1)$ and f, g be two $\alpha-$ differentiable at $c \in D_f \cap D_g \subseteq \mathbb{R}^n$ in the direction of a non-zero vectoru. Then:*

1. $D_u^\alpha (f \pm g) = D_u^\alpha (f) \pm D_u^\alpha (g)$.

2. $D_u^\alpha (\gamma f) = \gamma D_u^\alpha (f)$, for all constant γ.

3. $D_u^\alpha (k) = 0$, for all constant functions $k : \mathbb{R}^n \to \mathbb{R}^m$.

Example 2. Let $f(x,y) = xy$ and $u = (2,1)$. Then

$$D_{(2,1)}^{\frac{1}{2}} f((1,2)) = \lim_{\in\to 0} \frac{f((1,2) + \in 2(2,1)) - f((1,2))}{\in}$$

$$=\lim_{\in\to 0}\frac{f\left(\left(1+4\in,2+2\in\right)\right)-f\left(\left(1,2\right)\right)}{\in}$$
$$=\lim_{\in\to 0}\frac{\left(1+4\in\right)\left(2+2\in\right)-2}{\in}=\lim_{\in\to 0}\frac{10\in+8\in^{2}}{\in}=10.$$

Clearly $D_u f\left(c\right)=\left(2,1\right)\cdot\left(2,1\right)=5$, so $D_u^{\alpha} f\left(c\right)=10=\left|c\cdot u\right|^{1-\frac{1}{2}}D_u f\left(c\right).$

Example 3. Let $f\left(x,y\right)=\sqrt{x}+y$. To find $D^{\frac{1}{2}}_{(1,1)}f\left(\left(0,1\right)\right)$ we shall find $D^{\frac{1}{2}}_{(1,1)}f\left(c_n\right)$, where $c_n\in\left\{(x,y):x>0\right\}$, and $c_n=\left(c_n^1,c_n^2\right)\to\left(0,1\right).$
Now

$$D^{\frac{1}{2}}_{(1,1)}f\left(c_n\right)=\lim_{\in\to 0}\frac{f\left(\left(c_n^1,c_n^2\right)+\in\left|\left(c_n^1+c_n^2\right)\right|^{\frac{1}{2}}\left(1,1\right)\right)-f\left(\left(c_n^1,c_n^2\right)\right)}{\in}$$
$$=\lim_{\in\to 0}\frac{\sqrt{c_n^1+\in\left|\left(c_n^1+c_n^2\right)\right|^{\frac{1}{2}}}+\in\left|\left(c_n^1+c_n^2\right)\right|^{\frac{1}{2}}-\sqrt{c_n^1}}{\in}$$
$$=\frac{\left|\left(c_n^1+c_n^2\right)\right|^{\frac{1}{2}}}{2\sqrt{c_n^1}}+\left|\left(c_n^1+c_n^2\right)\right|^{\frac{1}{2}}.$$

Taking the limit as $n\to\infty$, we have $D^{\frac{1}{2}}_{(1,1)}f\left(\left(0,1\right)\right)$ does not exist.

From Lemma 1. It is clear that if $D_u f\left(c\right)$ exist and $c\in u^{\perp}$, then $D_u^{\alpha}f\left(c\right)=0$. In the following example we shall find $D_u^{\alpha}f\left(c\right),\ c\in u^{\perp}$.

Example 4. Let $f\left(x,y\right)=\sqrt{x}+y$.To find $D^{\frac{1}{2}}_{(1,0)}f\left(\left(0,1\right)\right)$ we shall find $D^{\frac{1}{2}}_{(1,0)}f\left(\left(a,b\right)\right),\ (a,b)\in\left\{(x,y):x>0\right\}$,

$$D^{\frac{1}{2}}_{(1,0)}f\left(\left(a,b\right)\right)=\lim_{\in\to 0}\frac{f\left(\left(a,b\right)+\in\sqrt{a}\left(1,0\right)\right)-f\left(\left(a,b\right)\right)}{\in}$$
$$=\lim_{\in\to 0}\frac{f\left(\left(a+\in\sqrt{a},b\right)\right)-f\left(\left(a,b\right)\right)}{\in}=\lim_{\in\to 0}\frac{\sqrt{a+\in\sqrt{a}}-\sqrt{a}}{\in}$$
$$=\frac{\sqrt{a}}{2\sqrt{a}}=\frac{1}{2}.$$

4. Total Fractional Derivative

In this section we shall define total fractional derivative of functions on $\mathbb{R}^n$. Using the new definition the most important theorems of functions of several variables are still valid.

Definition 5. **.**[2]**.** Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be any function, and c an interior point ofD_f. We say that f is differentiable at c if there exist a linear map$T_c : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(c+v) = f(c) + T_c(v) + \|v\| E_c(v)$, where $\lim_{v\to 0} E_c(v) = 0$.

To define total fractional derivative of f, we replace v by $|c\cdot v|^{1-\alpha} v$.

Definition 6. $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be any function, $\alpha \in (0,1]$ and c an interior point of D_f. We say that the α fractional derivative of f at c exist, if there exist a linear map $T_c^\alpha : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f\left(c+|c\cdot v|^{1-\alpha} v\right) = f(c) + T_c^\alpha\left(|c\cdot v|^{1-\alpha} v\right) + \left\||c\cdot v|^{1-\alpha} v\right\| E_c(v), \qquad (*)$$

where $\lim_{v\to 0} E_c(v) = 0$.

Obviously $(*)$ can be expressed more compactly by writing

$$\lim_{\|v\|\to 0} \frac{\left\|f\left(c+|c\cdot v|^{1-\alpha} v\right) - f(c) - T_c^\alpha\left(|c\cdot v|^{1-\alpha} v\right)\right\|}{\left\||c\cdot v|^{1-\alpha} v\right\|} = 0.$$

Alternatively $(*)$ can be rephrased as. For any $\in > 0$, there exist $\delta(\in) > 0$ such that if $v \in \mathbb{R}^n$ and $\|v\| \le \delta(\in)$, then

$$\left\|f\left(c+|c\cdot v|^{1-\alpha} v\right) - f(c) - T_c^\alpha\left(|c\cdot v|^{1-\alpha} v\right)\right\| \le \in \left\||c\cdot v|^{1-\alpha} v\right\|. \qquad (**)$$

Such a linear map T_c^α is called the *fractional total derivative of* f **of order** α **at** c and we shall denote it by $D^\alpha f(c)$ and $D^\alpha f(c)(v)$ for the value of the linear map T_c^α at v. That is $D^\alpha f(c)(v) = T_c^\alpha(v)$.

From the above definition easily one can show.

Lemma 2. *1. If $\alpha = 1$, then $D^\alpha f(c)$ is $Df(c)$.*

2. If $v = 0$, then $T_c^\alpha(v) = 0$.

3. If $v \in c^\perp$, then $(*)$ is true so we assume $v \notin c^\perp$.

4. If f is constant, then for $z \in \mathbb{R}^n \setminus \{0\}$, we have $\|T_c^\alpha(v)\| \le \in \|v\|$, where $v = \frac{\delta(\in)}{\|z\|} z$, so $\|T_c^\alpha(z)\| \le \in \|z\|$ for all $\in > 0$. Hence $T_c^\alpha(z) = 0$.

Now one can easily prove the following lemma.

Lemma 3. *Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be any function, $\alpha \in (0,1)$ and c an interior point of D_f.*

1. If f is linear then the fractional total derivative of f of order α at c is the function f. That is $T_c^\alpha(v) = f(v)$.

2. If the total derivative of f at $c, Df(c)$ exists then

$$D^\alpha f(c)(v) = Df(c)\left(|c \cdot v|^{1-\alpha} v\right) = |c \cdot v|^{1-\alpha} Df(c)(v).$$

Moreover we have the following.

Lemma 4. *Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$, if $D^\alpha f(c)$ exists then it is unique.*

Proof. Let f has $T_{1c}^\alpha, T_{2c}^\alpha$ as fractional total derivative at c, then for any $\in > 0$, there exist $\delta(\in) > 0$ such that if $v \in \mathbb{R}^n$ and $\|v\| \leq \delta(\in)$, then

$$\left\| f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c) - T_{ic}\left(|c \cdot v|^{1-\alpha} v\right)\right\| \leq \in \left\| |c \cdot v|^{1-\alpha} v \right\|,$$

for $\|v\| \leq \delta(\in)$, $i = 1, 2$. Hence

$$\begin{aligned} 0 \leq & \left\| T_{1c}^\alpha\left(|c \cdot v|^{1-\alpha} v\right) - T_{2c}^\alpha\left(|c \cdot v|^{1-\alpha} v\right)\right\| \\ = & \left\| T_{1c}^\alpha\left(|c \cdot v|^{1-\alpha} v\right) - \left(f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c)\right)\right. \\ & \left. + \left(f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c)\right) - T_{2c}^\alpha\left(|c \cdot v|^{1-\alpha} v\right)\right\| \\ \leq & 2 \in \left\| |c \cdot v|^{1-\alpha} v \right\|. \end{aligned}$$

Therefore $0 \leq \|T_{1c}^\alpha(v) - T_{2c}^\alpha(v)\| \leq 2 \in \|v\|$ for all $v \in \mathbb{R}^n$.

To compleat the proof, we have two cases:

(i) If $z \in \mathbb{R}^n \setminus \{0\}$, let $z_0 = \frac{\delta(\in)}{\|z\|} z \in \mathbb{R}^n$, which implies $\|z_0\| \leq \delta(\in)$ and hence

$$\|T_{1c}^\alpha(z_0) - T_{2c}^\alpha(z_0)\| \leq 2 \in \|z_0\|.$$

So

$$\|T_{1c}^\alpha(z) - T_{2c}^\alpha(z)\| \leq 2 \in \|z\|,$$

for all $\in > 0$, and consequently $T_{1c}^\alpha(z) = T_{2c}^\alpha(z)$.

(ii) If $z = 0$, then by linearity of $T_{1c}^\alpha, T_{2c}^\alpha$ the result follows. □

It is known that a differentiable function is continuos. The following lemma show that this is still true if $D^\alpha f(c)$ exist for some $\alpha \in (0, 1)$, $c \in D_f^\circ$.

Lemma 5. *Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a given function. If $D^\alpha f(c)$ exist for some $\alpha \in (0,1)$, $c \in D_f^\circ$, then there exist strictly positive numbers δ, K such that if $\|v\| \leq \delta$, then*

$$\left\| f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c) \right\| \leq K \left\| |c \cdot v|^{1-\alpha} v \right\|.$$

In particular f is continuous at c.

Proof. Since every linear map on a finite dimensional normed space is bounded, there exists a positive constant M such that $\|T_c^\alpha(u)\| \leq M\|u\|$. Now by Definition 3, it follows that for $\in = 1$, there exist $0 < \delta(1)$ such that for $v \in \mathbb{R}^n$ with $\|v\| < \delta(1)$. Then

$$\begin{aligned} &\left\| f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c) \right\| \\ &\leq \left\| f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c) - T_c^\alpha\left(|c \cdot v|^{1-\alpha} v\right) \right\| + \left\| T_c^\alpha\left(|c \cdot v|^{1-\alpha} v\right) \right\| \\ &\qquad \leq \left\| |c \cdot v|^{1-\alpha} v \right\| + \left\| T_c^\alpha\left(|c \cdot v|^{1-\alpha} v\right) \right\| \\ &\qquad\qquad \leq \left(\left\| |c \cdot v|^{1-\alpha} v \right\| + M \left\| |c \cdot v|^{1-\alpha} v \right\| \right). \end{aligned}$$

Hence

$$\left\| f\left(c + |c \cdot v|^{1-\alpha} v\right) - f(c) \right\| \leq (M+1) \left\| |c \cdot v|^{1-\alpha} v \right\|.$$

Now let u be a non-zero victor $v = tu$, if $|t| \leq \frac{\delta(1)}{\|u\|}$, then,

$$\left\| f\left(c + |c \cdot tu|^{1-\alpha} tu\right) - f(c) \right\| \leq (M+1) \left\| |c \cdot tu|^{1-\alpha} tu \right\|.$$

Hence

$$\lim_{t \to 0} \|f(c + tw) - f(c)\| = 0.$$

□

We know that if f is differentiable at c, then the directional derivative of f in the direction of u at c, $D_u f(c)$ exist and $D_u f(c) = Df(c)(u)$. So we shall end this section with the following Theorem.

Theorem 3. *Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$, if $D^\alpha f(c)$ exist for some $\alpha \in (0,1)$, $c \in D_f^\circ$, and $u \in \mathbb{R}^n$ be a non zero vector. Then the fractional directional derivative of f of order α in the direction of u at c, $D_u^\alpha f(c)$ exist and $D_u^\alpha f(c) = |c \cdot u|^{1-\alpha} D^\alpha f(c)(u)$.*

Proof. For $c \in u^{\perp}$, the result follows. Suppose $c \notin u^{\perp}$. Since the fractional derivative of order α of f at c exist, then given $\in > 0$, there exist $0 < \delta(\in)$, such that

$$\left\| f\left(c + |c \cdot v|^{\alpha-1} v\right) - f(c) - |c \cdot v|^{\alpha-1} D^{\alpha} f(c)(v) \right\| \leq \in |c \cdot v|^{\alpha-1} \|v\|$$

provided $\|v\| \leq \delta(\in)$.

Let u be a non zero vector, thus if $0 < |t| \leq \frac{\delta(\in)}{\|u\|}$, we have

$$\left\| f\left(c + t |c \cdot tu|^{\alpha-1} u\right) - f(c) - |c \cdot tu|^{\alpha-1} D^{\alpha} f(c)(tu) \right\| \leq \in |c \cdot tu|^{\alpha-1} \|tu\|,$$

thus

$$\left\| \frac{f\left(c + t |t|^{\alpha-1} |c \cdot u|^{\alpha-1} u\right) - f(c)}{t |t|^{\alpha-1} |c \cdot u|^{\alpha-1}} - D^{\alpha} f(c)(u) \right\| \leq \in \|u\|.$$

Therefore

$$\left\| \frac{f(c + ru) - f(c)}{r} - D^{\alpha} f(c)(u) \right\| \leq \in \|u\|.$$

This shows that $D_u f(c)$ exist and $D_u f(c) = D^{\alpha} f(c)(u)$. Hence $D_u^{\alpha} f(c) = |c \cdot u|^{1-\alpha} D_u f(c) = |c \cdot u|^{1-\alpha} D^{\alpha} f(c)(u)$. □

Corollary 1. *Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $c \in D_f^{\circ}$..*

If $D^{\alpha} f(c)$ exist for some $\alpha \in (0,1)$, then $D_i^{\alpha} f(c)$ the conformable fractional partial derivative of f at the variable x_i.exist.

If $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$, we have

$$D_u^{\alpha} f(c) = D^{\alpha} f(c)(u) = |c \cdot u|^{1-\alpha} \sum_{j=1}^{n} u_j D_j^{\alpha} f(c).$$

Proof. Let $u = e_i$, by Theorem 3 $D_i^{\alpha} f(c)$ exist for all $i = 1, ...n$, and if $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ then linearity of $D^{\alpha} f(c)$ implies

$$\begin{aligned} D_u^{\alpha} f(c) &= |c \cdot u|^{1-\alpha} D^{\alpha} f(c) \left(\sum_{j=1}^{n} u_j e_j \right) \\ &= |c \cdot u|^{1-\alpha} \sum_{j=1}^{n} u_j D^{\alpha} f(c)(e_j) = |c \cdot u|^{1-\alpha} \sum_{j=1}^{n} u_j D_j^{\alpha} f(c). \end{aligned}$$

□

Definition 7. Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$, if $D^\alpha f(c)$ exist for some $\alpha \in (0,1)$, $c \in D_f^\circ$. Then $\bigtriangledown_c^\alpha f$ the *fractional gradient* of f at c is defined by $\bigtriangledown_c^\alpha f = (D_1^\alpha f(c), D_2^\alpha f(c), ...D_n^\alpha f(c))$.

From the theory of functions of several variables for $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where both $n > 1, m > 1$. In this case we can represent $y = f(x)$ by a system of m functions of n variables

$$\begin{aligned} y_1 &= f_1(x_1, x_2, ..., x_n), \\ y_2 &= f_2(x_1, x_2, ..., x_n), \\ &\vdots \\ y_m &= f_m(x_1, x_2, ..., x_n). \end{aligned}$$

If f is differentiable at a point c,then $Df(c)$ is the linear mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$ determined by the $n \times m$ matrix

$$\begin{bmatrix} D_1 f_1(c) & D_2 f_1(c) & ... & D_n f_1(c) \\ D_1 f_2(c) & D_2 f_2(c) & ... & D_n f_2(c) \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_m(c) & D_2 f_m(c) & ... & D_n f_m(c) \end{bmatrix}.$$

In the following theorem we shall show that a similar result is obtained if we use *total fractional derivative.*

Theorem 4. *Let $f : D_f \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where $n > 1$, $m > 1$.*
If $D^\alpha f(c)$ exist for some $\alpha \in (0,1)$, $c \in D_f^\circ$.. Then $D^\alpha f(c)$ is the linear mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$ determined by the $n \times m$ matrix

$$\begin{bmatrix} D_1^\alpha f_1(c) & D_2^\alpha f_1(c) & ... & D_n^\alpha f_1(c) \\ D_1^\alpha f_2(c) & D_2^\alpha f_2(c) & ... & D_n^\alpha f_2(c) \\ \vdots & \vdots & \ddots & \vdots \\ D_1^\alpha f_m(c) & D_2^\alpha f_m(c) & ... & D_n^\alpha f_m(c) \end{bmatrix}.$$

Proof. Let us present $y = f(x)$ by the system,

$$\begin{aligned} y_1 &= f_1(x_1, x_2, ..., x_n), \\ y_2 &= f_2(x_1, x_2, ..., x_n), \\ &\vdots \\ y_m &= f_m(x_1, x_2, ..., x_n). \end{aligned}$$

If $D^{\alpha}f(c)$ exist for some $\alpha \in (0,1)$, $c \in D_f^{\circ}$, then $D^{\alpha}f_i(c)$, exist for $i = 1, ...m$.

Now $D^{\alpha}f(c)$ maps the point $(u_1, u_2, ...u_n)$ of $\mathbb{R}^n$ into the point $w = (w_1, w_2, ...w_m)$ of $\mathbb{R}^m$ given by

$$\begin{aligned}
w_1 =& D_1^{\alpha}f_1(c)\,u_1 + D_2^{\alpha}f_1(c)\,u_2 + \cdots + D_n^{\alpha}f_1(c)\,u_n,\\
w_2 =& D_1^{\alpha}f_2(c)\,u_1 + D_2^{\alpha}f_2(c)\,u_2 + \cdots + D_n^{\alpha}f_2(c)\,u_n,\\
&\vdots\\
w_m =& D_1^{\alpha}f_m(c)\,u_1 + D_2^{\alpha}f_m(c)\,u_2 + \cdots + D_n^{\alpha}f_m(c)\,u_n.
\end{aligned}$$

So the fractional derivative $D^{\alpha}f(c)$ determined by the $n \times m$ matrix whose elements are

$$\begin{bmatrix}
D_1^{\alpha}f_1(c) & D_2^{\alpha}f_1(c) & ... & D_n^{\alpha}f_1(c)\\
D_1^{\alpha}f_2(c) & D_2^{\alpha}f_2(c) & ... & D_n^{\alpha}f_2(c)\\
\vdots & \vdots & \ddots & \vdots\\
D_1^{\alpha}f_m(c) & D_2^{\alpha}f_m(c) & ... & D_n^{\alpha}f_m(c)
\end{bmatrix}. \qquad \square$$

This matrix is as usual called the *fractional Jacobi matrix* and denoted by $J_f^{\alpha}(c)$.

5. Application

In his paper [5] U. Katugampola, wrote " one of the limitation of this version of the fractional derivative is that it assumes that the variable $t > 0$. So the question is, wether we can relax this condition on a special class of functions?, if so, what it is?."

Using our definition of total fractional derivatives we shall give a simple *modification* of the definition of *conformable fractional derivative* of f, in this modification we don't assume $t > 0$. Also we shall show that all the results obtained in [6], are still valid using the modified *conformable fractional derivative* of f.

It is known that if f is a function of several variable, the directional derivative of f in the direction of a vector u, at a point c is defined by $D_u f(c) = \lim\limits_{h \to 0} \frac{f(c+hu)-f(c)}{h}$, so if we let $u = 1$ and $c = x_0$, then we obtain $f'(x_0)$. Using this idea we can modify the definition given in [6].

Let $f : D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, $\alpha \in (0,1)$ and c a non zero interior point of D_f. Let $u = 1$ in 3. Then we have,

Definition 8. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be any function,and $t\in\mathbb{R}\backslash\{0\},\alpha\in(0,1)$.Then define the *modified conformable fractional derivative* of f of order α, T_α or f^α by. $f^{(\alpha)}(c)=T_\alpha(f)(c)=\lim\limits_{\in\to 0}\frac{f\left(c+\in|c|^{1-\alpha}\right)-f(c)}{\in}$. If f is $\alpha-$ differentiable in some $(-a,a)\setminus\{0\}$, $a>0$, and $\lim\limits_{c\to 0^+}f^{(\alpha)}(c)$ exist, then $f^{(\alpha)}(0)$ is defined by $f^{(\alpha)}(0)=\lim\limits_{c\to 0^+}f^{(\alpha)}(c)$.

Clearly, if $\alpha=1$ the modified definition coincide with the classical definition of derivative.

Example 5. Let $n=3$, then for $u=i=(1,0,0)$, we have:

1. $D_i^\alpha f(c)=\lim\limits_{t\to 0}\frac{f\left(c+t|c\cdot u|^{1-\alpha}u\right)-f(c)}{t}=\lim\limits_{t\to 0}\frac{f\left(c_1+t|c_1|^{1-\alpha},c_2,\ldots,c_n\right)-f(c)}{t}=f_x^{(\alpha)}(c)$, the modified partial conformable fractional derivatives f with respect to x.

2. Similarly $D_j^\alpha f(c)$, $D_k^\alpha f(c)$ the modified partial conformable fractional derivative of f with respect to y, z respectively.

Remark 2. Let $\{e_1,e_2,\ldots,e_n\}$ be the standard basis of $\mathbb{R}^n$. Then $D_{e_i}^\alpha f(c)=D_i^\alpha f(c)$ is the modified conformable fractional partial derivative of f with respect to the variable x_i.

Definition 9. Let $\alpha\in(1,\infty)$, $f:\mathbb{R}\longrightarrow\mathbb{R}$ be $(\lceil\alpha\rceil-1)$differentiable at $t\in\mathbb{R}\backslash\{0\}$,where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α. Then the *modified conformable fractional derivative of* f of order α, T_α or f^α is defined by

$$f^{(\alpha)}(t)=T_\alpha(f)(t)=\lim_{\in\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\in|t|^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\in}.$$

Remark 3. As a consequence of Definition 9 it is easy to show that for $\alpha\in(1,\infty)$.

1. $T_\alpha(f)(t)=T_{1+\alpha-\lceil\alpha\rceil}\left(f^{(\lceil\alpha\rceil-1)}\right)(t)$.

2. If f is $\lceil\alpha\rceil$-differentiable at $t\in\mathbb{R}\backslash\{0\}$, then $T_\alpha(f)(t)=|t|^{(\lceil\alpha\rceil-\alpha)}f^{\lceil\alpha\rceil}(t)$.

Theorem 5. *If $f:\mathbb{R}\longrightarrow\mathbb{R}$, if the modified conformable the fractional derivative of f of order α exists at $t_0\in\mathbb{R}\backslash\{0\},\alpha\in(0,1]$, then f is continuous at t_0*

Proof. We want to show that $\lim\limits_{h\to 0}f(t_0+h)=f(t_0)$. Let $\in=\frac{h}{|t_0|^{1-\alpha}}$, then

$$\lim_{h\to 0} f(t_0+h) - f(t_0) = \lim_{\in\to 0} f\left(t_0 + \in |t_0|^{1-\alpha}\right) - f(t_0)$$
$$= \lim_{\in\to 0} \frac{f\left(t_0 + \in |t_0|^{1-\alpha}\right) - f(t_0)}{\in} \in = f^{(\alpha)}(t_0).0 = 0. \quad \square$$

It is clear that if $\alpha \in (1,\infty)$ and $T_\alpha(f)(t)$ exists then f is continuous. The following theorem is an easy consequence of the modified definition.

Theorem 6. *Let $\alpha \in (0,1]$ and $f.g$ be α- differentiable at $t \in \mathbb{R}\backslash\{0\}$. Then:*

1.- $T_\alpha(af+bg) = aT_\alpha(f) + bT_\alpha(g)$ for all $a,b \in \mathbb{R}$.

2. $T_\alpha(|t|^p) = p\,|t|^{p-\alpha}\, signt$. for all $p \in \mathbb{R}$, where $signt = \left\{\begin{array}{cc} 1 & t>0 \\ -1 & t<0 \end{array}\right\}$.

3. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

4. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

5. $T_\alpha(\lambda) = 0$, for all constant function λ.

6. If, in addition f is differentiable, then $T_\alpha(f)(t) = |t|^{1-\alpha}\frac{df}{dt}$.

In the following example the fractional derivative of a well known functions are given using the modified definition.

Example 6. Let $\alpha \in (0,1]$ and $c,b \in \mathbb{R}$. Then:

1. $T_\alpha\left(e^{ct}\right) = c\,|t|^{1-\alpha}\, e^{ct}$.

2. $T_\alpha(\sin bx) = b\,|t|^{1-\alpha}\cos bt$.

3. $T_\alpha(\cos bx) = -b\,|t|^{1-\alpha}\sin bt$.

4. Let $t \neq 0$, then $T_\alpha\left(\frac{1}{\alpha}|t|^\alpha\right) = signt$.

Using the modified definition, the proof of Theorem 2.3 in [6] (Rolle's Theorem for conformable fractional derivative) is the same, but Theorem 2.4 in [6] (Mean Value Theorem for Conformable Fractional Differentiable Functions) needs a simple modification.

Theorem 7. *(Rolle's Theorem for Conformable Fractional Differentiable Functions). Let $0 \notin [a,b]$, and $f:[a,b] \to \mathbb{R}$ be a given function that satisfies:*

(i) f is continuous on $[a,b]$.

(ii) f is $\alpha-$differentiable for some $\alpha \in (0,1]$.

(iii) $f(a) = f(b)$.

Then, there exists $c \in [a,b]$, *such that* $f^{(\alpha)}(c) = 0$.

Theorem 8. *(Mean Value Theorem for Conformable Fractional Differentiable Functions). Let* $0 \notin [a,b]$ *,and* $f : [a,b] \to \mathbb{R}$ *be a given function that satisfies,*

(i) f *is continuous on* $[a,b]$.

(ii) f *is* $\alpha-$*differentiable for some* $\alpha \in (0,1]$.

Then, there exists $c \in [a,b]$, *such that* $f^{(\alpha)}(c) = sgnc \frac{f(b)-f(a)}{\frac{1}{\alpha}|b|^{\alpha} - \frac{1}{\alpha}|a|^{\alpha}}$.

Proof. Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{1}{\alpha}|b|^{\alpha} - \frac{1}{\alpha}|a|^{\alpha}} \left(\frac{1}{\alpha}|x|^{\alpha} - \frac{1}{\alpha}|a|^{\alpha} \right).$$

Then the function g satisfies the condition of 7. Hence there exists $c \in [a,b]$, such that $0 = g^{(\alpha)}(c) = f^{(\alpha)}(c) - \frac{f(b)-f(a)}{\frac{1}{\alpha}|b|^{\alpha} - \frac{1}{\alpha}|a|^{\alpha}} sgnc.$ □

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