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# ON THE DIOPHANTINE EQUATION $x^4 + y^4 = p^k z^7$

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**Abstract:** This paper solves the Diophantine equation $x^4+y^4 = p^k z^7$ nontrivially in the case of $x = y$ where $p$ is prime and $k \in \mathbb{Z}^+$. The parametric solutions are formulated using number theory theorems, especially those concerning divisibility of integers, linear Diophantine equations, properties of prime numbers, and properties of congruence. There exist infinitely many nontrivial integral solutions to this Diophantine equation, where the parametric solutions found solve completely for different values of $p$ and $k$ in the case of $x = y$.

**AMS Subject Classification:** 11D41
**Key Words:** Diophantine equation, congruence, septic degree

## 1. Introduction

In 2011, Ismail (see [2]) sought the integral solutions for $x^4 + y^4 = p^k z^3$ where $p$ is prime, $2 \leq p \leq 13$, and $k \in \mathbb{Z}^+$. In her studies, she claimed to have found all integral solutions to this equation. However, there are solutions that cannot be obtained using her parametric solutions; thus not a complete solution to her choice of Diophantine equation. For example, in the case of $x^4 + y^4 = 3^4 z^3$ where $x = y$, her parametric solution could not find $(12, 12, 8)$, $(96, 96, 128)$, $(324, 324, 648)$, $(768, 768, 2048)$, and $(1500, 1500, 5000)$ which are also solutions to the equation.

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This is due to her assumption in her proofs that $z$ must always contain the prime $p$ when represented as product of primes. Although her parametric solutions can fulfill her Diophantine equation, they would not yield the complete solutions to the equation whenever $k \equiv 1 \pmod 4$ (for $p = 2$) and $k \equiv 0 \pmod 4$ (for $p > 2$) are concerned. Having identified this shortcoming, this motivates the study of

$$x^4 + y^4 = p^k z^7 \tag{1}$$

where $p$ is any prime and $k \in \mathbb{Z}^+$, where the prime concern is to seek nontrivial parametric solutions to this Diophantine equation by taking congruence consideration into account. This paper focuses on the case where $x = y$. The idea of solving this Diophantine equation can be used on other similar Diophantine equations for $x = y$. This includes Ismail's; thus closing the gap in her parametric solutions for such case.

This paper is organized as follows: Section 2 gives the parametric solutions for (1) where $x = y$ and $p = 2$. It considers three cases of $k$; $k = 1$, $k > 1$ and $k \equiv 1 \pmod 4$, and $k > 1$ and $k \not\equiv 1 \pmod 4$. Section 3 gives the parametric solutions for (1) where $x = y$ and $p > 2$. It considers two cases of $k$; $k \equiv 0 \pmod 4$ and $k \not\equiv 0 \pmod 4$. Finally, Section 4 concludes the paper with a summary of all parametric solutions for (1) where $x = y$.

## 2. On the Diophantine Equation $x^4 + y^4 = p^k z^7$, where $x = y$ and $p = 2$

The following theorem gives the nontrivial parametric solutions to (1) where $x = y$ and $p = 2$.

**Theorem 1.** *Suppose that $(x_0, y_0, z_0)$ is a nontrivial integral solution to $x^4 + y^4 = 2^k z^7$ where $x_0 = y_0$ and $k \in \mathbb{Z}^+$. If $k = 1$, then $(x_0, y_0, z_0) = (\pm n^7, \pm n^7, n^4)$ where $n \in \mathbb{Z}^+$. If $k > 1$ and $k \equiv 1 \pmod 4$, then $(x_0, y_0, z_0) = (\pm 2^v n^7, \pm 2^v n^7, n^4)$ where $n \in \mathbb{Z}^+$ and $4v + 1 = k$. If $k > 1$ and $k \not\equiv 1 \pmod 4$, then $(x_0, y_0, z_0) = (\pm 2^{2k-2} n^7, \pm 2^{2k-2} n^7, 2^{k-1} n^4)$ where $n \in \mathbb{Z}^+$.*

*Proof.* Let $(x_0, y_0, z_0)$ be a solution to $x^4 + y^4 = 2^k z^7$ where $x_0 = y_0$. Then

$$x_0^4 = 2^{k-1} z_0^7. \tag{2}$$

$x_0$ is of quartic degree. Thus, without loss of generality, let $x_0$ be a positive integer. By Fundamental Theorem of Arithmetic, let $x_0$ and $z_0$ be represented as a product of primes in their canonical forms, respectively;

$$x_0 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} = \prod_{i=1}^{r} p_i^{\alpha_i} \tag{3}$$

$$z_0 = q_1^{\beta_1} q_2^{\beta_2} \dots q_s^{\beta_s} = \prod_{j=1}^{s} q_j^{\beta_j} \tag{4}$$

where $p_i$ and $q_j$ are primes, $p_1 < p_2 < \cdots < p_r$, $q_1 < q_2 < \cdots < q_s$, $r, s$ are nonnegative integers, and $\alpha_i, \beta_j$ are positive integers.

Plugging (3) and (4) into (2),

$$\prod_{i=1}^{r} p_i^{4\alpha_i} = 2^{k-1} \prod_{j=1}^{s} q_j^{7\beta_j}. \tag{5}$$

There are two cases to be considered: $k = 1$ and $k > 1$.

***Case 1***: Suppose $k = 1$. Then (5) becomes

$$\prod_{i=1}^{r} p_i^{4\alpha_i} = \prod_{j=1}^{s} q_j^{7\beta_j}. \tag{6}$$

In this case, due to the uniqueness of canonical representation of integers, $r = s$, $p_i = q_j$, and $4\alpha_i = 7\beta_j$ for $1 \leq i = j \leq r = s$. Note that $7|4\alpha_i$, $4|7\beta_j$, and $\gcd(4,7) = 1$. We must have positive integers $v_i$ and $w_j$ such that $\alpha_i = 7v_i$ and $\beta_j = 4w_j$. Thus $4(7v_i) = 7(4w_j)$ and hence $v_i = w_j$. So, (3) and (4) become

$$x_0 = y_0 = \prod_{i=1}^{r} p_i^{7v_i} = \left(\prod_{i=1}^{r} p_i^{v_i}\right)^7 \tag{7}$$

$$z_0 = \prod_{j=1}^{s} q_j^{4w_j} = \left(\prod_{j=1}^{s} q_j^{w_j}\right)^4. \tag{8}$$

Let $n = \prod_{i=1}^{r} p_i^{v_i} = \prod_{j=1}^{s} q_j^{w_j}$, then from (7) and (8), $x_0 = y_0 = n^7$ and $z_0 = n^4$. Note that $n$ is any positive integer. Also, $x_0$ being positive or negative has no effect on the balance of the equation since it is of quartic degree in (2). Thus we have $(x_0, y_0, z_0) = (\pm n^7, \pm n^7, n^4)$ where $n \in \mathbb{Z}^+$.

***Case 2***: Suppose $k > 1$. Then from (2), $2^{k-1}|x_0^4$. Thus, $x_0 = \prod_{i=1}^{r} p_i^{\alpha_i}$ must contain a prime $p_1 = 2$. (3) becomes

$$x_0 = 2^{\alpha_1} \prod_{i=2}^{r} p_i^{\alpha_i} \tag{9}$$

and (5) becomes

$$2^{4\alpha_1} \prod_{i=2}^{r} p_i^{4\alpha_i} = 2^{k-1} \prod_{j=1}^{s} q_j^{7\beta_j}. \tag{10}$$

Comparing the indices of 2 on both sides of (10), $4\alpha_1 = k-1$ suggests that only positive integers $k$ where $k \equiv 1 \pmod 4$ allow (10) to be consistent. However, in the case of $k \not\equiv 1 \pmod 4$, (10) still holds if $z_0$ contains a prime factor 2. Hence, there are two cases that need to be considered: $k \not\equiv 1 \pmod 4$ and $k \equiv 1 \pmod 4$.

***Case a***: Suppose $k \not\equiv 1 \pmod 4$. (10) will not be consistent unless there exists a prime factor $q_1 = 2$ in $z_0$. Thus $z_0 = 2^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j}$ and (10) becomes

$$2^{4\alpha_1} \prod_{i=2}^{r} p_i^{4\alpha_i} = 2^{k-1+7\beta_1} \prod_{j=2}^{s} q_j^{7\beta_j}. \tag{11}$$

We now have a linear Diophantine problem to be solved in order for (11) to be consistent: $4\alpha_1 - 7\beta_1 = k-1$. It is easy to check that $(\alpha_1, \beta_1) = (2k-2, k-1)$ is a particular solution to this equation, and its general solution is thus $\alpha_1 = 2k-2-7t$ and $\beta_1 = k-1-4t$ where $t \leq 0$. Any integer represented in its canonical form is unique. Thus $r = s$, $p_i = q_j$, and $4\alpha_i = 7\beta_j$ for $2 \leq i = j \leq r = s$. Note that $7|4\alpha_i$, $4|7\beta_j$, and gcd(4,7)=1. We must have positive integers $v_i$ and $w_j$ such that $\alpha_i = 7v_i$ and $\beta_j = 4w_j$. Thus $4(7v_i) = 7(4w_j)$ and hence $v_i = w_j$. Using these new forms, we have

$$x_0 = y_0 = 2^{2k-2} \left( 2^{-t} \prod_{i=2}^{r} p_i^{v_i} \right)^7 \tag{12}$$

$$z_0 = 2^{k-1} \left( 2^{-t} \prod_{j=2}^{s} q_j^{w_j} \right)^4. \tag{13}$$

Let $n = 2^{-t} \prod_{i=2}^{r} p_i^{v_i} = 2^{-t} \prod_{j=2}^{s} q_j^{w_j}$. Then (12) and (13) become $x_0 = y_0 = 2^{2k-2}n^7$ and $z_0 = 2^{k-1}n^4$, respectively. $x_0$ being positive or negative has no effect on the balance of the equation since it is of quartic degree in (2). Thus, in the case of $k \not\equiv 1 \pmod 4$, $(x_0, y_0, z_0) = (\pm 2^{2k-2} n^7, \pm 2^{2k-2} n^7, 2^{k-1} n^4)$ where $n \in \mathbb{Z}^+$.

***Case b***: Suppose $k \equiv 1 \pmod 4$. Then there are another two subcases to be considered: $z_0$ contains a prime factor $q_1 = 2$ and $z_0$ does not contain a prime factor 2.

***Case b1***: Suppose $z_0$ contains a prime factor $q_1 = 2$. Then we have $z_0 = 2^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j}$ and (10) becomes (11). This subcase is solved using the same steps in *Case a*, yielding the same forms of $(x_0, y_0, z_0)$ in the said case.

***Case b2***: Suppose $z_0$ does not contain a prime factor 2. Then $z_0$ remains as (4) and in order for (10) to be balanced, $s = r - 1$. Thus (10) becomes

$$2^{4\alpha_1} \prod_{i=2}^{r} p_i^{4\alpha_i} = 2^{k-1} \prod_{j=2}^{r} q_j^{7\beta_j}. \tag{14}$$

Looking at indices of 2 on both sides of (14), we have $4\alpha_1 = k - 1$. $4|(k-1)$, so there exists a positive integer $v$ such that $k = 4v + 1$. Thus, $4\alpha_1 = (4v + 1) - 1$ which leads to $\alpha_1 = v$. Again, due to the uniqueness of canonical form of integers, $p_i = q_j$ for $2 \leq i = j \leq r$. $7|4\alpha_i$, $4|7\beta_j$, and gcd(4,7)=1. We must have positive integers $v_i$ and $w_j$ such that $\alpha_i = 7v_i$ and $\beta_j = 4w_j$. This leads to $v_i = w_j$. Using these new forms, we have

$$x_0 = y_0 = 2^v \left( \prod_{i=2}^{r} p_i^{v_i} \right)^7 \tag{15}$$

$$z_0 = \left( \prod_{j=2}^{r} q_j^{w_j} \right)^4. \tag{16}$$

Let $n = \prod_{i=2}^{r} p_i^{v_i} = \prod_{j=2}^{r} q_j^{w_j}$, and thus (15) and (16) become $x_0 = y_0 = 2^v n^7$ and $z_0 = n^4$, respectively. $x_0$ being positive or negative has no effect on the balance of the equation since $x_0$ is of quartic degree in (2). Thus we have $(x_0, y_0, z_0) = (\pm 2^v n^7, \pm 2^v n^7, n^4)$ where $4v + 1 = k$ and $n \in \mathbb{Z}^+$. This form of solution covers that of *Case b1* where $z_0$ contains prime factor 2 and $k \equiv 1 \pmod 4$. It is easy to show that for a choice of $n = n_1$ in *Case b1*, the same solution can be obtained at $n = 2^v n_1$ in *Case b2*. Thus, in the case of $k \equiv 1 \pmod 4$, the parametric form in *Case b2* is sufficient to solve for all $(x_0, y_0, z_0)$. □

## 3. On the Diophantine Equation $x^4 + y^4 = p^k z^7$, where $x = y$ and $p > 2$

The following theorem gives the nontrivial parametric solutions to (1) where $x = y$ and $p > 2$.

**Theorem 2.** *Suppose that* $(x_0, y_0, z_0)$ *is a nontrivial integral solution to* $x^4 + y^4 = p^k z^7$ *where* $x_0 = y_0$, $k \in \mathbb{Z}^+$, *and* $p$ *is a prime where* $p > 2$. *If* $k \equiv 0 \pmod 4$, *then* $(x_0, y_0, z_0) = (\pm 2^5 p^v n^7, \pm 2^5 p^v n^7, 2^3 n^4)$ *where* $n \in \mathbb{Z}^+$ *and* $4v = k$. *If* $k \not\equiv 0 \pmod 4$, *then* $(x_0, y_0, z_0) = (\pm 2^5 p^{2k} n^7, \pm 2^5 p^{2k} n^7, 2^3 p^k n^4)$ *where* $n \in \mathbb{Z}^+$.

*Proof.* Let $(x_0, y_0, z_0)$ be a solution to (1) where $x_0 = y_0$. Then,

$$2x_0^4 = p^k z_0^7. \tag{17}$$

$x_0$ is of quartic degree. Thus, without loss of generality, let $x_0$ be a positive integer. By Fundamental Theorem of Arithmetic, let $x_0$ and $z_0$ be represented as a product of primes $\prod_{i=1}^{r} p_i^{\alpha_i}$ and $\prod_{j=1}^{s} q_j^{\beta_j}$, respectively, where $p_i$ and $q_j$ are primes, $r, s$ are nonnegative integers, and $\alpha_i, \beta_j$ are positive integers. Since gcd($2$,$p^k$)=1, $2|z_0^7$ and $p^k|x_0^4$. There must be a prime $p_i = p$ in $x_0$ and $q_j = 2$ in $z_0$, where $1 \le i \le r$ and $1 \le j \le s$. Let $p_2 = p$ and $q_1 = 2$. Then

$$x_0 = p^{\alpha_2} \prod_{i=1, i \boxminus 2}^{r} p_i^{\alpha_i} \tag{18}$$

$$z_0 = 2^{\beta_1} \prod_{j=2}^{s} q_j^{\beta_j}. \tag{19}$$

Plugging (18) and (19) into (17),

$$2p^{4\alpha_2} \prod_{i=1, i \boxminus 2}^{r} p_i^{4\alpha_i} = 2^{7\beta_1} p^k \prod_{j=2}^{s} q_j^{7\beta_j}. \tag{20}$$

Observe the indices of 2 in (20) that in order for (20) to be consistent, $7\beta_1 = 1$ but $\beta_1 \notin \mathbb{Z}^+$ (contradiction). Thus, $x_0$ must contain a prime $p_i = 2$ where $1 \le i \le r$, $i \ne 2$. Let $p_1 = 2$ and we have $x_0 = 2^{\alpha_1} p^{\alpha_2} \prod_{i=3}^{r} p_i^{\alpha_i}$. Then (20) becomes

$$2^{4\alpha_1 + 1} p^{4\alpha_2} \prod_{i=3}^{r} p_i^{4\alpha_i} = 2^{7\beta_1} p^k \prod_{j=2}^{s} q_j^{7\beta_j}. \tag{21}$$

As for the indices of $p$ in (21), $4\alpha_2 = k$ suggests that only positive integers $k$ where $k \equiv 0 \pmod 4$ allow (21) to be consistent. However, in the case of $k \not\equiv 0 \pmod 4$, (21) still holds if $z_0$ contains a prime factor $p$. Hence, there are two cases that need to be considered: $k \not\equiv 0 \pmod 4$ and $k \equiv 0 \pmod 4$.

***Case a***: Suppose that $k \not\equiv 0 \pmod 4$. Then (21) will not be consistent unless there exists a prime $q_j = p$ where $2 \leq j \leq s$. Let $q_2 = p$ and we have $z_0 = 2^{\beta_1} p^{\beta_2} \prod_{j=3}^{s} q_j^{\beta_j}$. Then (21) becomes

$$2^{4\alpha_1+1} p^{4\alpha_2} \prod_{i=3}^{r} p_i^{4\alpha_i} = 2^{7\beta_1} p^{7\beta_2+k} \prod_{j=3}^{s} q_j^{7\beta_j}. \tag{22}$$

We now have two linear Diophantine problems to be solved in order for (22) to be consistent:

$$-4\alpha_1 + 7\beta_1 = 1 \tag{23}$$

$$4\alpha_2 - 7\beta_2 = k. \tag{24}$$

For (23), its solution is $\alpha_1 = 5 - 7t_1$ and $\beta_1 = 3 - 4t_1$ where $t_1 \leq 0$. For (24), its solution is $\alpha_2 = 2k - 7t_2$ and $\beta_2 = k - 4t_2$ where $t_2 \leq 0$.

Let $\sigma$ and $\gamma$ be permutation functions defined respectively as

$$\sigma : \{3, 4, \ldots, r\} \longmapsto \{3, 4, \ldots, r\} \tag{25}$$

where $\sigma(i) = i'$, $p_{i'} < p_{i'+1}$, $i' < i' + 1$, and

$$\gamma : \{3, 4, \ldots, s\} \longmapsto \{3, 4, \ldots, s\} \tag{26}$$

where $\gamma(j) = j'$, $q_{j'} < q_{j'+1}$, $j' < j' + 1$.

Applying them on the products of primes in (22), they arrange the primes in $\prod_{i=3}^{r} p_i^{4\alpha_i}$ and $\prod_{j=3}^{s} q_j^{7\beta_j}$ into their canonical forms:

$$2^{4\alpha_1+1} p^{4\alpha_2} \prod_{i'=3}^{r} p_{i'}^{4\alpha_{i'}} = 2^{7\beta_1} p^{7\beta_2+k} \prod_{j'=3}^{s} q_{j'}^{7\beta_{j'}}. \tag{27}$$

Any integer represented in its canonical form is unique; $r = s$ and $p_{i'} = q_{j'}$ for $3 \leq i' = j' \leq r = s$ for (27) to be consistent. Consequently, $4\alpha_{i'} = 7\beta_{j'}$. $7|4\alpha_{i'}$, $4|7\beta_{j'}$, and gcd(4,7)=1. We must have positive integers $v_{i'}$ and $w_{j'}$ such that

$\alpha_{i'} = 7v_{i'}$ and $\beta_{j'} = 4w_{j'}$. Thus, $4(7v_{i'}) = 7(4w_{j'})$ and hence $v_{i'} = w_{j'}$. Using these new forms, we have

$$x_0 = y_0 = 2^5 p^{2k} \left( 2^{-t_1} p^{-t_2} \prod_{i'=3}^{r} p_{i'}^{v_{i'}} \right)^7 \tag{28}$$

$$z_0 = 2^3 p^k \left( 2^{-t_1} p^{-t_2} \prod_{j'=3}^{s} q_{j'}^{w_{j'}} \right)^4 . \tag{29}$$

Let $n = 2^{-t_1} p^{-t_2} \prod_{i'=3}^{r} p_{i'}^{v_{i'}} = 2^{-t_1} p^{-t_2} \prod_{j'=3}^{s} q_{j'}^{w_{j'}}$, and thus (28) and (29) become $x_0 = y_0 = 2^5 p^{2k} n^7$ and $z_0 = 2^3 p^k n^4$, respectively. Thus, in the case of $k \not\equiv 0 \pmod 4$, we have $(x_0, y_0, z_0) = (\pm 2^5 p^{2k} n^7, \pm 2^5 p^{2k} n^7, 2^3 p^k n^4)$ where $n \in \mathbb{Z}^+$.

***Case b***: Suppose that $k \equiv 0 \pmod 4$. Then there are another two subcases to be considered: $z_0$ contains a prime factor $q_j = p$ where $2 \le j \le s$ and $z_0$ does not contain a prime factor $p$ (see (21)).

***Case b1***: Suppose that $z_0$ contains a prime factor $q_j = p$ where $2 \le j \le s$. Let $q_2 = p$ and we have $z_0 = 2^{\beta_1} p^{\beta_2} \prod_{j=3}^{s} q_j^{\beta_j}$ and (21) becomes (22). This subcase is solved using the same steps in *Case a*, yielding the same forms of $(x_0, y_0, z_0)$ in the said case.

***Case b2***: Suppose that $z_0$ does not contain a prime factor $p$. Then $z_0$ is in the form of (19) and in order for (21) to be consistent, $s = r - 1$. Thus (21) becomes

$$2^{4\alpha_1+1} p^{4\alpha_2} \prod_{i=3}^{r} p_i^{4\alpha_i} = 2^{7\beta_1} p^k \prod_{j=3}^{r} q_j^{7\beta_j}. \tag{30}$$

Comparing the indices of 2 and $p$ on both sides in (30), we have two linear Diophantine problems to solve:

$$-4\alpha_1 + 7\beta_1 = 1 \tag{31}$$

$$4\alpha_2 = k. \tag{32}$$

(31) is the same as (23) in *Case a* with the solutions of $\alpha_1 = 5 - 7t$ and $\beta_1 = 3 - 4t$ where $t \le 0$. For (32), note that $4|k$, so there exists a positive integer $v$ such that $k = 4v$. So $4\alpha_2 = 4v$ and hence $\alpha_2 = v$. Applying (25)

and (26) (where the domain and codomain of (26) are now $\{3, 4, \ldots, r\}$) on the products of primes in (30) to arrange them into canonical forms, we get

$$2^{4\alpha_1+1} p^{4\alpha_2} \prod_{i'=3}^{r} p_{i'}^{4\alpha_{i'}} = 2^{7\beta_1} p^k \prod_{j'=3}^{r} q_{j'}^{7\beta_{j'}}. \tag{33}$$

Any integer represented in its canonical form is unique, so $p_{i'} = q_{j'}$ and $4\alpha_{i'} = 7\beta_{j'}$ for $3 \leq i' = j' \leq r$. $7|4\alpha_{i'}$, $4|7\beta_{j'}$, and gcd(4,7)=1. We must have positive integers $v_{i'}$ and $w_{j'}$ such that $\alpha_{i'} = 7v_{i'}$ and $\beta_{j'} = 4w_{j'}$. So, $4(7v_{i'}) = 7(4w_{j'})$ and hence $v_{i'} = w_{j'}$. Using these new forms, we have

$$x_0 = y_0 = 2^5 p^v \left(2^{-t} \prod_{i'=3}^{r} p_{i'}^{v_{i'}}\right)^7 \tag{34}$$

$$z_0 = 2^3 \left(2^{-t} \prod_{j'=3}^{r} q_{j'}^{w_{j'}}\right)^4. \tag{35}$$

Let $n = 2^{-t} \prod_{i'=3}^{r} p_{i'}^{v_{i'}} = 2^{-t} \prod_{j'=3}^{r} q_{j'}^{w_{j'}}$, and thus (34) and (35) become $x_0 = y_0 = 2^5 p^v n^7$ and $z_0 = 2^3 n^4$, respectively. Thus, in the case of $k \equiv 0 \pmod 4$, we have $(x_0, y_0, z_0) = (\pm 2^5 p^v n^7, \pm 2^5 p^v n^7, 2^3\ n^4)$ where $4v = k$ and $n \in \mathbb{Z}^+$. This form of solution covers that of *Case b1* where $z_0$ contains prime factor $p$ and $k \equiv 0 \pmod 4$. It is easy to show that for a choice of $n = n_1$ in *Case b1*, the same solution can be obtained at $n = p^v n_1$ in *Case b2*. Thus, in the case of $k \equiv 0 \pmod 4$, the parametric form in *Case b2* is sufficient to solve for all $(x_0, y_0, z_0)$.

□

## 4. Conclusion

There exist infinitely many nontrivial integral solutions to (1) in the case of $x = y$, where the parametric solutions are $(x, y, z) = (\pm n^7, \pm n^7, n^4)$ for $p = 2$ and $k = 1$, $(x, y, z) = (\pm 2^v n^7, \pm 2^v n^7, n^4)$ where $4v+1 = k$ for $p = 2$, $k > 1$, and $k \equiv 1 \pmod 4$, $(x, y, z) = (\pm 2^{2k-2} n^7, \pm 2^{2k-2} n^7, 2^{k-1} n^4)$ for $p = 2$, $k > 1$, and $k \not\equiv 1 \pmod 4$, $(x, y, z) = (\pm 2^5 p^v n^7, \pm 2^5 p^v n^7, 2^3 n^4)$ where $4v = k$ for $p > 2$ and $k \equiv 0 \pmod 4$, and $(x, y, z) = (\pm 2^5 p^{2k} n^7, \pm 2^5 p^{2k} n^7, 2^3 p^k n^4)$ for $p > 2$ and $k \not\equiv 0 \pmod 4$; $n \in \mathbb{Z}^+$. These parametric solutions solve (1) completely in such case.

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