L-FUZZY \((K, E)\)-SOFT TOPOLOGIES AND L-FUZZY \((K, E)\)-SOFT CLOSURE OPERATORS

Yong Chan Kim\(^1\), A.A. Ramadan\(^2\)

\(^1\)Department of Mathematics
Gangneung-Wonju University
Gangneung, Gangwondo, 210-702, KOREA

\(^2\)Department of Mathematics
Faculty of Science
Beni-Suef University
Beni-Suef, EGYPT

Abstract: In this paper, we investigate the properties of fuzzy soft sets and fuzzy soft maps in stsc-quantales. We define a \(L\)-fuzzy \((K, E)\)-soft topology and a \(L\)-fuzzy \((K, E)\)-soft closure spaces as a Höhle’s sense. We study the relations between \(L\)-fuzzy \((K, E)\)-soft topologies and \(L\)-fuzzy \((K, E)\)-soft closure spaces. We give their examples.

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1. Introduction

In 1999, Molodtsov [13] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. In [14], Molodtsov applied successfully in directions such as, smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability and theory of measurement. Maji et al. [10,11] gave the first

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\(^\S\)Correspondence author

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices. Höhle [9] introduced $L$-fuzzy topologies with algebraic structure $L$(cqms, quantales, $MV$-algebra). It has developed in many directions [16-18]. Ramadan et al. [17] define the the concept of $L$-fuzzy soft topogenous orders, $L$-fuzzy soft uniform spaces, $L$-fuzzy soft topological spaces in strictly two sided commutative quantales and investigated the relation between them.

In this paper, we define a $L$-fuzzy $(K,E)$-soft topology and a $L$-fuzzy $(K,E)$-soft closure spaces as a Aygınoglu et.al [2] in stsc-quantales. We study the relations between $L$-fuzzy $(K,E)$-soft topologies and $L$-fuzzy $(K,E)$-soft closure spaces. We give their examples.

2. Preliminaries

Let $L = (L, \leq, \lor, \land, 0, 1)$ be a completely distributive lattice with the least element $0$ and the greatest element $1$ in $L$.

**Definition 1.** [8,9,17] A complete lattice $(L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \odot)$ is a commutative semigroup,
(L2) $x = x \odot 1$, for each $x \in L$ and $1$ is the universal upper bound,
(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee x_i) \odot y = \bigvee (x_i \odot y).

There exists a further binary operation $\to$ (called the implication operator or residuated) satisfying the following condition

\[ x \to y = \bigvee \{z \in L | x \odot z \leq y\}. \]

Then it satisfies Galois correspondence; i.e, $(x \odot z) \leq y$ iff $z \leq (x \to y)$.

In this paper, we always assume that $(L, \leq, \odot, \to, \oplus, \ast)$ is a stsc-quantales with an order reversing involution $\ast$ which is defined by

$\bigoplus y = (x^* \odot y^*)^*, \quad x^* = x \to 0$
unless otherwise specified.

**Remark 2.** Every completely distributive lattice \((L, \leq, \land, \lor, *)\) with order reversing involution \(*\) is a stsc-quantale \((L, \leq, \circ, \oplus, *)\) with a strong negation \(*\) where \(\circ = \land\) and \(\oplus = \lor\).

**Lemma 3.** [8,9,17] For each \(x, y, z, x_i, y_i, w \in L\), we have the following properties.

1. \(1 \rightarrow x = x, 0 \circ x = 0\),
2. If \(y \leq z\), then \(x \circ y \leq x \circ z, x \oplus y \leq x \oplus z, x \rightarrow y \leq x \rightarrow z\) and \(z \rightarrow x \leq y \rightarrow x\),
3. \(x \leq y\) iff \(x \rightarrow y = 1\),
4. \((\bigwedge_i y_i)^* = \bigvee_i y_i^*, (\bigvee_i y_i)^* = \bigwedge_i y_i^*\),
5. \(x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)\),
6. \((\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)\),
7. \(x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)\),
8. \((\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)\),
9. \((x \circ y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)\),
10. \(x \circ y = (x \rightarrow y)^*\) and \(x \oplus y = x^* \rightarrow y\),
11. \((x \rightarrow y) \circ (z \rightarrow w) \leq (x \circ z) \rightarrow (y \circ w)\),
12. \(x \rightarrow y \leq (x \circ z) \rightarrow (y \circ z)\) and \((x \rightarrow y) \circ (y \rightarrow z) \leq x \rightarrow z\),
13. \((x \rightarrow y) \circ (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)\).
14. \(x \rightarrow y = y^* \rightarrow x^*\).
15. \((x \lor y) \circ (z \lor w) \leq (x \lor z) \lor (y \circ w) \leq (x \oplus z) \lor (y \circ w)\).
16. \(\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\) and \(\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)\),
17. \((x \circ y) \circ (z \oplus w) \leq (x \circ z) \oplus (y \circ w)\).

Throughout this paper, \(X\) refers to an initial universe, \(E\) and \(K\) are the sets of all parameters for \(X\), and \(L^X\) is the set of all \(L\)-fuzzy sets on \(X\).

**Definition 4.** [4] A map \(f\) is called an \(L\)-fuzzy soft set on \(X\), where \(f\) is a mapping from \(E\) into \(L^X\), i.e., \(f_e := f(e)\) is an \(L\)-fuzzy set on \(X\), for each \(e \in E\). The family of all \(L\)-fuzzy soft sets on \(X\) is denoted by \((L^X)^E\). Let \(f\) and \(g\) be two \(L\)-fuzzy soft sets on \(X\).
(1) $f$ is an $L$-fuzzy soft subset of $g$ and we write $f \subseteq g$ if $f_e \leq g_e$, for each $e \in E$. $f$ and $g$ are equal if $f \subseteq g$ and $g \subseteq f$.

(2) The intersection of $f$ and $g$ is an $L$- fuzzy soft set $h = f \cap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of $f$ and $g$ is an $L$- fuzzy soft set $h = f \cup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An $L$- fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An $L$- fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an $L$- fuzzy soft sets on $X$ is denoted by $f^*$, where $f^* : E \to L^X$ is a mapping given by $f^*_e = (f_e)^*$, for each $e \in E$.

(7) $f$ is called a null $L$- fuzzy soft set and is denoted by $0_X$, if $f_e(x) = 0$, for each $e \in E$, $x \in X$.

(8) $f$ is called an absolute $L$- fuzzy soft set and is denoted by $1_X$, if $f_e(x) = 1$, for each $e \in E$, $x \in X$ and $(1_x)_e(x) = 1$.

**Definition 5.** Let $\varphi : X \to Y$ and $\psi : E_1 \to E_2$ be two mappings, where $E_1$ and $E_2$ are parameters sets for the crisp sets $X$ and $Y$, respectively. Then $\varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2}$ is called a fuzzy soft mapping.

(1) For $f \in (L^X)^{E_1}$, the image of $f$ under the fuzzy soft mapping $\varphi_\psi$ defined by, $\forall k \in K, \forall y \in Y$,

$$\varphi(f)_{e_2}(y) = \left\{ \begin{array}{ll}
\bigvee_{\psi(e_1) = y} \bigvee_{\varphi(x) = y} f_{e_1}(x), & \text{if } x \in \varphi^{-1}(y), e_1 \in \psi^{-1}(e_2) \\
0, & \text{otherwise.}
\end{array} \right.$$ 

(2) For $f \in (L^X)^{E_1}$, the pre-image of $g$ defined by

$$\varphi^{-1}_\psi(g)(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.$$ 

(3) The soft mapping $\varphi_\psi : (L^X)^{E_1} \to (L^Y)^{E_2}$ is called injective (resp. surjective, bijective) if $f$ and $\varphi$ are both injective (resp. surjective, bijective).

**Definition 6.** [2,15] A mapping $T : K \to L^{(L^X)^E}$ (where $T_k := T(k) : (L^X)^E \to L$ is a mapping for each $k \in K$) is called an $L$-fuzzy $(K, E)$-soft topology on $X$ if it satisfies the following conditions for each $k \in K$. 
The pair $(X, \mathcal{T})$ is called an $L$-fuzzy $(K, E)$-soft topological space. Let $(X, \mathcal{T}^1)$ be an $L$-fuzzy $(K_1, E_1)$-soft topological space and $(Y, \mathcal{T}^2)$ be an $L$-fuzzy $(K_2, E_2)$-soft topological space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from $(X, \mathcal{T}^1)$ into $(Y, \mathcal{T}^2)$ is called $L$-fuzzy soft continuous if

$$T_{\eta(k)}^2(f) \leq T_{k}^1(\varphi_{\psi}^{-1}(f)) \quad \forall f \in (L^Y)^{E_2}, k \in K_1.$$ 

**Definition 7.** [5] A map $C : K \times (L^X)^E \times L_0 \to (L^X)^E$ is called an $L$-fuzzy $(K, E)$-soft closure operator if it satisfies the following conditions;

(C1) $C(k, 0_X, r) = 0_X$,

(C2) $C(k, f, r) \sqsubseteq f$,

(C3) If $f_1 \sqsubseteq f_2$, then $C(k, f_1, r) \sqsubseteq C(k, f_2, r))$,

(C4) If $r_1 \leq r_2$, then $C(k, f, r_1) \sqsubseteq C(k, f, r_2)$,

(C5) $C(k, f_1 \oplus f_2, r \circ s) \sqsubseteq C(k, f_1, r) \oplus C(k, f_2, s)$.

The pair $(X, C)$ is called an $L$-fuzzy $(K, E)$-soft closure space. An $L$-fuzzy $(K, E)$-soft closure operator is called topological if

(T) $C(k, C(k, f, r), r) \sqsubseteq C(k, f, r)$.

Let $C_1$ and $C_2$ be $L$-fuzzy $(K, E)$-soft closure operators on $X$. Then $C_1$ is finer than $C_2$ if $C_1(k, f, r) \sqsubseteq C_2(k, f, r)$, for all $f \in (L^X)^E, r \in L_0$.

Let $(X, C_X)$ be $L$-fuzzy $(K_1, E_1)$-soft closure spaces and $(Y, C_Y)$ be $L$-fuzzy $(K_2, E_2)$-soft closure spaces. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be maps. Then $\varphi_{\psi, \eta}$ is called an $L$-fuzzy soft closed map if, for each $k \in K_1, f \in (L^X)^{E_1}, r \in L_0$,

$$\varphi_{\psi, \eta}(C_X(k, f, r)) \sqsubseteq C_Y(\eta(k), \varphi_\psi(f), r).$$
3. \(L\)-Fuzzy \((K, E)\)-Soft Topologies and \(L\)-fuzzy \((K, E)\)-Soft Closure Operators

**Lemma 8.** Let \(\varphi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}\) be a soft mapping. Then we have the following properties. For \(f, f_i \in (L^X)^{E_1}\) and \(g, g_i \in (L^Y)^{E_2}\),

1. \(g \supseteq \varphi \psi^{-1}(g)\) with equality if \(\varphi \psi\) is surjective,
2. \(f \supseteq \varphi \psi^{-1}(f)\) with equality if \(\varphi \psi\) is injective,
3. if \(\varphi \psi\) is injective,

\[
\varphi(f)_{e_2}(y) = \begin{cases} f_{e_1}(x), & \text{if } x \in \varphi^{-1}(y), e_1 \in \psi^{-1}(e_2) \\ 0, & \text{otherwise} \end{cases}
\]

4. \(\varphi \psi^{-1}(g^*) = (\varphi \psi^{-1}(g))^*\),
5. \(\varphi \psi^{-1}(\bigvee_{i \in I} g_i) = \bigvee_{i \in I} \varphi \psi^{-1}(g_i)\),
6. \(\varphi \psi^{-1}(\bigwedge_{i \in I} g_i) = \bigwedge_{i \in I} \varphi \psi^{-1}(g_i)\),
7. \(\varphi \psi(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} \varphi \psi(f_i)\),
8. \(\varphi \psi(\bigwedge_{i \in I} f_i) \subseteq \bigwedge_{i \in I} \varphi \psi(f_i)\) with equality if \(\varphi \psi\) is injective,
9. \(\varphi \psi^{-1}(g_1 \odot g_2) = \varphi \psi^{-1}(g_1) \odot \varphi \psi^{-1}(g_2)\),
10. \(\varphi \psi^{-1}(g_1 \oplus g_2) = \varphi \psi^{-1}(g_1) \oplus \varphi \psi^{-1}(g_2)\),
11. \(\varphi \psi(f_1 \odot f_2) \supseteq \varphi \psi(f_1) \odot \varphi \psi(f_2)\) with equality if \(\varphi \psi\) is injective,
12. \(\varphi \psi(f_1 \oplus f_2) \supseteq \varphi \psi(f_1) \oplus \varphi \psi(f_2)\) with equality if \(\varphi \psi\) is injective.

**Proof.** (1) If \(\psi^{-1}\{e_2\} \neq \emptyset\) and \(\varphi^{-1}\{y\} \neq \emptyset\), then

\[
\varphi \psi(\varphi \psi^{-1}(g))(e_2)(y) = \bigvee_{\varphi(x)=y} \bigwedge_{\psi(e_1)=e_2} \varphi(\varphi^{-1}(g))(e_1)(x) = g(\psi(e_1))(\varphi(x)) = g(e_2)(y).
\]

If \(\psi^{-1}\{e_2\} = \emptyset\) or \(\varphi^{-1}\{y\} = \emptyset\), then \(\varphi \psi(\varphi \psi^{-1}(f))(e_2)(y) = 0\). Hence the result holds.

(2)

\[
\varphi \psi^{-1}(\varphi \psi(f))(e_1)(x) = \varphi \psi(f)(\psi(e_1))(\varphi(x)) = \bigvee_{\varphi(z)=\varphi(x)} \bigwedge_{\psi(e)=\psi(e_1)} \varphi \psi^{-1}(g)(e)(z) \geq \varphi(e_1)(x).
\]

If \(\varphi \psi\) is injective, the equality holds.
(3) If $\psi^{-1}\{e_2\} \neq \emptyset$ and $\varphi^{-1}\{y\} \neq \emptyset$, there exist unique $e_1 \in \psi^{-1}\{e_2\}$ and $x \in \varphi^{-1}\{y\}$. Hence the result holds.

(4) 
\[
\varphi^{-1}_{\psi}(g^*)(e_1)(x) = g^*(\psi(a))(\varphi(x)) = (g(\psi(e_1))(\varphi(x))^* = (\varphi^{-1}_{\psi}(g)(e_1)(x))^*.
\]

(11) 
\[
\varphi_{\psi}(f_1)(e_2)(y) \odot \varphi_{\psi}(f_2)(e_2)(y) \\
= V_{\varphi(x)=y} V_{\psi(e_1)=e_2} f_1(e_1)(x) \odot V_{\varphi(z)=y} V_{\psi(e_3)=e_2} f_2(e_3)(z) \\
\geq V_{\varphi(x)=y} V_{\psi(e_1)=e_2} (f_1(e_1)(x) \odot f_2(e_1)(x)) \\
= \varphi_{\psi}((f_1 \odot f_2))(e_2)(y).
\]

If $\varphi_{\psi}$ is injective, by (3), If $\psi^{-1}\{e_2\} \neq \emptyset$ and $\varphi^{-1}\{y\} \neq \emptyset$, there exist unique $e_1 \in \psi^{-1}\{e_2\}$ and $x \in \varphi^{-1}\{y\}$.

\[
\varphi_{\psi}(f_1)(e_2)(y) \odot \varphi_{\psi}(f_2)(e_2)(y) = f_1(e_1)(x) \odot f_2(e_1)(x) = \varphi_{\psi}((f_1 \odot f_2))(e_2)(y).
\]

If $\psi^{-1}\{e_2\} = \emptyset$ and $\varphi^{-1}\{y\} = \emptyset$,

\[
\varphi_{\psi}(f_1)(e_2)(y) \odot \varphi_{\psi}(f_2)(e_2)(y) = 0 = \varphi_{\psi}((f_1 \odot f_2))(e_2)(y).
\]

Other cases are similarly proved.

**Example 9.** Let $X = \{h_i \mid i = \{1, \ldots, 5\}\}$ with $h_i$=house and $E_X = \{e, b, w, c\}$ with $e$=expensive, $b$= beautiful, $w$=wooden, $c$= creative.

Define a binary operation $\odot$ on $[0,1]$ by

\[
x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}
\]

\[
x \odot y = \min\{1, x + y\}, \quad x^* = 1 - x.
\]

Then $([0,1], \odot, \rightarrow, 0, 1)$ is a stsc-quantale (ref.[8,9,18]). Let $E_1 = \{e, b, w\} \subseteq E_X$, $f_1, f_2 \in (E_1)^X$ as follows:

\[
(f_1)_e = (0.8, 0.1, 0.5, 0.9, 0.6), \quad (f_1)_b = (0.7, 0.9, 0.4, 0.5, 0.7) \quad (f_1)_w = (0.4, 0.7, 0.5, 0.6, 0.5)
\]

\[
(f_2)_e = (0.5, 0.9, 0.4, 0.8, 0.4), \quad (f_2)_b = (0.3, 1.0, 0.2, 0.4, 0.5) \quad (f_2)_w = (0.5, 0.4, 0.8, 0.5, 0.1)
\]
\[(f_1 \circ f_2)_e = (0.3, 0, 0, 0.7, 0), (f_1 \circ f_2)_b = (0, 0.9, 0, 0, 0.2)\]
\[(f_1 \circ f_2)_w = (0, 0.1, 0.3, 0.1, 0)\]
\[(f_1 \oplus f_2)_e = (1, 1, 0.9, 1, 1), (f_1 \oplus f_2)_b = (1, 1, 0.6, 0.9, 1)\]
\[(f_1 \oplus f_2)_w = (0.9, 1, 1, 1, 0.6)\]

(1) Let \(Y = \{y_1, y_2, y_3\}\) and \(E_2 = \{b_1, b_2\}\). Define \(\varphi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}\) as follows:

\[\varphi(h_1) = \varphi(h_2) = y_1, \varphi(h_3) = \varphi(h_4) = y_2, \varphi(h_5) = y_3\]
\[\psi(e) = \psi(b) = b_1, \psi(w) = b_2.\]

Since

\[\varphi(\psi(f_1)(b_1)(y_1) = \bigvee_{\varphi(x) = y_1} \bigvee_{\psi(a) = b_1} f_1(a)(x) = f(e)(h_1) \lor f(b)(h_1) \lor f(e)(h_2) \lor f(b)(h_2) = 0.8 \lor 0.7 \lor 0.1 \lor 0.9.\]

\[\varphi(\psi(f_1)(b_1) = (0.9, 0.9, 0.7), \varphi(\psi(f_1)(b_2) = (0.7, 0.6, 0.5)\]
\[\varphi(\psi(f_2)(b_1) = (1.0, 0.8, 0.5), \varphi(\psi(f_2)(b_2) = (0.5, 0.8, 0.1)\]

\[\varphi(\psi(f_1 \circ f_2)(b_1) = (0.9, 0.7, 0.2), \varphi(\psi(f_1 \circ f_2)(b_2) = (0.1, 0.3, 0)\]

\[\varphi(\psi(f_1 \oplus f_2)(b_1) = (1, 1, 1), \varphi(\psi(f_1 \oplus f_2)(b_2) = (1, 1, 0.6)\]

\[\varphi(\psi(f_1 \circ f_2)(b_1) = (0.9, 0.7, 0.2)\]
\[\varphi(\psi(f_1 \circ f_2)(b_2) = (0.2, 0.4, 0)\]

\[\varphi^{-1}(\psi(f_1)) = (0.9, 0.9, 0.9, 0.9, 0.7)\]
\[\varphi^{-1}(\psi(f_1)) = (0.9, 0.9, 0.9, 0.9, 0.7)\]
\[\varphi^{-1}(\psi(f_1)) = (0.7, 0.7, 0.6, 0.6, 0.5)\]

Since \(f\) is not injective,

\[\varphi(\psi(f_1) \circ \varphi(\psi(f_2) \neq \varphi(\psi(f_1) \circ f_2), \varphi^{-1}(\psi(f_1)) \neq f_1.\]

But \(\varphi(\psi(f_1) \oplus \varphi(\psi(f_2) = \varphi(\psi(f_1) \oplus f_2)\) and \(\varphi\) is not injective.

Let \(g \in ([0, 1]^Y)^{E_2}\) as follows:

\[g_{b_1} = (0.5, 0.9, 0.6), g_{b_1} = (0.7, 0.4, 0.3)\]
\[
\begin{align*}
\varphi_\psi^{-1}(g)_e &= (0.5, 0.5, 0.9, 0.9, 0.6) \\
\varphi_\psi^{-1}(g)_b &= (0.5, 0.5, 0.9, 0.9, 0.6) \\
\varphi_\psi^{-1}(g)_w &= (0.7, 0.7, 0.1, 0.1, 0.1)
\end{align*}
\]

Since \( \varphi_\psi \) is surjective, we have \( \varphi_\psi(\varphi_\psi^{-1}(g)) = g \).

(2) Let \( Z = \{ z_1, z_2, \ldots, z_6 \} \) and \( E_3 = \{ c_1, c_2, c_3, c_4 \} \). Define \( \pi_\psi : ([0,1]^X)^{E_1} \rightarrow ([0,1]^Z)^{E_3} \) as follows:

\[
\pi(h_i) = z_i, \, \psi(e) = c_1, \, \psi(b) = c_2, \, \psi(w) = c_3.
\]

\[
\begin{align*}
\pi_\psi(f_1)(c_1) &= (0.8, 0.1, 0.5, 0.9, 0.6, 0) \\
\pi_\psi(f_1)(c_2) &= (0.7, 0.9, 0.4, 0.5, 0.7, 0) \\
\pi_\psi(f_1)(c_3) &= (0.4, 0.7, 0.5, 0.6, 0.5, 0) \\
\pi_\psi(f_1)(c_4) &= (0, 0, 0, 0, 0, 0) \\
\pi_\psi(f_2)(c_1) &= (0.5, 0.9, 0.4, 0.8, 0.4, 0) \\
\pi_\psi(f_2)(c_2) &= (0.3, 1.0, 0.2, 0.4, 0.5, 0) \\
\pi_\psi(f_2)(c_3) &= (0.5, 0.4, 0.8, 0.5, 0.1, 0) \\
\pi_\psi(f_2)(c_4) &= (0, 0, 0, 0, 0, 0)
\end{align*}
\]

Since \( \pi \) is injective,

\[
\pi_\psi f_1 \circ \pi_\psi f_2 = \pi_\psi (f_1 \circ f_2)
\]

\[
\pi_\psi f_1 \oplus \pi_\psi f_2 = \pi_\psi (f_1 \oplus f_2), \quad \pi_\psi^{-1}(\pi_\phi(f_1)) = f_1.
\]

Let \( p \in ([0,1]^Z)^{E_3} \) as follows:

\[
\begin{align*}
p_{c_1} &= (0.5, 0.3, 0.9, 0.1, 0.6, 0.2) \\
p_{c_2} &= (0.3, 0.7, 0.5, 0.4, 0.3, 0.1) \\
p_{c_3} &= (0.3, 0.1, 0.2, 0.7, 0.6, 0.9) \\
p_{c_4} &= (0.8, 0.2, 0.4, 0.5, 0.6, 0.1, 0.3)
\end{align*}
\]

\[
\begin{align*}
\pi_\psi^{-1}(p)_e &= (0.5, 0.3, 0.9, 0.1, 0.6) \\
\pi_\psi^{-1}(p)_b &= (0.2, 0.7, 0.5, 0.4, 0.3) \\
\pi_\psi^{-1}(p)_w &= (0.3, 0.4, 0.2, 0.7, 0.6)
\end{align*}
\]
\[
\pi_\psi(\pi_\psi^{-1}(p))(c_1) = (0.5, 0.3, 0.9, 0.1, 0.6, 0)
\]
\[
\pi_\psi(\pi_\psi^{-1}(p))(c_2) = (0, 0.2, 0.7, 0.5, 0.4, 0.3)
\]
\[
\pi_\psi(\pi_\psi^{-1}(p))(c_3) = (0.3, 0.4, 0.2, 0.7, 0.6, 0)
\]
\[
\pi_\psi(\pi_\psi^{-1}(p))(c_4) = (0, 0, 0, 0, 0, 0)
\]

Since \( \varphi_\psi \) is injective, we have \( \varphi_\psi(\varphi_\psi^{-1}(p)) \subseteq p \). \(\square\)

**Theorem 10.**  (1) Let \((X, T)\) be an \(L\)-fuzzy \((K, E)\)-soft topological space. Define \(\mathcal{C}_T : K \times (L^X)^E \times L_0 \to (L^X)^E\) as
\[
\mathcal{C}_T(k, f, r) = \bigwedge \{ g \in (L^X)^E \mid g \supseteq f, \mathcal{T}_k(g^*) \geq r \}.
\]
Then (1) \(\mathcal{C}_T\) is a topological \(L\)-fuzzy \((K, E)\)-soft closure operator.

(2) Let \((X, \mathcal{C})\) be an \(L\)-fuzzy \((K, E)\)-soft closure space. Define \(\mathcal{T}_C : (L^X)^E \to L\) as
\[
(\mathcal{T}_C)(k)(f) = \bigvee \{ r \in L \mid \mathcal{C}(k, f^*, r) \subseteq f^* \}.
\]
Then \(\mathcal{T}_C\) is an \(L\)-fuzzy \((K, E)\)-soft topology on \(X\).

(3) Let \((X, T)\) be an \(L\)-fuzzy \((K, E)\)-soft topological space. Then \(T = \mathcal{T}_C\).

**Proof.** (1) \((C1), (C2), (C3)\) and \((C4)\) are easily proved.

\((C5)\)
\[
\mathcal{C}_T(k, f, r) \oplus \mathcal{C}_T(k, g, s) = \bigwedge \{ f_1 \in (L^X)^E \mid f_1 \supseteq f, \mathcal{T}_k(f_1^*) \geq r \}
\]
\[
\oplus \bigwedge \{ g_1 \in (L^X)^E \mid g_1 \supseteq g, \mathcal{T}_k(g_1^*) \geq s \}
\]
\[
\supseteq \bigwedge \{ f_1 \oplus g_1 \in (L^X)^E \mid f_1 \oplus g_1 \supseteq f \oplus g, \mathcal{T}_k(f_1^* \oplus g_1^*) \geq r \oplus s \}
\]
\[
\supseteq \mathcal{C}_T(k, f \oplus g, r \oplus s)
\]

(T) Suppose there exist \(k \in K, f \in (L^X)^E\) and \(r \in L_0\) such that
\[
\mathcal{C}_T(k, \mathcal{C}_T(k, f, r), r) \not\subseteq \mathcal{C}_T(k, f, r).
\]
By the definition of \(\mathcal{C}_T(k, f, r)\), there exists \(g \in (L^X)^E\) with \(f \supseteq g\) and \(\mathcal{T}_k(g^*) \geq r\) such that
\[
\mathcal{C}_T(k, \mathcal{C}_T(k, f, r), r) \not\subseteq g.
\]
On the other hand, since \(f \supseteq g\) and \(\mathcal{T}(g^*) \geq r\), \(\mathcal{C}_T(k, f, r) \subseteq \mathcal{C}_T(k, g, r) = g\). Thus,
\[
\mathcal{C}_T(k, \mathcal{C}_T(k, f, r), r) \subseteq g.
\]
It is a contradiction. Hence \(\mathcal{C}_T(k, \mathcal{C}_T(k, f, r), r) \subseteq \mathcal{C}_T(k, f, r)\).
(2) \[
(\mathcal{T}_C)_k(f) \odot (\mathcal{T}_C)_k(g) \\
= \bigvee \{r \in L \mid C(k, f^*, r) \sqsubseteq f^*\} \odot \bigvee \{s \in L \mid C(k, g^*, s) \sqsubseteq g^*\} \\
\leq \bigvee \{r \odot s \mid C(k, f^*, r) \sqcup C(k, g^*, s) \sqsubseteq f^* \oplus g^*\} \\
\leq \bigvee \{r \odot s \mid C(k, f^* \oplus g^*, r \odot s) \sqsubseteq f^* \oplus g^*\} \\
\leq (\mathcal{T}_C)_k(f \odot g).
\]

Let \( \bigwedge_{i \in I} (\mathcal{T}_C)_k(f_i) \geq r \); i.e. \((\mathcal{T}_C)_k(f_i) \geq r\) for all \(i \in I\). For each \(s < r\), by the definition of \((\mathcal{T}_C)_k\), \(C(k, f_i^*, s) \sqsubseteq f_i^*\) for all \(i \in I\). Hence

\[
C(k, \bigwedge_{i \in I} f_i^*, s) \sqsubseteq \bigwedge_{i \in I} C(k, f_i^*, s) \sqsubseteq \bigwedge_{i \in I} f_i^*.
\]

Thus, \((\mathcal{T}_C)_k(\bigvee_{i \in I} f_i) \geq s\). It implies

\[
(\mathcal{T}_C)_k(\bigvee_{i \in I} f_i) \geq \bigwedge_{i \in I} (\mathcal{T}_C)_k(f_i).
\]

Thus \(\mathcal{T}_C\) is an \(L\)-fuzzy \((K, E)\)-soft topology on \(X\). We only show that \(\mathcal{T}_{C_T} = \mathcal{T}\).

Suppose that there exists \(f \in (L^X)^E\) such that

\[
(\mathcal{T}_{C_T})_k(f) \not\leq \mathcal{T}_k(f).
\]

Then there exists \(r \in L_0\) with \(C_T(k, f^*, r) = f^*\) such that

\[
r \not\leq \mathcal{T}_k(f).
\]

On the other hand, since \(C_T(k, f^*, r) = f^*\), we have \(\mathcal{T}_k(f) \geq r\) from the definition of \(C_T\). It is a contradiction. Hence \((\mathcal{T}_{C_T})_k \leq \mathcal{T}_k\).

Suppose that there exists \(g \in (L^X)^E\) such that

\[
(\mathcal{T}_{C_T})_k(g) \not\geq \mathcal{T}_k(g).
\]

Then there exists \(r \in L_0\) such that

\[
(\mathcal{T}_{C_T})_k(g) \not\geq \mathcal{T}_k(g) = r.
\]

On the other hand, since \(\mathcal{T}_k(g) \geq r\), we have

\[
C_T(k, g^*, r) = \bigwedge \{f \mid f \sqsupseteq g^* \mathcal{T}_k(f^*) \geq r\} = g^*.
\]

Thus, \((\mathcal{T}_{C_T})_k(g) \geq r\). It is a contradiction. Hence \((\mathcal{T}_{C_T})_k \geq \mathcal{T}_k\). \(\square\)
Theorem 11. Let \((X, \mathcal{T}_X)\) and \((Y, \mathcal{T}_Y)\) be \(L\)-fuzzy \((K_1, E_1)\)-soft and \(L\)-fuzzy \((K_2, E_2)\)-soft topological spaces, respectively. Let \(\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}\) be a soft map. If \(\varphi_\psi, \eta : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)\) is an \(L\)-fuzzy soft continuous map, then \(\varphi_\psi, \eta : (X, \mathcal{C}_\mathcal{T}_X) \rightarrow (Y, \mathcal{C}_\mathcal{T}_Y)\) is an \(L\)-fuzzy soft closure map.

Proof. Let \((T_Y)_\eta(g))(k) \leq (T_X)_k(\varphi_\psi^{-1}(g))\) for all \(g \in (L^Y)^{E_2}\). Then

\[
\begin{align*}
\mathcal{C}_{T_Y}(\eta(k), \varphi_\psi(f), r)) & \subset \{g_1 \in (L^X)^{E_2} \mid \varphi_\psi(f) \subset g_1, (T_Y)_\eta(g_1) \geq r\} \\
& \ni \{\varphi_\psi(\varphi_\psi^{-1}(g_1)) \in (L^Y)^{E_2} \mid \varphi_\psi^{-1}(\varphi_\psi(f)) \subset \varphi_\psi^{-1}(g_1)\}, \\
(T_X)_k(\varphi_\psi^{-1}(g_1)) \geq r & \text{ (by Lemma 8 (1))} \\
& \ni \{\varphi_\psi(\varphi_\psi^{-1}(g_1)) \in (L^X)^{E} \mid f \subset \varphi_\psi^{-1}(g_1)\}, \\
(T_X)_k(\varphi_\psi^{-1}(g_1)) \geq r & \text{ (by Lemma 8 (8))} \\
& \ni \varphi_\psi(\mathcal{C}_{T_X}(k, f, r)).
\end{align*}
\]

\(\square\)

Theorem 12. Let \((X, \mathcal{C}_X)\) and \((Y, \mathcal{C}_Y)\) be \(L\)-fuzzy \((K_1, E_1)\)-soft and \(L\)-fuzzy \((K_2, E_2)\)-soft closure spaces, respectively. Let \(\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}\) be a soft map. Then \(\varphi_\psi, \eta : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)\) is an \(L\)-fuzzy soft closure map iff \(\mathcal{C}_X(k, \varphi_\psi^{-1}(g), r) \ni \varphi_\psi^{-1}(\mathcal{C}_Y(\eta(k), g, r))\).

Proof. Let \(\varphi_\psi, \eta(\mathcal{C}_X(k, f, r)) \ni \mathcal{C}_Y(\eta(k), \varphi_\psi(f), r)\). Put \(f = \varphi_\psi^{-1}(g)\). Then

\[
\begin{align*}
\varphi_\psi, \eta(\mathcal{C}_X(k, \varphi_\psi^{-1}(g), r)) & \ni \mathcal{C}_Y(\eta(k), \varphi_\psi^{-1}(g), r) \\
& \ni \mathcal{C}_Y(\eta(k), g, r)
\end{align*}
\]

Hence \(\mathcal{C}_X(k, \varphi_\psi^{-1}(g), r) \ni \varphi_\psi^{-1}(\mathcal{C}_Y(\eta(k), g, r))\).

Let \(\mathcal{C}_X(k, \varphi_\psi^{-1}(g), r) \ni \varphi_\psi^{-1}(\mathcal{C}_Y(\eta(k), g, r))\). Put \(g = \varphi_\psi(f)\).

\[
\begin{align*}
\mathcal{C}_X(k, f, r) & \ni \mathcal{C}_X(k, \varphi_\psi^{-1}(\varphi_\psi(f)), r) \\
& \ni \varphi_\psi^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f), r))
\end{align*}
\]

Hence \(\varphi_\psi, \eta(\mathcal{C}_X(k, f, r)) \ni \mathcal{C}_Y(\eta(k), \varphi_\psi(f), r)\). \(\square\)

Theorem 13. Let \((X, \mathcal{C}_X)\) and \((Y, \mathcal{C}_Y)\) be \(L\)-fuzzy \((K_1, E_1)\)-soft and \(L\)-fuzzy \((K_2, E_2)\)-soft closure spaces, respectively. Let \(\varphi_\psi : (L^X)^{E} \rightarrow (L^Y)^{E_2}\) be a soft map. Then the following properties;
(1) If $\varphi_\psi, \eta : (X, C_X) \to (Y, C_Y)$ is an $L$-fuzzy soft closure map, then $\varphi_\psi, \eta : (X, T_{C_X}) \to (Y, T_{C_Y})$ is an $L$-fuzzy soft continuous map.

(2) $\varphi_\psi, \eta : (X, A, T_X) \to (Y, B, T_Y)$ is an $L$-fuzzy soft continuous map iff $\varphi_\psi, \eta : (X, A, C_{T_X}) \to (Y, B, C_{T_Y})$ is an $L$-fuzzy soft closure map.

**Proof.** (1) Let $\varphi_\psi, \eta$ be a closed soft map. By Theorem 11,

$$\varphi_\psi^{-1}(C_Y(\eta(k), g, r)) \supseteq C_X(k, \varphi_\psi^{-1}(g), r)$$

for all $g \in (L^Y)^E$. Then

$$(T_{C_Y})_k(g) = \bigvee \{r \in L \mid g^* \supseteq C_Y(\eta(k), g^*, r)\}$$

$$\leq \bigvee \{r \in L \mid \varphi_\psi^{-1}(g^*) \supseteq \varphi_\psi^{-1}(C_Y(\eta(k), g^*, r))\}$$

$$\leq \bigvee \{r \in L \mid \varphi_\psi^{-1}(g^*) \supseteq C_X(k, \varphi_\psi^{-1}(g^*), r)\}$$

$$\leq (T_{C_X})_k(\varphi_\psi^{-1}(g)).$$

(2) Let $\varphi_\psi, \eta : (X, C_{T_X}) \to (Y, C_{T_Y})$ be a closure soft map. Since $T_{C_T} = T$ and $T_{C_T} = T$ from Theorem 10(3), by (1), $\varphi_\psi : (X, T_X) \to (Y, T_Y)$ is an $L$-fuzzy soft continuous map.

**Example 14.** Let $X$ and $E_X$ be given as Example 9. Define a binary operation $\land$ on $[0, 1]$ by

$$x \land y = \min\{x, y\}, \quad x^* = 1 - x, \quad x \lor y = (x^* \land y^*)^*$$

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise}. \end{cases}$$

Then $([0, 1], \land, \to, 0, 1)$ is a stsc-quantale (ref. [8,9,18]). Let $f_1$ and $f_2$ as Example 9. Then we obtain:

$$(f_1 \lor f_2)_e = (0.8, 0.9, 0.5, 0.9, 0.6)$$

$$(f_1 \lor f_2)_b = (0.7, 1.0, 0.4, 0.5, 0.7)$$

$$(f_1 \lor f_2)_w = (0.5, 0.7, 0.8, 0.6, 0.5)$$

$$(f_1 \land f_2)_e = (0.5, 0.1, 0.4, 0.8, 0.4)$$

$$(f_1 \land f_2)_b = (0.3, 0.9, 0.2, 0.4, 0.5)$$

$$(f_1 \land f_2)_w = (0.4, 0.4, 0.5, 0.5, 0.1)$$
For $K = \{k_1, k_2\}$, we define a $[0,1]$-fuzzy $(K, E)$-soft topology $\mathcal{T}_X : K \to [0,1]^{([0,1]^X)^E_1}$ as follows:

$$(\mathcal{T}_X)_{k_1}(f) = \begin{cases} 
1, & \text{if } f = 0_X \text{ or } 1_X, \\
0.7, & \text{if } f = f_1 \\
0.4, & \text{if } f = f_2 \\
0.5, & \text{if } f = f_1 \lor f_2 \\
0.6, & \text{if } f = f_1 \land f_2 \\
0, & \text{otherwise}.
\end{cases}$$

$$(\mathcal{T}_X)_{k_2}(f) = \begin{cases} 
1, & \text{if } f = 0_X \text{ or } 1_X, \\
0.5, & \text{if } f = f_1 \\
0, & \text{otherwise}.
\end{cases}$$

From Theorem 10(1), we obtain a $[0,1]$-fuzzy $(K, E)$-soft closure operator $\mathcal{C}_{\mathcal{T}_X} : K \times ([0,1]^X)^E_1 \times (0,1) \to ([0,1]^X)^E_1$ as follows:

$$
\mathcal{C}_{\mathcal{T}_X}(k_1, f, r) = \begin{cases} 
1_X, & \text{if } f = 1_X, r \in (0,1], \\
f_1^*, & \text{if } f \subseteq f_1^*, f \not\sqsubseteq f_1^* \land f_2^*, r \leq 0.7, \\
f_2^*, & \text{if } f \subseteq f_2^*, f \not\sqsubseteq f_1^* \land f_2^*, r \leq 0.4, \\
f_1^* \land f_2^* & \text{if } f \subseteq f_1^* \land f_2^*, r \leq 0.5 \\
f_1^* \lor f_2^* & \text{if } f \subseteq f_1^* \lor f_2^*, \\
f_1^*, f \not\sqsubseteq f_2^*, r \leq 0.6, \\
0_X, & \text{otherwise}.
\end{cases}
$$

$$
\mathcal{C}_{\mathcal{T}_X}(k_2, f, r) = \begin{cases} 
1_X, & \text{if } f = 1_X, r \in (0,1], \\
f_1^*, & \text{if } f \subseteq f_1^*, r \leq 0.5, \\
0_X, & \text{otherwise}.
\end{cases}
$$

\[\square\]

**Example 15.** Let $X$ and $E_X$ be given as Example 14. Let $([0,1], \odot, \oplus, \to, ^*, 0, 1)$ be a complete residuated lattice as Example 9. Let $E = \{b, w, c\} \subset E_X$, $f \in ([0,1]^X)^E$ be a fuzzy soft set as follows:

$$
f_b = (0.5, 0.3, 0.5, 0.6, 0.2) \\
f_c = (0.1, 0.2, 0.6, 0.5, 0.5) \\
f_w = (0.4, 0.4, 0.5, 0.6, 0.6)
$$

$$
(f \odot f)_b = (0.0, 0.0, 0.0, 0.2, 0.0) \\
(f \odot f)_c = (0.0, 0.0, 0.2, 0.0, 0.0) \\
(f \odot f)_w = (0.0, 0.0, 0.0, 0.2, 0.2)
$$
Let $K = \{k_1, k_2\}$ be given. Define a $[0, 1]$-fuzzy $(K, E)$-soft topology $T_X : K \to [0, 1]^{([0, 1]^X)^E}$ as follows:

\[
(T_X)_{k_1}(h) = \begin{cases} 
1, & \text{if } h = 0_X \text{ or } h = 1_X, \\
0.6, & \text{if } h = f \\
0.3, & \text{if } h = f \odot f \\
0, & \text{otherwise.}
\end{cases}
\]

\[
(T_X)_{k_2}(h) = \begin{cases} 
1, & \text{if } h = 0_X \text{ or } h = 1_X, \\
0.4, & \text{if } h = f \\
0, & \text{otherwise.}
\end{cases}
\]

We obtain a $[0, 1]$-fuzzy $(K, E)$-soft closure operator $C_{T_X} : K \times ([0, 1]^X)^E \times (0, 1) \to ([0, 1]^X)^E$ as follows:

\[
C_{T_X}(k_1, h, r) = \begin{cases} 
1_X, & \text{if } h = 1_X, r \in (0, 1], \\
f^*, & \text{if } h \subseteq f^*, r \leq 0.6 \\
f^* \oplus f^* & \text{if } h \subseteq f^* \oplus f^* \\
0_X, & \text{otherwise.}
\end{cases}
\]

\[
C_{T_X}(k_2, h, r) = \begin{cases} 
1_X, & \text{if } h = 1_X, r \in (0, 1], \\
f^*, & \text{if } h \subseteq f^*, r \leq 0.4 \\
0_X, & \text{otherwise.}
\end{cases}
\]

References


