International Journal of Pure and Applied Mathematics
Volume 107 No. 4 2016, 903-908
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v107i4.8
ijpam.eu

ON DETERMINANTS OF TRIDIAGONAL MATRICES WITH ALTERNATING PAIRS OF $1's$ AND $-1's$ ON THE DIAGONAL CONNECTED WITH FIBONACCI NUMBERS

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Abstract: We will concentrate on some special tridiagonal matrices connected with Fibonacci numbers. In the previous paper we generalized one of the results in Strang's book, as we derived that determinants of some tridiagonal matrices with alternating $1's$ and $-1's$ on the diagonal or the superdiagonal are connected with Fibonacci numbers. This paper is devoted to a generalization of that paper, we show determinants of tridiagonal matrices with alternating pairs of $1's$ and $-1's$ on the diagonal are related to Fibonacci numbers too.

AMS Subject Classification: 11B39, 11B36, 11A07
Key Words: tridiagonal matrix, recurrence, Fibonacci number, determinant

1. Introduction

The Fibonacci sequence (or sequence of Fibonacci numbers) $(F_n)_{n\geq 0}$ is the sequence of positive integers satisfying the recurrence $F_{n+2} = F_{n+1} + F_n$ with the initial conditions $F_0 = 0$ and $F_1 = 1$.

The Fibonacci numbers have many amazing properties (see e. g. [4]). Let α and β be the roots of the characteristic equation $x^2 - x - 1 = 0$ (thus $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$), then the Binet formula for the Fibonacci numbers has the form

Received: February 7, 2016
Published: May 5, 2016

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

The recurrence relation for the Fibonacci numbers can be used to extend the sequence backward

$$F_{-1} = F_1 - F_0 = 1, F_{-2} = F_0 - F_{-1} = -1, F_{-3} = F_{-1} - F_{-2} = 2, \dots,$$

thus

$$F_{-n} = (-1)^{n+1} F_n$$

for any positive integer n. A tridiagonal matrix is a square matrix $\mathbb{A} = (a_{jk})$ of the order n, where $a_{jk} = 0$ for $|k - j| > 1$ and $1 \le j, k \le n$, i. e.

$$\mathbb{A}(n) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \ddots & \vdots \\ 0 & a_{3,2} & a_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

Now we turn our attention to the relation of determinants of special tridiagonal matrices with Fibonacci numbers. Probably the first example was done by Strang in [8], where he showed, that the determinant of $n \times n$ matrix

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 1 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

is equal to F_n for $n \ge 1$. Cahil et. al. [1] found some types tridiagonal matrices whose determinants are equal to Fibonacci numbers. Kiliç and Tasci [3] showed that the determinant of the following tridiagonal matrix

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & -1 & -1 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -1 \end{pmatrix}$$

is equal to $(-1)^n F_{n+1}$ for $n \geq 1$. The author [10] proved that the determinant of the following tridiagonal matrix

$$\begin{pmatrix} (-1)^{1+\delta} & 1 & 0 & \cdots & 0 \\ 1 & (-1)^{2+\delta} & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & (-1)^{n-1+\delta} & 1 \\ 0 & \cdots & \cdots & 1 & (-1)^{n+\delta} \end{pmatrix} \tag{1}$$

is equal to $(-1)^{\frac{n}{2}(n+1-2\delta)} F_{n+1}$, where $\delta \in \{0, 1\}$, for $n \geq 1$.

Many authors derived the similar types of matrices which determinants or permanents are related to Fibonacci numbers or different kinds of their generalizations, e. g. k-generalized Fibonacci numbers, see [5], [7] [2], [6], [9] and [11].

Now we turn our attention to the relation of determinants of special tridiagonal matrices with Fibonacci numbers. We show that matrix in (1) can be changed into a sequence of matrices, whose determinants are equal to the Fibonacci numbers.

2. Main Results

We formulate the following theorem on determinants of sequences of tridiagonal matrices with alternating couples of $1's$ and $-1's$ on the diagonal.

Theorem 1. *Let* $\{\mathbb{B}^\delta(n) = (b^\delta_{jk})_{1 \leq j,k \leq n}, n = 1, 2, 3, \dots\}$, *where* $\delta \in \{0, 1\}$, *be a sequence of tridiagonal matrices in the form*

$$b_{jk}^{\delta} = \begin{cases} (-1)^{\binom{j-1}{2}+\delta}, & j = k; \\ 1, & j = k \pm 1; \\ 0, & otherwise, \end{cases}$$

i.e.,

$$\mathbb{B}^{\delta}(n) = \begin{pmatrix} (-1)^{\delta} & 1 & 0 & \cdots & 0 & 0 \\ 1 & (-1)^{\delta} & 1 & 0 & \vdots & \vdots \\ 0 & 1 & (-1)^{1+\delta} & 1 & \ddots & 0 \\ \vdots & 0 & 1 & (-1)^{3+\delta} & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & (-1)^{\binom{n-1}{2}+\delta} \end{pmatrix}.$$

Then

$$\det \mathbb{B}^{\delta}(n) = \begin{cases} F_{\frac{n-2}{2}}, & n \equiv 0 \pmod 2; \\ (-1)^{\frac{n-1+2\delta}{2}} F_{\frac{n+1}{2}}, & n \equiv 1 \pmod 2). \end{cases} \tag{2}$$

Proof. We use the mathematical induction on n. The assertion holds for $n = 1$ and $n = 2$ as

$$\begin{aligned} \det \mathbb{B}^{\delta}(1) &= (-1)^{\delta} = (-1)^{\frac{0+2\delta}{2}} F_{\frac{2}{2}}, \\ \det \mathbb{B}^{\delta}(2) &= 0 = F_{\frac{2-2}{2}}. \end{aligned}$$

For $n \geq 3$ using cofactor expansion on the last row and then on the last column of matrix $\mathbb{B}^{\delta}(n)$ we obtain

$$\begin{vmatrix} (-1)^{\delta} & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & (-1)^{\binom{n-3}{2}+\delta} & 1 & 0 \\ \vdots & \ddots & 1 & (-1)^{\binom{n-2}{2}+\delta} & 1 \\ 0 & \cdots & 0 & 1 & (-1)^{\binom{n-1}{2}+\delta} \end{vmatrix}$$

$$= (-1)^{\binom{n-1}{2}+\delta} \begin{vmatrix} (-1)^\delta & 1 & 0 & 0 & 0 \\ 1 & (-1)^\delta & 1 & \vdots & \vdots \\ 0 & 1 & (-1)^{1+\delta} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 1 & (-1)^{\binom{n-2}{2}+\delta} \end{vmatrix}$$

$$+ (-1)^{2n-1} \begin{vmatrix} (-1)^\delta & 1 & 0 & 0 & 0 \\ 1 & (-1)^\delta & 1 & \vdots & \vdots \\ 0 & 1 & (-1)^{1+\delta} & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ \vdots & \cdots & \cdots & 1 & (-1)^{\binom{n-3}{2}+\delta} \end{vmatrix}.$$

Thus we get the following recurrence relation

$$\det \mathbb{B}^\delta(n) = (-1)^{\binom{n-1}{2}+\delta} \det \mathbb{B}^\delta(n-1) - \det \mathbb{B}^\delta(n-2) \tag{3}$$

for $n \geq 3$.

Suppose that (2) holds for every k, $3 \leq k < n$ and we show, using recurrence (3), that (2) holds for n too. We consider the following two cases.

- Let n be odd.

Using identity (2) we have

$$\begin{aligned} \det \mathbb{B}^\delta(n) &= (-1)^{\frac{(n-1)(n-2)}{2}+\delta} \det \mathbb{B}^\delta(n-1) - \det \mathbb{B}^\delta(n-2) \\ &= (-1)^{\frac{(n-1)(n-2)}{2}+\delta} F_{\frac{(n-1)-2}{2}} - (-1)^{\frac{(n-2)-1+2\delta}{2}} F_{\frac{(n-2)+1}{2}} \\ &= (-1)^{\frac{n-1}{2}+\delta} \left((-1)^{\frac{(n-1)(n-3)}{2}} F_{\frac{n-3}{2}} - (-1)^{-1} F_{\frac{n-1}{2}} \right) \\ &= (-1)^{\frac{n-1}{2}+\delta} \left(F_{\frac{n-3}{2}} + F_{\frac{n-1}{2}} \right) = (-1)^{\frac{n-1}{2}+\delta} F_{\frac{n+1}{2}}, \end{aligned}$$

where we use clear fact, that $4 \mid (n-1)(n-3)$ for an odd n.

- Let n be even.

Using identity (2) we obtain

$$\det \mathbb{B}^\delta(n) = (-1)^{\frac{(n-1)(n-2)}{2}+\delta} \det \mathbb{B}^\delta(n-1) - \det \mathbb{B}^\delta(n-2)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}+\delta}(-1)^{\frac{(n-1)-1+2\delta}{2}} F_{\frac{(n-1)+1}{2}} - F_{\frac{(n-2)-2}{2}}$$
$$= (-1)^{\frac{n(n-2)}{2}} F_{\frac{n}{2}} - F_{\frac{n-4}{2}} = F_{\frac{n}{2}} - F_{\frac{n-4}{2}} = F_{\frac{n-2}{2}},$$

where we use obvious fact, that $4 \mid n(n-2)$ for an even n. □

Acknowledgements

The author thanks to Specific research PrFUHKCZ2016/No.2114 for financial support.

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