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ON THE BANACH ALGEBRAS AND STABILITY

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Abstract: In this paper we investigate the stability of derivations on special Banach algebras.

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1. Introduction

In the theory of functional equations, if the question:

Is it true that a given function which approximately satisfies a functional equation must be close to an exact solution of the equation?

accepts a solution, we say that the equation is stable. The notion of stability of mathematical theorems considered from a rather general point of view: When is it true that the solution of an equation differing slightly from a given one, should be necessarily close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near to the exact solutions of that equation?

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The stability problem of functional equations had been first raised by Ulam and Hyers ([2, 4]). Then Th. M. Rassias provided a generalization of the Hyers' theorem by proving the existence of unique linear mappings near approximate to additive mappings ([3]). The result of Rassias has provided a lot of influence in the development of generalization of the Hyers-Ulam stability concept which is now called the Hyers-Ulam-Rassias stability theory for functional equations. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta ([1]).

Theorem 1.1. *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that:

(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;

(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(c) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Recall that a $\mathbb{C}$-linear self mapping D on a Banach algebra B is called a *derivation* on B if D satisfies

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in B$. Also, a Banach algebra A, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on A is called a Lie Banach algebra. A linear self mapping D on a Lie Banach algebra A is called a Lie derivation if

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in A$.

2. Main Result

We will investigate the Hyers-Ulam stability of Lie derivations for a functional equations on Banach algebras.

Theorem 2.1. *Suppose that A is an Banach algebra and $\mathrm{f} : A^2 \to A^2$ is a mapping such that $\mathrm{f}(0,0) = (0,0)$. Also, let $\varphi : A^2 \to [0,\infty)$ be a function satisfying the conditions:*

(1) $\|2\mu\,\mathrm{f}\,(\frac{a+c}{2}, \frac{b+d}{2}) - \mathrm{f}\,(\mu a, \mu b) - \mathrm{f}\,(\mu c, \mu d)\| \leq \varphi(a+c, b+d)$

for all a, b in A.

(2) $\|\mathrm{f}\,([(a,b),(c,d)]) - [\mathrm{f}\,(a,b),(c,d)] - [(a,b),\mathrm{f}\,(c,d)]\| \leq \varphi\,(ac, bd)$

for all a, b, c, d in A. Suppose that there exists $0 < r < 1$ such that

(3) $\varphi\,(a,b) \leq \frac{r}{2}\,\varphi\,(2a, 2b)$

for all a, b in A. Then there exists a unique Lie derivative $D : A^2 \to A^2$ satisfying the following condition:

(4) $\|\mathrm{f}\,(a,b) - D\,(a,b)\| \leq \frac{1}{1-r}\,\varphi(a,b)$

for all $a, b \in A$.

Proof. First note that if A is a Lie Banach algebra, then $D : A^2 \to A^2$ satisfying

$$D([(a,b),(c,d)]) = [D(a,b),(c,d)] + [(a,b),D(c,d)]$$

for all $a, b, c, d \in A$, is a Lie derivation on A^2. As a standard method, consider X as the set of all functions $g : A^2 \to A^2$ such that $g(0,0) = 0$, and define a generalized metric d on X by

$$d(\mathrm{f}, g) = \inf\{c \in [0,\infty] : \|\mathrm{f}\,(a,b) - g\,(a,b)\| \leq c\;\varphi(a,b) \;\; \forall a, b \in A\}.$$

Clearly (X, d) is a complete generalized metric on X. Define $J : X \to X$ by

$$Jg(a,b) = 2\,g\,(\frac{a}{2}, \frac{b}{2})$$

for all $a, b \in A$. We have

$$\|J\,g(a,b) - J\,h(a,b)\| \;\; \leq \;\; 2\,d(g,h)\,\varphi\,(\frac{a}{2}, \frac{b}{2})$$

$$\leq \quad r\, d(g,h)\, \varphi(a,b)$$

for all $g, h \in X$ and all $a, b \in A$. Thus

$$d(J\,g, J\,h) \leq r\, d\,(g,h)$$

for all g, h in X and so J is a contraction with constant at most r. If $\mu = 1$ and $c = d = 0$ in (1), we get

$$\|2\,\mathrm{f}\,(\frac{a}{2}, \frac{b}{2}) - \mathrm{f}\,(a,b)\| \leq \varphi\,(a,b)$$

for all $a, b \in A$. Thus $d(J\mathrm{f}, \mathrm{f}) \leq 1$. By Theorem 1.1, there exists a unique fixed point function $D : A^2 \to A^2$ of J in the set

$$\Omega = \{g \in X : d(\mathrm{f}, g) < \infty\}.$$

Since $JD = D$, thus $D(\frac{a}{2}, \frac{b}{2}) = \frac{1}{2}D(a,b)$ for all $a, b \in A$. Note that

$$\|\mathrm{f}(a,b) - D(a,b)\| \leq s\,\varphi(a,b)$$

where $s = d(f, D) \in (0, \infty)$ and $d(J^n f, D) \to 0$ as $n \to \infty$.

Since

$$J^n\mathrm{f}\,(a,b) = 2^n\mathrm{f}\,(\frac{a}{2^n}, \frac{b}{2^n}),$$

we have

$$D(a,b) = \lim_{n\to\infty} 2^n\mathrm{f}\,(\frac{a}{2^n}, \frac{b}{2^n}).$$

Now we can see that $d\,(\mathrm{f}, D) \leq \frac{1}{1-r}\, d\,(\mathrm{f}, J\mathrm{f})$ and

$$\|\mathrm{f}\,(a,b) - D\,(a,b)\| \leq d\,(\mathrm{f}, D)\, \varphi\,(a,b),$$

so $d\,(\mathrm{f}, D) \leq \frac{1}{1-r}$. By using the relations (2) and (3) we obtain

$$\begin{aligned} \|D\,([(a,b),(c,d)]) \quad &- \quad [D\,(a,b),(c,d)] - [(a,b), D\,(c,d)]\| \\ &\leq \quad \lim_{n\to\infty} (r^n)^2\, \varphi(ac, bd) = 0 \end{aligned}$$

for all $a, b, c, d \in A$. This completes the proof. □

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