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SOME COMMON FIXED POINT RESULTS FOR COMPATIBLE MAPPINGS OF TYPES IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we proved the some common fixed point results for compatible mappings of types in multiplicative metric spaces.

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1. Introduction and Preliminaries

It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, in 2008, Bashirov et al. [1] introduced the concept of multiplicative metric spaces as follows:

Definition 1.1. Let X be a nonempty set. A multiplicative metric is a

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mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping d together with X, that is, (X, d) is a multiplicative metric space.

Example 1.2. ([2]) Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d^* : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $|\cdot|^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}^n_+, d^*)$ is a multiplicative metric space.

One can refer to [2] for detailed a multiplicative metric topology.

Definition 1.3. Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

In 2012, Özavsar and Çevikel [2] gave the concept of multiplicative contractive mappings and proved some fixed point theorem of such mappings in a multiplicative metric space.

Definition 1.4. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative contraction* if there exists a real number $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^{\lambda}(x, y) \quad \text{for all } x, y \in X.$$

In 2015, Kang et al. [3] introduced the notion of compatible mappings and its variants as follows:

Definition 1.5. Let f and g be mappings of a multiplicative metric space (X, d) Then f and g are called

(1) *compatible* if

$$\lim_{n \to \infty} d(fgx_n, gfx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$,

(2) *compatible of type* (A) if

$$\lim_{n \to \infty} d(fgx_n, g^2x_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} d(gfx_n, f^2x_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$,

(3) *compatible of type* (B) if

$$\lim_{n \to \infty} d(fgx_n, g^2x_n) \leq \Big[\lim_{n \to \infty} d(fgx_n, ft) \cdot \lim_{n \to \infty} d(ft, f^2x_n) \Big]^{1/2}$$

and

$$\lim_{n \to \infty} d(gfx_n, f^2x_n) \leq \Big[\lim_{n \to \infty} d(gfx_n, gt) \cdot \lim_{n \to \infty} d(gt, g^2x_n) \Big]^{1/2},$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$,

(4) *compatible of type* (C) if

$$\lim_{n \to \infty} d(fgx_n, g^2x_n) \leq \Big[\lim_{n \to \infty} d(fgx_n, ft) \cdot \lim_{n \to \infty} d(ft, f^2x_n) \cdot \lim_{n \to \infty} d(ft, g^2x_n) \Big]^{1/3}$$

and

$$\lim_{n \to \infty} d(gfx_n, f^2x_n) \leq \Big[\lim_{n \to \infty} d(gfx_n, gt) \cdot \lim_{n \to \infty} d(gt, g^2x_n) \cdot \lim_{n \to \infty} d(gt, f^2x_n) \Big]^{1/3},$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$,

(5) *compatible of type* (P) if

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Now we give some properties related to compatible mappings and its variants in a multiplicative metric space, see [3].

Proposition 1.6. *Let f and g be compatible mappings of type (A) of a multiplicative metric space (X, d) into itself. If one of f and g is continuous, then f and g are compatible.*

Proposition 1.7. *Let f and g compatible mappings of type (B) of a multiplicative metric space (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = ggt = gft$.*

Proposition 1.8. *Let f and g compatible mappings of type (B) of a multiplicative metric space (X, d) itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then:*

(i) $\lim_{n\to\infty} ggx_n = ft$ *if f is continuous at t.*

(ii) $\lim_{n\to\infty} ffx_n = gt$ *if g is continuous at t.*

(iii) $fgt = gft$ *and* $ft = gt$ *if f and g are continuous at t.*

Remark 1.9. (1) In Proposition 1.7, assume that f and g are compatible mappings of type (C) or of type (P) instead of type (B). Then the conclusion of Proposition 1.7 still holds.

(2) In Proposition 1.8, assume that f and g are compatible mappings of type (C) or of type (P) instead of type (B). Then the conclusion of Proposition 1.8 still holds.

2. Main Results

Now we give the following theorem for compatible mappings of type (A).

Theorem 2.1. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) satisfying the following conditions*

(C_1)
$$SX \subset BX, \quad TX \subset AX;$$

(C_2)
$$d(Sx, Ty) \leq M^{\lambda}(x, y)$$

for each $x, y \in X$ and $\lambda \in (0, 1/2)$, where

$$\begin{aligned}
&M(x,y)\\
&= \max\Big\{d(Ax,Sx), d(By,Ty), d(By,Ax), (d(Ax,Ty)\cdot d(By,Sx))^{1/2},\\
&\quad \min\Big\{\frac{d(Ax,Sx)\cdot d(By,Ty)}{d(Ax,By)}, \frac{d(Ax,Ty)\cdot d(By,Sx)}{d(Ax,By)},\\
&\quad \frac{d(Ax,Ty)\cdot d(By,Sx)}{d(Sx,Ty)}\Big\}\Big\};
\end{aligned}$$

(C_3) *one of A, B, S and T is continuous;*

Assume that the pairs A, S and B, T are compatible of type (A). Then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that S is continuous. Since A and S are compatible of type (A), by Proposition 1.6, A and S are compatible. The result easily follows from [4, Theorem 2.1].

Similarly, we can complete the proof when A or B or T is continuous. $\square$

Next we give the following theorem for compatible mappings of type (B).

Theorem 2.2. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the conditions (C_1), (C_2) and (C_3).*

Assume that the pairs A, S and B, T are compatible of type (B). Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $SX \subset BX$ and $TX \subset AX$, there exists $x_1 \in X$ such that $Sx_0 = Bx_1 = y_0$ and for this point x_1, there exists $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way, we can construct a sequence $\{y_n\}$ such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}; \quad y_{2n} = Sx_{2n} = Bx_{2n+1}.$$

From the proof of [4, Theorem 2.1], $\{y_n\}$ is a multiplicative Cauchy sequence in X and hence it converges to some point $z \in X$. Further the subsequence $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converges to z.

Suppose that S is continuous. Then we have

$$SSx_n \to Sz, \quad SAx_{2n} \to Sz \quad \text{as } n \to \infty.$$

Since A and S are compatible of type (B), by Proposition 1.8, Hence we have

$$AAx_{2n} \to Sz \quad \text{as } n \to \infty.$$

Now putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C_2), we have

$$d(SAx_{2n}, Tx_{2n+1}) \leq M^{\lambda}(Ax_{2n}, x_{2n+1}),$$

where

$$\begin{aligned}
&M(Ax_{2n}, x_{2n+1})\\
&= \max\Big\{d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, AAx_{2n}),\\
&\qquad (d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n}))^{1/2},\\
&\qquad \min\Big\{\frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(AAx_{2n}, Bx_{2n+1})},\\
&\qquad\quad \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n})}{d(AAx_{2n}, Bx_{2n+1})},\\
&\qquad\quad \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n})}{d(SAx_{2n}, Tx_{2n+1})}\Big\}\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n \to \infty} M(Ax_{2n}, x_{2n+1})\\
&= \max\{1, 1, d(z, Sz), (d(Sz, z) \cdot d(z, Sz))^{1/2},\\
&\qquad \min\{1/d(Sz, z), d(z, Sz), d(z, Sz)\}\}\\
&= d(Sz, z).
\end{aligned}$$

Hence we have

$$d(Sz, z) \leq d^{\lambda}(Sz, z),$$

which implies that $Sz = z$. Since $SX \subset BX$ there exists a point $u \in X$ such that $z = Sz = Bu$.

Consider $x = Ax_{2n}$ and $y = u$ in (C_2), we have

$$d(SAx_{2n}, Tu) \leq M^{\lambda}(Ax_{2n}, u),$$

where

$$\begin{aligned}
&M(Ax_{2n}, u) \\
&= \max\Bigg\{d(AAx_{2n}, SAx_{2n}), d(Bu, Tu), d(Bu, AAx_{2n}), \\
&\qquad (d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n}))^{1/2}, \min\left\{\frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bu, Tu)}{d(AAx_{2n}, Bu)},\right. \\
&\qquad\quad \left.\frac{d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n})}{d(AAx_{2n}, Bu)}, \frac{d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n})}{d(SAx_{2n}, Tu)}\right\}\Bigg\}.
\end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n \to \infty} M(Ax_{2n}, u) \\
&= \max\{1, d(Sz, Tu), 1, d^{1/2}(Sz, Tu), \min\{d(Sz, Tu), d(Sz, Tu), 1\}\} \\
&= d(Sz, Tu).
\end{aligned}$$

Hence we have

$$d(Sz, Tu) \leq d^{\lambda}(Sz, Tu),$$

which implies that $Tu = Sz = z$. Since B and T are compatible of type (B) and $Bu = z = Tu$, by Proposition 1.7, we have $TBu = BTu$ and so $Bz = BTu = TBu = Tz$.

Next putting $x = x_{2n}$ and $y = z$ in (C_2), we have

$$d(Sx_{2n}, Tz) \leq M^{\lambda}(x_{2n}, z),$$

where

$$\begin{aligned}
&M(x_{2n}, z) \\
&= \max\Bigg\{d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Bz, Ax_{2n}), \\
&\qquad (d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n}))^{1/2}, \min\left\{\frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Tz)}{d(Ax_{2n}, Bz)},\right. \\
&\qquad\quad \left.\frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Sx_{2n}, Tz)}\right\}\Bigg\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}&\lim_{n\to\infty} M(x_{2n}, z)\\ &= \max\{1, 1, d(Tz, z), d(z, Tz), \min\{1/d(z, Tz), d(z, Tz), d(Tz, z))\}\}\\ &= d(z, Tz).\end{aligned}$$

Hence we have

$$d(z, Tz) \le d^{\lambda}(z, Tz),$$

which implies $Tz = z$. Since $TX \subset AX$, there exists a point $v \in X$ such that $z = Tz = Av$.

Now putting $x = v$ and $y = z$ in (C_2), we have

$$d(Sv, Tz) \le M^{\lambda}(v, z),$$

where

$$\begin{aligned}&M(v, z)\\ &= \max\Big\{d(Av, Sv), d(Bz, Tz), d(Bz, Av), (d(Av, Tz)\cdot d(Bz, Sv))^{1/2},\\ &\qquad \min\Big\{\frac{d(Av, Sv)\cdot d(Bz, Tz)}{d(Av, Bz)}, \frac{d(Av, Tz)\cdot d(Bz, Sv)}{d(Av, Bz)},\\ &\qquad \frac{d(Av, Tz)\cdot d(Bz, Sv)}{d(Sv, Tz)}\Big\}\Big\}\\ &= \max\{d(Tz, Sv), 1, 1, d^{1/2}(Tz, Sv), \min\{d(Tz, Sv), d(Tz, Sv), 1\}\}\\ &= d(Tz, Sv).\end{aligned}$$

Hence we have

$$d(Sv, Tz) \le d^{\lambda}(Tz, Sv),$$

which implies that $Sv = Tz = z$. Since A and S are compatible of type (B) and $Sv = z = Av$, it follows from Proposition 1.7 that $Sz = SAv = ASv = Az$. Therefore, $Az = Bz = Sz = Tz = z$ and hence z is a common fixed point of A, B, S and T.

Similarly, we can complete the proof when T is continuous.

Now suppose that A is continuous. Then we have

$$AAx_{2n} \to Az, \quad ASx_{2n} \to Az \quad \text{as } n \to \infty.$$

Since A and S are compatible of type (B), it follows from Proposition 1.8 that

$$SSx_{2n} \to Az \quad \text{as } n \to \infty.$$

Now putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C_2), we have

$$d(SSx_{2n}, Tx_{2n+1}) \leq M^{\lambda}(Sx_{2n}, x_{2n+1}),$$

where

$$\begin{aligned}
&M(Sx_{2n}, x_{2n+1}) \\
&= \max\Bigg\{d(ASx_{2n}, SSx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, ASx_{2n}), \\
&\qquad (d(ASx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n}))^{1/2}, \\
&\qquad \min\Bigg\{\frac{d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(ASx_{2n}, Bx_{2n+1})}, \\
&\qquad\quad \frac{d(ASx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n})}{d(ASx_{2n}, Bx_{2n+1})}, \\
&\qquad\quad \frac{d(ASx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n})}{d(SSx_{2n}, Tx_{2n+1})}\Bigg\}\Bigg\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(Sx_{2n}, x_{2n+1}) \\
&= \max\{1, 1, d(z, Az), d(Az, z), \min\{1/d(Az, z), d(Az, z), d(Az, z)\}\} \\
&= d(Az, z).
\end{aligned}$$

This implies that $Az = z$.

Now putting $x = z$ and $y = x_{2n+1}$ in (C_2), we have

$$d(Sz, Tx_{2n+1}) \leq M^{\lambda}(z, x_{2n+1}),$$

where

$$\begin{aligned}
&M(z, x_{2n+1}) \\
&= \max\Bigg\{d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, Az), \\
&\qquad (d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz))^{1/2}, \\
&\qquad \min\Bigg\{\frac{d(Az, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(Az, Bx_{2n+1})}, \frac{d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz)}{d(Az, Bx_{2n+1})}, \\
&\qquad\quad \frac{d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz)}{d(Sz, Tx_{2n+1})}\Bigg\}\Bigg\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} M(z,z) &= \max\{d(Az,Sz), 1, d(z,Az), d^{1/2}(z,Sz), \\ &\qquad \min\{d(z,Sz), d(z,Sz), 1\}\} \\ &= d(Sz,z). \end{aligned}$$

This implies that $Sz = z$. Since $SX \subset BX$, there exists a point $w \in X$ such that $z = Sz = Bw$.

Next putting $x = z$ and $y = w$ in (C_2), we have

$$d(z,Tw) = d(Sz,Tw) \leq M^{\lambda}(z,w),$$

where

$$\begin{aligned} &M(z,w) \\ &= \max\Bigg\{d(Az,Sz), d(Bw,Tw), d(Bw,Az), (d(Az,Tw)\cdot d(Bw,Sz))^{1/2}, \\ &\qquad \min\Bigg\{\frac{d(Az,Sz)\cdot d(Bw,Tw)}{d(Az,Bw)}, \frac{d(Az,Tw)\cdot d(Bw,Sz)}{d(Az,Bw)}, \\ &\qquad \frac{d(Az,Tw)\cdot d(Bw,Sz)}{d(Sz,Tw)}\Bigg\}\Bigg\} \\ &= d(z,Tw). \end{aligned}$$

This implies that $z = Tw$. Since B and T are compatible of type (B) and $Bw = z = Tw$, by Proposition 1.7, $TBw = BTw$ and so $Bz = BTw = TBw = Tz$.

Now putting $x = z$ and $y = z$ in (C_2), we have

$$d(z,Tz) = d(Sz,Tz) \leq M^{\lambda}(z,z),$$

where

$$\begin{aligned} M(z,z) &= \max\Bigg\{d(Az,Sz), d(Bz,Tz), d(Bz,Az), \\ &\qquad (d(Az,Tz).d(Bz,Sz))^{1/2}, \min\Bigg\{\frac{d(Az,Sz)\cdot d(Bz,Tz)}{d(Az,Bz)}, \\ &\qquad \frac{d(Az,Tz)\cdot d(Bz,Sz)}{d(Az,Bz)}, \frac{d(Az,Tz)\cdot d(Bz,Sz)}{d(Sz,Tz)}\Bigg\}\Bigg\} \\ &= d(z,Tz). \end{aligned}$$

This implies that $z = Tz$. Therefore, z is a common fixed point of A, B, S and T.

Similarly, we can complete the proof when B is continuous.

Finally, suppose that z and w ($z \neq w$) are two common fixed points. Now putting $x = z$ and $y = w$ in (C_2), we have

$$d(z, w) = d(Sz, Tw) \leq M^{\lambda}(z, w),$$

where

$$\begin{aligned}
&M(z, w) \\
&= \max\Big\{d(Az, Sz), d(Bw, Tw), d(Bw, Az), (d(Az, Tw) \cdot d(Bw, Sz))^{1/2}, \\
&\qquad \min\Big\{\frac{d(Az, Sz) \cdot d(Bw, Tw)}{d(Az, Bw)}, \frac{d(Az, Tw) \cdot d(Bw, Sz)}{d(Az, Bw)}, \\
&\qquad \frac{d(Az, Tw) \cdot d(Bw, Sz)}{d(Sz, Tw)}\Big\}\Big\} \\
&= d(z, w),
\end{aligned}$$

which implies that $z = w$. Therefore, z is a unique common fixed point of A, B, S and T. This completes the proof. □

Next, we give the theorem for compatible mappings of type (C).

Theorem 2.3. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the conditions (C_1), (C_2) and (C_3).*

Assume that the pairs A, S and B, T are compatible of type (C). Then A, B, S and T have a unique common fixed point in X.

Proof. From the proof of Theorem 2.2, $\{y_n\}$ is a multiplicative Cauchy sequence in X and hence it converges to some point $z \in X$. Further the subsequence $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converges to z.

Suppose that S is continuous. Then

$$SSx_n \to Sz, \quad SAx_{2n} \to Sz \quad \text{as } n \to \infty.$$

Since A and S are compatible of type (C), by Remark 1.9(2),

$$AAx_{2n} \to Su \quad \text{as } n \to \infty.$$

Now putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C_2), we have

$$d(SAx_{2n}, Tx_{2n+1}) \leq M^{\lambda}(Ax_{2n}, x_{2n+1}),$$

where

$$\begin{aligned}
&M(Ax_{2n}, x_{2n+1})\\
&= \max\Big\{d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, AAx_{2n}),\\
&\quad (d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n}))^{1/2},\\
&\quad \min\Big\{\frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(AAx_{2n}, Bx_{2n+1})},\\
&\qquad \frac{d(AAx_{2n}, Ty) \cdot d(Bx_{2n+1}, SAx_{2n})}{d(AAx_{2n}, Bx_{2n+1})},\\
&\qquad \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n})}{d(SAx_{2n}, Tx_{2n+1})}\Big\}\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(Ax_{2n}, x_{2n+1})\\
&= \max\{1, 1, d(z, Sz), d(Sz, z), \min\{1/d(Sz, z), d(Sz, z), d(Sz, z)\}\}\\
&= d(Sz, z),
\end{aligned}$$

which implies that $Sz = z$. Since $SX \subset BX$, there exists a point $u \in X$ such that $z = Sz = Bu$.

Consider $x = Ax_{2n}$ and $y = u$ in (C_2), we have

$$d(SAx_{2n}, Tu) \le M^{\lambda}(Ax_{2n}, u),$$

where

$$\begin{aligned}
&M(Ax_{2n}, u)\\
&= \max\Big\{d(AAx_{2n}, SAx_{2n}), d(Bu, Tu), d(Bu, AAx_{2n}),\\
&\quad (d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n}))^{1/2},\\
&\quad \min\Big\{\frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bu, Tu)}{d(AAx_{2n}, Bu)}, \frac{d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n})}{d(AAx_{2n}, Bu)},\\
&\qquad \frac{d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n})}{d(SAx_{2n}, Tu)}\Big\}\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(Ax_{2n}, u)\\
&= \max\{1, d(Sz, Tu), 1, d^{1/2}(Tu, Sz), \min\{d(Tu, Sz), d(Tu, Sz), 1\}\}\\
&= d(Sz, Tu).
\end{aligned}$$

Hence we have

$$d(Sz, Tu) \leq d^\lambda(Sz, Tu),$$

which implies that $Tu = Sz = z$. Since B and T are compatible of type (C) and $Bu = z = Tu$, by Remark 1.9(1), we get $TBu = BTu$ and so $Bz = BTu = TBu = Tz$.

Next putting $x = x_{2n}$ and $y = z$ in (C_2), we have

$$d(Sx_{2n}, Tz) \leq M^\lambda(x_{2n}, z),$$

where

$$\begin{aligned}
&M(x_{2n}, z)\\
&= \max\Big\{d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Bu, Ax_{2n}),\\
&\qquad (d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n}))^{1/2}, \min\Big\{\frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bz, Tz)}{d(Ax_{2n}, Bz)},\\
&\qquad \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Tz) \cdot d(Bz, Sx_{2n})}{d(Sx_{2n}, Tz)}\Big\}\Big\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(x_{2n}, z)\\
&= \max\Big\{d(z, z), d(Bz, Tz), d(Bz, z), (d(z, Tz) \cdot d(Bz, z))^{1/2},\\
&\qquad \min\Big\{\frac{d(z, z) \cdot d(Bz, Tz)}{d(z, Bz)}, \frac{d(z, Tz) \cdot d(Bz, z)}{d(z, Bz)}, \frac{d(z, Tz) \cdot d(Bz, z)}{d(z, Tz)}\Big\}\Big\}\\
&= \max\{1, 1, d(Tz, z), d(Tz, z), \min\{1/d(z, Tz), d(Tz, z), d(Tz, z)\}\}\\
&= d(Tz, z).
\end{aligned}$$

This implies that $Tz = z$. Since $TX \subset AX$, there exists a point $v \in X$ such that $z = Tz = Av$.

Now putting $x = v$ and $y = z$ in (C_2), we have

$$d(Sv, Tz) \leq M^\lambda(v, z),$$

where

$$
\begin{aligned}
&M(v,z)\\
&= \max\Big\{ d(Av,Sv), d(Bz,Tz), d(Bz,Av), (d(Av,Tz)\cdot d(Bz,Sv))^{1/2},\\
&\qquad \min\Big\{ \frac{d(Av,Sv)\cdot d(Bz,Tz)}{d(Av,Bz)}, \frac{d(Av,Tz)\cdot d(Bz,Sv)}{d(Av,Bz)},\\
&\qquad\quad \frac{d(Av,Tz)\cdot d(Bz,Sv)}{d(Sv,Tz)}\Big\}\Big\}\\
&= \max\{d(Tz,Sv), 1, 1, d^{1/2}(Tz,Sv), \min\{d(Tz,Sv),1,1\}\}\\
&= d(Tz,Sv).
\end{aligned}
$$

Hence we have

$$d(Sv,Tz) \leq d^{\lambda}(Tz,Sv),$$

which implies that $Sv = Tz = z$. Since A and S are compatible of type (C) and $Sv = z = Av$, it follows from Remark 1.9(1) that $Sz = SAv = ASv = Az$. Therefore $Bz = Az = Tz = Sz = z$ and hence z is a common fixed point of A, B, S and T.

Similarly, we can complete the proof when A or B or T is continuous.

Uniqueness follows easily. Therefore A, B, S and T have a unique common fixed point in X. This completes the proof. □

Finally, we give the theorem for compatible mappings of type (P).

Theorem 2.4. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the conditions (C_1), (C_2) and (C_3).*

Assume that the pairs A, S and B, T are compatible of type (P). Then A, B, S and T have a unique common fixed point in X.

Proof. From the proof of Theorem 2.2, $\{y_n\}$ is a multiplicative Cauchy sequence in X and hence it converges to some point $z \in X$. Further the subsequence $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ of $\{y_n\}$ converges to z.

Suppose that S is continuous. Then

$$SSx_{2n} \to Sz, \quad SAx_{2n} \to Sz \quad \text{as } n \to \infty.$$

Since A and S are compatible of type (P), it follows from Remark 1.9(2) that

$$SSx_{2n} \to Az \quad \text{as } n \to \infty.$$

Now putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C_2), we have

$$d(SAx_{2n}, Tx_{2n+1}) \leq M^{\lambda}(Ax_{2n}, x_{2n+1}),$$

where

$$\begin{aligned}
&M(Ax_{2n}, x_{2n+1})\\
&= \max\Big\{d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(Bx_{2n+1}, AAx_{2n}),\\
&\quad (d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n}))^{1/2},\\
&\quad \min\Big\{\frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{d(AAx_{2n}, Bx_{2n+1})},\\
&\quad \frac{(d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n})}{d(AAx_{2n}, Bx_{2n+1})},\\
&\quad \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n})}{d(SAx_{2n}, Tx_{2n+1})}\Big\}\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(Ax_{2n}, x_{2n+1})\\
&= \max\{d(Sz, Sz), d(z, z), d(z, Sz), (d(Sz, z) \cdot d(z, Sz))^{1/2},\\
&\qquad \min\{1/d(Sz, z), d(Sz, z), d(Sz, z)\}\}\\
&= d(Sz, z).
\end{aligned}$$

Hence we have

$$d(Sz, z) \leq d^{\lambda}(Sz, z),$$

which implies that $Sz = z$. Since $SX \subset BX$, there exists a point $u \in X$ such that $z = Sz = Bu$.

Consider $x = Ax_{2n}$ and $y = u$ in (C_2), we have

$$d(SAx_{2n}, Tu) \leq M^{\lambda}(Ax_{2n}, u),$$

where

$$\begin{aligned}
&M(Ax_{2n}, u)\\
&= \max\Big\{d(AAx_{2n}, SAx_{2n}), d(Bu, Tu), d(Bu, AAx_{2n}),\\
&\quad (d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n}))^{1/2},\\
&\quad \min\Big\{\frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bu, Tu)}{d(AAx_{2n}, Bu)}, \frac{d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n})}{d(AAx_{2n}, Bu)},\\
&\quad \frac{d(AAx_{2n}, Tu) \cdot d(Bu, SAx_{2n})}{d(SAx_{2n}, Tu)}\Big\}\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(Ax_{2n}, u)\\
&= \max\Big\{d(Sz, Sz), d(Sz, Tu), d(Tu, Sz), (d(Sz, Tu)\cdot d(Bu, Sz))^{1/2},\\
&\qquad \min\Big\{\frac{d(Sz, Sz)\cdot d(Sz, Tu)}{d(Sz, Bu)}, \frac{d(Sz, Tu)\cdot d(Tu, Sz)}{d(Sz, Bu)},\\
&\qquad \frac{d(Sz, Tu)\cdot d(Sz, Tu)}{d(Sz, Tu)}\Big\}\Big\}\\
&= d(Sz, Tu).
\end{aligned}$$

Hence we have

$$d(Sz, Tu) \le d^{\lambda}(Sz, Tu),$$

which implies that $Tu = Sz = z = Bu$. Since $B$and T are compatible of type (P), by Remark 1.9(1), we have $Bz = BBu = TTu = Tz$.

Now putting $x = x_{2n}$ and $y = z$ in (C_2), we have

$$d(Sx_{2n}, Tz) \le M^{\lambda}(x_{2n}, z),$$

where

$$\begin{aligned}
&M(x_{2n}, z)\\
&= \max\Big\{d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Bz, Ax_{2n}),\\
&\qquad (d(Ax_{2n}, Tz)\cdot d(Bz, Sx_{2n}))^{1/2}, \min\Big\{\frac{d(Ax_{2n}, Sx_{2n})\cdot d(Bz, Tz)}{d(Ax_{2n}, Bz)},\\
&\qquad \frac{d(Ax_{2n}, Tz)\cdot d(Bz, Sx_{2n})}{d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Tz)\cdot d(Bz, Sx_{2n})}{d(Sx_{2n}, Tz)}\Big\}\Big\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
&\lim_{n\to\infty} M(x_{2n}, z)\\
&= \max\Big\{d(z, z), d(Bz, Tz), d(Bz, z), (d(z, Tz)\cdot d(Bz, z))^{1/2},\\
&\qquad \min\Big\{\frac{d(z, z)\cdot d(Bz, Tz)}{d(z, Bz)}, \frac{d(z, Tz)\cdot d(Bz, z)}{d(z, Bz)}, \frac{d(z, Tz)\cdot d(Bz, z)}{d(z, Tz)}\Big\}\Big\}\\
&= \max\{1, 1, d(Tz, z), d(Tz, z), \min\{1/d(z, Tz), d(Tz, z), d(Tz, z)\}\}\\
&= d(Tz, z).
\end{aligned}$$

Hence we have

$$d(z, Tz) \leq d^{\lambda}(Tz, z),$$

which implies that $Bz = Tz = z$. Since $TX \subset AX$, so there exists a point $v \in X$ such that $z = Tz = Av$.

On putting $x = v$ and $y = z$ in (C_2), we have

$$d(Sv, Tz) \leq M^{\lambda}(v, z),$$

where

$$\begin{aligned} M(v, z) = \max \Bigg\{ & d(Av, Sv), d(Bz, Tz), d(Bz, Av), \\ & (d(Av, Tz) \cdot d(Bz, Sv))^{1/2}, \min \left\{ \frac{d(Av, Sv) \cdot d(Bz, Tz)}{d(Av, Bz)}, \right. \\ & \left. \frac{d(Av, Tz) \cdot d(Bz, Sv)}{d(Av, Bz)}, \frac{d(Av, Tz) \cdot d(Bz, Sv)}{d(Sv, Tz)} \right\} \Bigg\} \\ = & \max\{d(Tz, Sv), 1, 1, d^{1/2}(Tz, Sv), \min\{d(Tz, Sv), 1, 1\}\} \\ = & d(Tz, Sv). \end{aligned}$$

Hence we have

$$d(Sv, Tz) \leq d^{\lambda}(Tz, Sv),$$

which implies that $Sv = Tz = z = Av$. Since A and S are compatible of type (P), by Remark 1.9(1), we have $Sz = SSv = AAv = Az$. Since $Az = Bz = Sz = Tz = z$, we get z is a common fixed point of A, B, S and T.

Similarly, we can complete the proof when A or B or T is continuous.

Uniqueness follows easily. Therefore A, B, S and T have a unique common fixed point in X. This completes the proof. □

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