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ON THE HYPERBOLIC WEIGHTED COMPOSITION OPERATORS

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Abstract: In this present paper, we investigate the hypercyclicity of a hyperbolic weighted composition operator acting on some Banach spaces of holomorphic functions on the open unit ball in $\mathbb{C}^N$.

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1. Introduction

Suppose that $\mathcal{X}$ is a separable Banach space of analytic functions on the open unit ball B_N. The functional of evaluation at λ, $e_\lambda : \mathcal{X} \to \mathcal{C}$ is defined by $e_\lambda(f) = f(\lambda)$ for all $f \in \mathcal{X}$. A complex valued function φ on B_N for which $\varphi\mathcal{X} \subseteq \mathcal{X}$ is called a multiplier of $\mathcal{X}$. The set of all multipliers of $\mathcal{X}$ is denoted by $M(\mathcal{X})$ and it is well-known that $M(\mathcal{X}) \subseteq H^\infty(B_N)$.

For the algebra $\mathcal{B}(\mathcal{X})$ of all bounded linear operators on a Banach space $\mathcal{X}$, the weak operator topology (WOT) is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in \mathcal{X}$. Also, the strong operator topology (SOT) is the one in which a net A_α converges to A if $A_\alpha x \to Ax$, $x \in \mathcal{X}$.

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For $z = (z_1, ..., z_N)$ and $w = (w_1, ..., w_N)$ in $\mathbf{C}^N$, write $< z, w >$ for the Euclidean inner product $\sum_{j=1}^{N} z_j \bar{w}_j$ and let $|z| =< z, z >^{1/2}$. With this notation, the unit ball in $\mathbf{C}^N$ is the set $B_N = \{z \in \mathbf{C}^N : \ |z| < 1\}$ and the unit sphere in $\mathbf{C}^N$ is the set $S_N = \{z \in \mathbf{C}^N : \ |z| = 1\}$, analogously to the unit disc and circle for $N = 1$. The space $H(B_N)$, is the set of all holomorphic fuctions on B_N, can be made into a F-space by a complete metric for which a sequence $\{f_n\}$ in $H(B_N)$ converges to $f \in H(B_N)$ if and only if $f_n \longrightarrow f$ uniformly on every compact subset of B_N. Each $\varphi \in H(B_N)$ and holomorphic self-map ψ of B_N induces a linear weighted composition operator $C_{\varphi,\psi} : H(B_N) \longrightarrow H(B_N)$ by $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$ for every $f \in H(B_N)$ and $z \in B_N$. Indeed, $C_{\varphi,\psi} = M_\varphi C_\psi$ where M_φ denotes the operator of multiplication by φ and C_ψ is a composition operator by means of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(B_N)$.

A bounded linear operator T on a F-space X is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X and in this case we refer to x as a hypercyclic vector for T.

The holomorphic self maps of B_N are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in B_N. It is well known that this map is conjugate to a rotation.

In the following, by ψ_n we denote the nth iterate of ψ.

Theorem 1.1. (see [3]) *Suppose ψ is a holomorphic self-map of the open unit ball B_N without interior fixed point. Then there is a point $w \in \partial B_N$ such that $\psi_n \xrightarrow{k} w$ and $0 < d(w) \leq 1$ where $d(w) = \lim_{|z|\to 1^-} \inf \frac{1 - |\psi(z)|^2}{1 - |z|^2}$.*

The boundary point w is called the Denjoy-Wollf point of ψ. Recall that a holomorphic self-map ψ of B_N is called hyperbolic whenever $d(w) < 1$. A weighted composition operator $C_{\varphi,\psi}$ is called a hyperbolic weighted composition operator whenever the compositional symbol ψ is hyperbolic.

The next section of the present paper shows that weighted composition operators with non-constant weight function and hyperbolic compositional symbol can be hypercyclic on $H(B_N)$. For some sources see [1–7].

2. Main Result

In this section ψ will denote a holomorphic self-map of B_N and φ is a nonzero holomorphic map on B_N.

Theorem 2.1. *Suppose that $\mathcal{X} \subset \mathcal{H}(\mathcal{B_N})$ is a separable Banach space such that $\mathcal{X}$ contains constants, the multiplication operator by the variable z is a contraction on $\mathcal{X}$, and for all $\lambda \in B_N$ the functional of evaluation at λ is bounded on $\mathcal{X}$. Let φ be a nonzero holomorphic map on B_N and ψ be a hyperbolic map of B_N with w the Denjoy-Wolff point such that $\varphi(w) \neq 0$ and $|\varphi(w) - \varphi(\psi_n(z))| \leq d(w)^{n/2}|w - \psi_n(z)|$ for all n and $z \in B_N$. If $|\varphi \circ \psi_n(z)| \leq |\varphi(w)|$ eventually for all n or $\|\psi\|_{B_N} < 1$, then $C^*_{\varphi,\psi}$ fails to be hypercyclic, but $C_{\varphi,\psi}$ is hypercyclic whenever C_ψ is hypercyclic and φ never vanishes on B_N, and also $|\varphi(w)| = 1$.*

Proof. First we show that $M(\mathcal{X}) = \mathcal{H}^\infty(\mathcal{B_N})$.Let $f \in H^\infty(B_N)$. Then by the Farrell-Rubel-Shields Theorem, there is a sequence $\{p_n\}_n$ of polynomials converging to f pointwise and for all n, $\|p_n\|_{B_N} \leq M$ for some $M > 0$. Since $||M_z|| \leq 1$ on H, we get $\|M_q\| \leq \|q\|_{B_N}$ for all polynomials q. Hence we obtain $\|M_{p_n}\| \leq M$ for all n. But ball $B(\mathcal{X})$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(\mathcal{X})$, $M_{p_n} \longrightarrow A$ in the weak operator topology. Using the fact that $M^*_{p_n} \longrightarrow A^*$ in the weak operator topology and acting these operators on e_λ we get $p_n(\lambda)e_\lambda = M^*_{p_n}e_\lambda \longrightarrow A^*e_\lambda$ weakly. Since $p_n(\lambda) \longrightarrow f(\lambda)$, we see that $A^*e_\lambda = f(\lambda)e_\lambda$ from which we can conclude that $A = M_f$ and this implies that $f \in M(\mathcal{X})$. Thus $H^\infty(B_N) \subset M(\mathcal{X})$ and indeed, $M(\mathcal{X}) = \mathcal{H}^\infty(\mathcal{B_N})$.

Now let K be a compact subset of B_N. By Julia's Lemma in B_N ([3]), there exists a constant $c > 0$ such that

$$|1- <\psi_n(z), w>|^2 \leq c(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. But

$$|1- <\psi_n(z), w>|^2 = |w - \psi_n(z)|^2,$$

thus

$$|w - \psi_n(z)|^2 \leq c(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. On the otherhand we note that

$$\begin{aligned} |\varphi(w) - \varphi(\psi_n(z))| \leq & d(w)^{n/2}|w - \psi_n(z)| \\ = & d(w)^{n/2}|1- <\psi_n(z), w>| \\ \leq & c^{1/2}d(w)^{n/2}(1 - |\psi_n(z)|^2)^{1/2}, \end{aligned} \qquad (*)$$

for all n and $z \in B_N$. Also, note that that

$$\frac{|1- <\psi_n(z), w>|^2}{1 - |\psi_n(z)|^2} \leq d(w)^n \frac{|1- <z, w>|^2}{1 - |z|^2},$$

for every $z \in B_N$ and $n \in \mathbb{N}$. Since K is compact, then there exists a constant $\beta > 0$ such that

$$4|1 - < z, w > |^2 < \beta(1 - |z|^2)$$

for all z in K. By a method used in [7], we get

$$\begin{aligned} 1 - |\psi_n(z)|^2 &\leq 2|1 - < \psi_n(z), w > | \\ &\leq 4\frac{|1 - < z, w > |^2}{1 - |z|^2} d(w)^n \\ &< \beta\, d(w)^n. \end{aligned}$$

Now by using the relation $(*)$, we obtain

$$\begin{aligned} |1 - \frac{1}{\varphi(w)}\varphi(\psi_n(z))| &< \frac{c^{\frac{1}{2}} d(w)^{\frac{n}{2}}}{|\varphi(w)|}(1 - |\psi_n(z)^2|)^{1/2} \\ &\leq \frac{c^{\frac{1}{2}}\beta^{\frac{1}{2}}}{|\varphi(w)|} d(w)^n. \end{aligned}$$

Since $0 < d(w) < 1$, $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z))$ converges uniformly on K. Define

$$g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z)).$$

Clearly g is a nonzero holomorphic function on B_N. If $|\varphi \circ \psi_n(z)| \leq |\varphi(w)|$ eventually for all n, then we can see that $g \in H^\infty(B_N)$ and so $g \in \mathcal{X}$. In the case of $\|\psi\|_U < 1$, note that $C^n_{\varphi,\psi} g = \varphi(w)^n g$. Thus

$$g = \varphi(w)^{-n} \prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n$$

which implies that $g \in \mathcal{X}$. Note that since $C_{\varphi,\psi} g = \varphi(w) g$, $C^*_{\varphi,\psi}$ fails to be hypercyclic. Also, note that

$$C_{\varphi,\psi} M_g = M_g(\varphi(w) C_\psi),$$

and g has no zero in B_N whenever φ never vanishes. Thus, M_g is one to one and has dense range and so $C_{\varphi,\psi}$ is quasisimilar to $\varphi(w)C_\psi$. Now if $|\varphi(w)| = 1$ and C_ψ is hypercyclic, then $\varphi(w)C_\psi$ and so $C_{\varphi,\psi}$ is also hypercyclic on $H(B_N)$. This completes the proof. □

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