ON THE HYPERBOLIC WEIGHTED COMPOSITION OPERATORS

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Abstract: In this present paper, we investigate the hypercyclicity of a hyperbolic weighted composition operator acting on some Banach spaces of holomorphic functions on the open unit ball in $\mathbb{C}^N$.

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1. Introduction

Suppose that $\mathcal{X}$ is a separable Banach space of analytic functions on the open unit ball $B_N$. The functional of evaluation at $\lambda$, $e_\lambda : \mathcal{X} \to \mathbb{C}$ is defined by $e_\lambda(f) = f(\lambda)$ for all $f \in \mathcal{X}$. A complex valued function $\varphi$ on $B_N$ for which $\varphi\mathcal{X} \subseteq \mathcal{X}$ is called a multiplier of $\mathcal{X}$. The set of all multipliers of $\mathcal{X}$ is denoted by $M(\mathcal{X})$ and it is well-known that $M(\mathcal{X}) \subseteq H^\infty(B_N)$.

For the algebra $\mathcal{B}(\mathcal{X})$ of all bounded linear operators on a Banach space $\mathcal{X}$, the weak operator topology (WOT) is the one in which a net $A_\alpha$ converges to $A$ if $A_\alpha x \to Ax$ weakly, $x \in \mathcal{X}$. Also, the strong operator topology (SOT) is the one in which a net $A_\alpha$ converges to $A$ if $A_\alpha x \to Ax$, $x \in \mathcal{X}$.
For $z = (z_1, ..., z_N)$ and $w = (w_1, ..., w_N)$ in $\mathbb{C}^N$, write $<z, w>$ for the Euclidean inner product $\sum_{j=1}^N z_j \bar{w}_j$ and let $|z| = <z, z>^{1/2}$. With this notation, the unit ball in $\mathbb{C}^N$ is the set $B_N = \{z \in \mathbb{C}^N : |z| < 1\}$ and the unit sphere in $\mathbb{C}^N$ is the set $S_N = \{z \in \mathbb{C}^N : |z| = 1\}$, analogously to the unit disc and circle for $N = 1$. The space $H(B_N)$, is the set of all holomorphic functions on $B_N$, can be made into a F-space by a complete metric for which a sequence $\{f_n\}$ in $H(B_N)$ converges to $f \in H(B_N)$ if and only if $f_n \rightarrow f$ uniformly on every compact subset of $B_N$. Each $\varphi \in H(B_N)$ and holomorphic self-map $\psi$ of $B_N$ induces a linear weighted composition operator $C_{\varphi, \psi} : H(B_N) \rightarrow H(B_N)$ by $C_{\varphi, \psi}(f)(z) = \varphi(z)f(\psi(z))$ for every $f \in H(B_N)$ and $z \in B_N$. Indeed, $C_{\varphi, \psi} = M_\varphi C_\psi$ where $M_\varphi$ denotes the operator of multiplication by $\varphi$ and $C_\psi$ is a composition operator by means of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(B_N)$.

A bounded linear operator $T$ on a F-space $X$ is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $\text{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in $X$ and in this case we refer to $x$ as a hypercyclic vector for $T$.

The holomorphic self maps of $B_N$ are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $B_N$. It is well known that this map is conjugate to a rotation.

In the following, by $\psi_n$ we denote the nth iterate of $\psi$.

**Theorem 1.1.** (see [3]) Suppose $\psi$ is a holomorphic self-map of the open unit ball $B_N$ without interior fixed point. Then there is a point $w \in \partial B_N$ such that $\psi_n \rightarrow w$ and $0 < d(w) \leq 1$ where $d(w) = \lim_{|z| \rightarrow 1^-} \inf \frac{1 - |\psi(z)|^2}{1 - |z|^2}$.

The boundary point $w$ is called the Denjoy-Wolff point of $\psi$. Recall that a holomorphic self-map $\psi$ of $B_N$ is called hyperbolic whenever $d(w) < 1$. A weighted composition operator $C_{\varphi, \psi}$ is called a hyperbolic weighted composition operator whenever the compositional symbol $\psi$ is hyperbolic.

The next section of the present paper shows that weighted composition operators with non-constant weight function and hyperbolic compositional symbol can be hypercyclic on $H(B_N)$. For some sources see [1–7].

2. Main Result

In this section $\psi$ will denote a holomorphic self-map of $B_N$ and $\varphi$ is a nonzero holomorphic map on $B_N$.
**Theorem 2.1.** Suppose that $\mathcal{X} \subset \mathcal{H}(\mathcal{B}_N)$ is a separable Banach space such that $\mathcal{X}$ contains constants, the multiplication operator by the variable $z$ is a contraction on $\mathcal{X}$, and for all $\lambda \in B_N$ the functional of evaluation at $\lambda$ is bounded on $\mathcal{X}$. Let $\varphi$ be a nonzero holomorphic map on $B_N$ and $\psi$ be a hyperbolic map of $B_N$ with $w$ the Denjoy-Wolff point such that $\varphi(w) \neq 0$ and $|\varphi(w) - \psi_n(z)| \leq d(w)^{n/2}|w - \psi_n(z)|$ for all $n$ and $z \in B_N$. If $|\varphi \circ \psi_n(z)| \leq |\varphi(w)|$ eventually for all $n$ or $\|\psi\|_{B_N} < 1$, then $C_{\varphi, \psi}$ fails to be hypercyclic, but $C_{\varphi, \psi}$ is hypercyclic whenever $\varphi, \psi$ are contractive on $B_N$ and $\varphi$ never vanishes on $B_N$, and also $|\varphi(w)| = 1$.

**Proof.** First we show that $M(\mathcal{X}) = \mathcal{H}^\infty(\mathcal{B}_N)$. Let $f \in H^\infty(B_N)$. Then by the Farrell-Rubel-Shields Theorem, there is a sequence $\{p_n\}_n$ of polynomials converging to $f$ pointwise and for all $n$, $\|p_n\|_{B_N} \leq M$ for some $M > 0$. Since $\|M_z\| \leq 1$ on $H$, we get $\|M_q\| \leq \|q\|_{B_N}$ for all polynomials $q$. Hence we obtain $\|M_{p_n}\| \leq M$ for all $n$. But ball $B(\mathcal{X})$ is compact in the weak operator topology and so by passing to a subsequence if necessary, we may assume that for some $A \in B(\mathcal{X})$, $M_{p_n} \to A$ in the weak operator topology. Using the fact that $M_{p_n}^* \to A^*$ in the weak operator topology and acting these operators on $e_\lambda$ we get $p_n(\lambda)e_\lambda = M_{p_n}^*e_\lambda \to A^*e_\lambda$ weakly. Since $p_n(\lambda) \to f(\lambda)$, we see that $A^*e_\lambda = f(\lambda)e_\lambda$ from which we can conclude that $A = M_f$ and this implies that $f \in M(\mathcal{X})$. Thus $H^\infty(B_N) \subset M(\mathcal{X})$ and indeed, $M(\mathcal{X}) = \mathcal{H}^\infty(\mathcal{B}_N)$.

Now let $K$ be a compact subset of $B_N$. By Julia’s Lemma in $B_N$ ([3]), there exists a constant $C > 0$ such that

$$|1 - \psi_n(z), w|^2 \leq C(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. But

$$|1 - \psi_n(z), w|^2 = |w - \psi_n(z)|^2,$$

thus

$$|w - \psi_n(z)|^2 \leq C(1 - |\psi_n(z)|^2)$$

for every $z \in K$ and every $n \in \mathbb{N}$. On the other hand we note that

$$|\varphi(w) - \varphi(\psi_n(z))| \leq d(w)^{n/2}|w - \psi_n(z)|$$

$$= d(w)^{n/2}|1 - \psi_n(z), w|$$

$$\leq c^{1/2}d(w)^{n/2}(1 - |\psi_n(z)|^2)^{1/2}, \quad (*)$$

for all $n$ and $z \in B_N$. Also, note that that

$$\frac{|1 - \psi_n(z), w|^2}{1 - |\psi_n(z)|^2} \leq d(w)^n \frac{|1 - z, w|^2}{1 - |z|^2},$$
for every \( z \in B_N \) and \( n \in \mathbb{N} \). Since \( K \) is compact, then there exists a constant \( \beta > 0 \) such that

\[
4|1 - z, w|^2 < \beta (1 - |z|^2)
\]

for all \( z \) in \( K \). By a method used in [7], we get

\[
1 - |\psi_n(z)|^2 \leq 2|1 - \psi_n(z), w| \leq 4|1 - z, w|^2 d(w)^n < \beta d(w)^n.
\]

Now by using the relation (\(*\)), we obtain

\[
|1 - \frac{1}{\varphi(w)} \varphi(\psi_n(z))| < \frac{c^2 d(w)^n}{|\varphi(w)|} (1 - |\psi_n(z)|^2)^{1/2} \leq \frac{c^2 \beta^{1/2}}{|\varphi(w)|} d(w)^n.
\]

Since \( 0 < d(w) < 1 \), \( \prod_{n=0}^{\infty} \frac{1}{\varphi(w) \varphi(\psi_n(z))} \) converges uniformly on \( K \). Define

\[
g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w) \varphi(\psi_n(z))}.
\]

Clearly \( g \) is a nonzero holomorphic function on \( B_N \). If \( |\varphi \circ \psi_n(z)| \leq |\varphi(w)| \) eventually for all \( n \), then we can see that \( g \in H^\infty(B_N) \) and so \( g \in \mathcal{X} \). In the case of \( ||\psi||_U < 1 \), note that \( C_{\varphi,\psi}^n g = \varphi(w)^n g \). Thus

\[
g = \varphi(w)^{-n} \prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n
\]

which implies that \( g \in \mathcal{X} \). Note that since \( C_{\varphi,\psi} g = \varphi(w) g \), \( C_{\varphi,\psi}^* \) fails to be hypercyclic. Also, note that

\[
C_{\varphi,\psi} M_g = M_g (\varphi(w) C_\psi),
\]

and \( g \) has no zero in \( B_N \) whenever \( \varphi \) never vanishes. Thus, \( M_g \) is one to one and has dense range and so \( C_{\varphi,\psi} \) is quasisimilar to \( \varphi(w) C_\psi \). Now if \( |\varphi(w)| = 1 \) and \( C_\psi \) is hypercyclic, then \( \varphi(w) C_\psi \) and so \( C_{\varphi,\psi} \) is also hypercyclic on \( H(B_N) \). This completes the proof. \( \square \)
References


